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EL^2–hyperstructures Derived from (Partially) Quasi Ordered Hyperstructures

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Abstract. In this paper, we introduce a new class of (semi)hypergroup from a given (partially) quasi-ordered (semi)hypergroup as a generalization of *"El–hyperstructures"*. Then, we study some basic properties and important elements belong to this class.

Keywords: Ends lemma, *El*–hyperstructure, (semi)Hypergroup, Partial ordering, Quasi ordering.

2000 Mathematics subject classification: 16Y99, 20N20.

1. Motivation

Hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty [17]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [28]. A recent book on hyperstructures [6] points

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out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs.

Semihypergroups (hypergroups) have been found useful for dealing with problems in different areas of algebraic hyperstructures. Many authors studied different aspects of semihypergroups (hypergroups), for instance, P. Bonansinga and P. Corsini [3], Gutan [10], Onipchuk [24], Leoreanu [16], Davvaz and Leorenu [9] and Anvariyeh and et. al. [1, 2], and Hila [12]. The applications of semihypergroups (hypergroups) to areas such as optimization theory, graph theory, theory of discrete event dynamical systems, generalized fuzzy computation, automata theory, formal language theory, coding theory and analysis of computer programs have been extensively studied in the literature, see [6, 8].

This paper concerns a relationship between ordered sets and algebraic hyperstructures. The relation of ordered sets and algebraic hyperstructures first was studied by Vougiouklis in 1987 [27]. Then the connection between hyperstructures and ordered sets has been analyzed by many researchers, such as Vougiouklis [29], Corsini [7], Hoskova [13], Heidari and Davvaz [11] and Novak [19]. One special aspect of this issue, known as El–hyperstructures, was touched upon by Chvalina [4]. He investigated quasi ordered sets and hypergroups. Also, Rosenberg in [26], Hoskova in [14], Rackova in [25], Iampan in [15] and Novak in [23, 20, 22, 21] extended some results on the ordered semigroups and ordered groups connected with El–hyperstructures. El–hyperstructures, mainly studied by M. Novak, are hyperstructures constructed from a (partially) quasi-ordered (semi)groups. More exactly, Novak in [23] considered subhyperstructures of El–hyperstructures and in [20], he discussed some interesting results of important elements in this family of hyperstructures. Then, in [22] Novak studied some basic properties of El–hyperstructures like invertibility, normality, being closed (ultra closed) and etc. An interesting application of Ends Lemma can be also found in [11].

This paper aims at constructing an El–(semi)hypergroup based on a given (partially) quasi-ordered (semi)hypergroups unlike in [20, 22, 23], where the basis of El–(semi)hypergroups are single-valued structures. More precisely, we start from a (partially) quasi-ordered (semi)hypergroup and define a new hyperoperation using Ends lemma. To distinguish this concept from the one studied by Chvalina, Novak et al. we call these hyperstructures EL^2-hyperstructures. Then, we prove the associativity and study the circumstances needed for reproduction axiom to be hold. Also, we consider the subhyperstructures like subhypergroups, hyperideals, prime and minimal hyperideals. Finally, we focus on important elements, (partial or scalar) identities and inverses, in the given hyperstructure and the achieved one and the relations between them (if exist).

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2. Introduction and Preliminaries

In this part, we recall some basic definitions and properties which we consider later. A *hypergroupoid* is a pair $(H, \circ)$, where H is a non-empty set and $\circ : H \times H \longrightarrow \wp^*(H)$ is a *binary hyperoperation* on H. Symbol $\wp^*(H)$ denotes the the system of all nonempty subsets of H. If the *associativity axiom* $a \circ (b \circ c) = (a \circ b) \circ c$ holds for all $a, b, c \in H$, then the pair $(H, \circ)$ is called *semihypergroup*. If moreover the semihypergroup H satisfies $a \circ H = H = H \circ a$, for all $a \in H$, it is called a *hypergroup*. This condition is known as *reproduction axiom*.

In the above definition, if A an B are two non-empty subsets of H and $x \in H$, then $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$ and $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. Also, if for all $(a, b, c) \in H^3$ we have $a \circ (b \circ c) \bigcap (a \circ b) \circ c \neq \emptyset$, then the hyperoperation $\circ$ is called *weak associative* and the hyperstructure $(H, \circ)$ is called an H_v-*semigroup*. Moreover, an H_v-semigroup is called an H_v-*group* if the reproduction axiom holds. The hypergroup $(H, \circ)$ is called a *transposition* hypergroup if it satisfies the following transposition axiom: For all $a, b, c, d \in H$ the relation $a/b \cap c/d \neq \emptyset$ implies that $a \circ d \cap b \circ c \neq \emptyset$, where $a/b = \{x \in H; a \in x \circ b\}$ is called *left extension*. Similarly, *right extension* is defined as $a \setminus b = \{x \in H; b \in a \circ x\}$. A commutative transposition hypergroup is called a *join space* [18]. A nonempty subset $G \subseteq H$ is called a *subhypergroup* of $(H, \circ)$, if $a \circ G = G = G \circ a$ for all $a \in H$. A nonempty subset $I \subseteq H$ is called a left(right) *hyperideal* of $(H, \circ)$, if $H \circ I \subseteq I$ $(I \circ H \subseteq I)$. Also I is a hyperideal of H provided that it is both a left and right hyperideal. The hyperideal I is *minimal* if there is no non-trivial hyperideal J of H with the property $J \subsetneq I$. The hyperideal P is called *prime* if $I \circ J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$ for any hyperideals I and J of semihypergroup $(H, \circ)$. An element $e \in H$ is called a *partial identity* of $(H, \circ)$, if there exists $x \in H$ with the property $x \in x \circ e \cap e \circ x$. An element $e \in H$ is called an *identity* of $(H, \circ)$, if $x \in x \circ e \cap e \circ x$ for all $x \in H$. An element $e \in H$ is called a *scalar identity* of $(H, \circ)$, if $x = x \circ e = e \circ x$ for all $x \in H$. If $(H, \circ)$ is a hypergroup endowed with at least one identity, then an element $a' \in H$ is called an *inverse* of $a \in H$ if there is an identity $e \in H$ such that $e \in a \circ a' \cap a' \circ a$. The hypergroup $(H, \circ)$ is *regular* if it posses at least one identity and each element of H has at least one inverse. The regular hypergroup $(H, \circ)$ is *canonical* if

(1) it is commutative;
(2) it is reversible, which means that if $x \in y \circ z$ then there exist the inverse y^{-1} of y and z^{-1} of z such that $z \in y^{-1} \circ x$ and $y \in x \circ z^{-1}$.

For a deeper insight into the basic hyperstructure theory cf [6].

Since the theory of ordered structures is dealt with ordered relations, we need to recall some definitions in this respect. Binary relation R is called *quasi order* if it is *reflexive* and *transitive*. Also, if the binary relation R is reflexive, transitive and *anti symmetric*, then it is known as a partially order relation.

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By a (partially) quasi ordered (semi)group, we mean a triple $(G, \cdot, R)$, where $(G, \cdot)$ is a (semi)group and R is a (partially) quasi order relation on G such that for all $x, y, z \in G$ with the property xRy there holds $(x \cdot z)R(y \cdot z)$ and $(z \cdot x)R(z \cdot y)$. Moreover, the notation $[x)_R$ used below stands for the set $\{g \in G; xRg\}$ and also $[A)_R = \bigcup_{x \in A} [x)_R$. A nonempty subset I of a (partially) quasi ordered (semi)hypergroup $(H, \circ, R)$ is called a left (right) *ideal* of H if there holds:

(1) H is a left (right) hyperideal of H;
(2) if $b \in I$ and aRb, then $a \in I$ for every $b \in H$.

Finally, I is an ideal of H if it is a two sided ideal.

The *El*–hyperstructures or *Ends lemma* based hyperstructures are hyperstructures constructed from (partially) quasi (semi)groups using "Ends lemma". This concept was first introduced by Chvalina in 1995 [4].

Lemma 2.1. [4, 22]. *Let $(S, \cdot, \leq)$ be a partially ordered semigroup. Binary hyperoperation $\circ : S \times S \longrightarrow \wp^*(S)$ defined by $a \circ b = [a \cdot b)_{\leq} = \{x \in S, a \cdot b \leq x\}$ is associative. The semihypergroup $(S, \circ)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.*

Theorem 2.2. [4, 22]. *Let $(S, \cdot, \leq)$ be a partially ordered semigroup. The following conditions are equivalent:*
a) For any pair $(a, b) \in S^2$, there exists a pair $(c, c_1) \in S^2$ such that $b \cdot c \leq a$ and $c_1 \cdot b \leq a$.
b) The associated semihypergroup $(S, \circ)$ is a hypergroup.

The following theorem extending "Ends lemma" was proved by Rackova in [25].

Theorem 2.3. [25]. *Let $(S, \cdot, \leq)$ be a (partially) quasi ordered group and $(S, \circ)$ be the associated hypergroupoid. Then $(S, \circ)$ is the transposition hypergroup.*

Remark 1. Naturally, if $(S, \cdot)$ is commutative, then $(S, \circ)$ is a join space.

In some articles regarding this topic, mainly by M. Novak, the hyperstructure $(S, \circ)$ constructed in this way is called the *associated hyperstructure* to the single-valued structure $(S, \cdot)$ or an *Ends lemma-based hyperstructure* or an *El–hyperstructure*. Finally, note that the main result in [22] in which M. Novak proved that $(S, \circ)$ is not a canonical hypergroup.

Theorem 2.4. [22]. *Let $(S, \cdot, \leq)$ be a non-trivial quasi-ordered group, where the relation $\leq$ is not the identity relation, and let $(S, \circ)$ be its associated transposition hypergroup. Then $(S, \circ)$ does not have a scalar identity.*

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Corollary 2.5. [22]. *Let $(S, \cdot, \leq)$ be a non-trivial quasi-ordered group, where the relation $\leq$ is not the identity relation, and let $(S, \circ)$ be its associated transposition hypergroup. Then regardless of commutativity $(S, \circ)$ can not be a canonical hypergroup.*

3. Quasi (Partially) Ordered Hypergroups

In this section, we first define a new hyperoperation $*$ on a given (partially) quasi ordered hypergroupoid $(H, \circ, \leq)$ using the hyperstructure version of the Ends lemma and prove the (weak) associativity of $*$. Then, we consider the relation between the two hyperoperation $\circ$ and $*$ by some examples. Also, we focus on the circumstances needed for $(H, *)$ to be an/a H_v-group, hypergroup, transposition hypergroup and join space.

Definition 3.1. [8]. An algebraic hyperstructure $(H, \circ, \leq)$ is called a *(partially) quasi ordered* hypergroupoid if $(H, \circ)$ is a hypergroupoid and " $\leq$ " is a (partially) quasi order relation on H such that for all $a, b, c \in H$ with the property $a \leq b$ we have $a \circ c \leq b \circ c$ and $c \circ a \leq c \circ b$ (monotone condition), where if A and B are non-empty subsets of H, then we say $A \leq B$ whenever for all $a \in A$, there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that $a \leq b$.

Example 1. Let $(X, \leq)$ be a (partially) quasi ordered set and $\emptyset \neq Q \subset X$. If for every $x, y \in X$, we consider $x \circ y = Q$, then $(X, \circ, \leq)$ is a (partially) quasi ordered semihypergroup.

Example 2. Let $(S, \cdot, \leq)$ be a (partially) quasi ordered semigroup. If for every $x, y \in S$, set $x \circ y = \{x^i : i \in \mathbb{N}\}$, then $(S, \circ, \leq)$ is a (partially) quasi ordered semihypergroup.

Example 3. Let $(X, \leq)$ be a (partially) quasi ordered set. If for every $x, y \in X$, we consider $x \circ y = X$, then $(X, \circ, \leq)$ is a (partially) quasi ordered hypergroup.

Example 4. Let $(X, \leq)$ be a (partially) quasi ordered set. If for every $x, y \in X$, set $x \circ y = \{x, y\}$, then $(X, \circ, \leq)$ is a (partially) quasi ordered hypergroup.

Remark 2. In the Definition 1.12 in [11], the term regularly preordered hypergroup is used. Moreover, the Theorem 1.13 in [11] contained a non-trivial example of a (partially) quasi ordered hypergroup.

Definition 3.2. Suppose $(H, \circ)$ is a (partially) quasi ordered hypergroupoid. For $a, b \in H$, we define the new hyperoperation $* : H \times H \longrightarrow \wp^*(H)$ as follows:

$$a * b = [a \circ b)_{\leq} = \bigcup_{m \in a \circ b} [m)_{\leq}.$$

Remark 3. From now on, we name $(H, *)$ as the EL^2-hypergroupoid associated to (partially) quasi ordered hypergroupoid $(H, \circ, \leq)$.

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Proposition 3.3. *Let $(H, \circ, \leq)$ be a (partially) quasi ordered hypergroupoid and $(H, *)$ be its associated EL^2–hypergroupoid. Then $a \circ b \subseteq a * b$ for all $a, b \in H$.*

Proof. Let $t \in a \circ b$. Because $t \leq t$, we conclude that:

$$t \in [t)_{\leq} \subseteq \bigcup_{m \in a \circ b} [m)_{\leq} = a * b.$$

□

Theorem 3.4. *Let $(H, \circ, \leq)$ be a (partially) quasi ordered H_v-semigroup i.e. the hyperoperation $\circ$ is weak associative. Then, the hyperoperation $*$ on H, defined in Definition 3.2, is weak associative and therefore $(H, *)$ is an H_v-semigroup.*

Proof. For all $(a, b, c) \in H^3$, we have $(a \circ b) \circ c \bigcap a \circ (b \circ c) \neq \emptyset$. Now, by Proposition 3.3, we have $(a \circ b) \circ c \subseteq (a * b) * c$ and $a \circ (b \circ c) \subseteq a * (b * c)$, which implies that $(a * b) * c \bigcap a * (b * c) \neq \emptyset$. □

Corollary 3.5. *If $(H, \circ, \leq)$ is a (partially) quasi ordered H_v-group, then $(H, *)$ is a H_v-group.*

Proof. We need to show that $a * H = H = H * a$ for all $a \in H$. We show the first equality. Clearly $a * H \subseteq H$. To prove " $\supseteq$ ", suppose that $x \in H = a \circ H$. So $x \in a \circ h_1$ for some $h_1 \in H$. Hence,

$$x \in [x)_{\leq} \subseteq \bigcup_{m \in a \circ h_1} [m)_{\leq} = a * h_1 \subseteq \bigcup_{h \in H} a * h = a * H.$$

Therefore $H \subseteq a * H$. □

Remark 4. If $(H, \circ, \leq)$ is a (partially) quasi H_v-group, then the associated EL^2–hyperstructure $(H, *)$ need not be a hypergroup i.e. the weak associativity of the hyperoperation $\circ$ need not imply the associativity of the hyperoperation $*$. Consider the following example:

Example 5. Consider $(P = \{0, 1, 2, 3\}, \circ, \leq)$, where " $\leq$ " is ordinary " $\leq$ " relation and hyperoperation " $\circ$ " is given by the following table

then, $(H, \circ, \leq)$ is an H_v-group which is not a hypergroup. Indeed, we have $(1 \circ 2) \circ 3 = 3 \circ 3 = \{0, 3\}$ and $1 \circ (2 \circ 3) = 1 \circ 1 = \{0, 1\}$. Now, the associated EL^2–hyperstructure $(H, *)$ can be shown in the following table:

It can be easily checked that $(1 * 2) * 3 \neq 1 * (2 * 3)$ which implies that hyperoperation $*$ is not associative. So, $(H, *)$ is not a hypergroup. Finally, note that for all $(x, y) \in H^2$, there holds $x \circ y \subseteq x * y$.

In the next theorem, we show how the associativity of hyperoperation $\circ$ implies the associativity of hyperoperation $*$.

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$\circ$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{3\}$	$\{2\}$
2	$\{2\}$	$\{3\}$	$\{0,2\}$	$\{1\}$
3	$\{3\}$	$\{2\}$	$\{1\}$	$\{0,3\}$

TABLE 1

$*$	0	1	2	3
0	$\{0,1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$	$\{3\}$
1	$\{1,2,3\}$	$\{0,1,2,3\}$	$\{3\}$	$\{2,3\}$
2	$\{2,3\}$	$\{3\}$	$\{0,1,2,3\}$	$\{1,2,3\}$
3	$\{3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{0,1,2,3\}$

TABLE 2

Theorem 3.6. *Let $(H,\circ)$ be a (partially) quasi ordered semihypergroup. The hyperoperation $*$ on H, defined in Definition 3.2, is associative and therefore $(H,*)$ is a semihypergroup.*

Proof. Let $a,b,c\in H$. First, we claim that

$$\bigcup_{t\in a*b} t*c=\bigcup_{x\in b*c} a*x. \tag{1}$$

In order to show this equality, we first prove $\supseteq$. Suppose $z\in \bigcup_{x\in b*c} a*x$. So, there exists $x_0\in b*c$ such that $z\in a*x_0$. Hence,

$$x_0\in b*c=\bigcup_{m\in boc}[m)_{\leq}\Rightarrow \exists m_1\in b\circ c \text{ such that } x_0\in[m_1)_{\leq}$$

$$\Rightarrow m_1\leq x_0$$

and

$$z\in a*x_0=\bigcup_{n\in aox_0}[n)_{\leq}\Rightarrow \exists n_1\in a\circ x_0 \text{ such that } z\in[n_1)_{\leq}$$

$$\Rightarrow n_1\leq z.$$

Since $m_1\leq x_0$, for $a\in H$ we have $a\circ m_1\leq a\circ x_0$. Now, $n_1\in a\circ x_0$ implies that there exists $h_1\in a\circ m_1$ such that $h_1\leq n_1$ and so, due to transitivity of

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$\leq$, $h_1 \leq z$ which means that $z \in [h_1)_{\leq}$. On the other hand, $h_1 \in a \circ m_1 \subseteq a \circ (b \circ c) = (a \circ b) \circ c$. So, there exists $t_1 \in a \circ b$ such that $h \in t_1 \circ c$. Now,

$$z \in [h_1)_{\leq} \subseteq \bigcup_{h \in t_1 \circ c} [h)_{\leq} = t_1 * c \subseteq \bigcup_{t \in a * b} t * c.$$

Since, by the reflexive property of $\leq$, we have

$$t_1 \in [t_1)_{\leq} \subseteq \bigcup_{t \in a \circ b} [t)_{\leq} = a * b.$$

By the same argument, we can show that $\bigcup_{t \in a * b} t * c \subset \bigcup_{x \in b * c} a * x$. Finally, we show the associativity of $*$. Suppose $a, b, c \in H$, then considering (1)

$$a * (b * c) = \bigcup_{t \in a * b} t * c = \bigcup_{x \in b * c} a * x = (a * b) * c.$$

□

Corollary 3.7. *If $(H, \circ, \leq)$ is a (partially) quasi ordered hypergroup, then $(H, *)$ is a hypergroup.*

Proof. The proof is the same as the proof of Corollary 3.5. □

In the next to example we present two different relation " $\leq$ " and "$|$" on a single hypergroup $(H, \circ)$ and name the associated EL^2–hypergroups as $(H, *_{\leq})$ and $(H, *_{|})$ respectively in order to comare them.

Example 6. Consider $(H = \{1, 2, 3\}, \circ, \leq)$ where " $\leq$ " is ordinary " $\leq$ " relation and hyperoperation " $\circ$ " is given by the following table

$\circ$	0	1	2
0	$\{0\}$	$\{0, 1\}$	$\{0, 2\}$
1	$\{0, 1\}$	$\{1\}$	$\{1, 2\}$
2	$\{0, 2\}$	$\{1, 2\}$	$\{2\}$.

Table 3

It easy to check that $(H, \circ, \leq)$ is an ordered hypergroup. Also its associated EL^2–hypergroup $(H, *_{\leq})$, can be shown in this table

It can be seen that $a \circ b \subseteq a *_{\leq} b$ for all $a, b \in H$ as suggested by 3.3.

Example 7. If we replace the quasi order " $\leq$ " in Example 6, by "$|$" , the divisibility, then one can easily see that $(H, \circ, |)$ is an ordered hypergroup and the new hypergroup is $(H, *_{|})$.

Again it can be seen that $a \circ b \subseteq a *_{|} b$ for all $a, b \in H$. Note that $(H, *_{\leq})$ and $(H, *_{|})$ are completely different.

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$*_{\leq}$	0	1	2
0	H	H	H
1	H	$\{1,2\}$	$\{1,2\}$
2	H	$\{1,2\}$	$\{2\}$

TABLE 4

$*_{\mid}$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,2\}$
1	$\{0,1,2\}$	$\{1,2\}$	$\{0,1,2\}$
2	$\{0,2\}$	$\{0,1,2\}$	$\{2,1\}$

TABLE 5

EXAMPLE 8. Let $(H,.,\leq)$ be a (partially) quasi semigroup. If for every $x,y\in H$, we define $x\circ y=\{x^i : i\in\mathbb{N}\}$, then $(H,\circ,\leq)$ is a (partially) quasi semihypergroup [11]. Also, $(H,*)$ is a semihypergroup where the hyperoperation $*$ is defined as follows: for every $x,y\in H$, $x*y=\bigcup\limits_{m\in x\circ y}[m)_{\leq}=\bigcup\limits_{i\in\mathbb{N}}[x^i)_{\leq}$. Moreover, it is obvious that $x\circ y\subseteq x*y$ for each $x,y\in H$.

Let $(H,\circ,\leq)$ be a (partially) quasi ordered hypergroup. Then its EL^2–hypergroup $(H,*_{\leq})$ which is not necessarily a transposition hypergroup.

EXAMPLE 9. Suppose $(H=\mathbb{N}-\{1\},\circ,R)$, where $x\circ y=\{x,y\}$ and xRy iff $y|x$. It is easy to check that $(H,\circ,R)$ is a quasi ordered hypergroup. Also, $[x)_R=\{t\in H,xRt\}=\{t\in H,t|x\}$ and $x*y=\bigcup\limits_{m\in\{x,y\}}[m)_R=[x)_R\bigcup[y)_R=\{t,t|x\}\bigcup\{t,t|y\}$. Now, for 3, 5, 7 and 13 in H we have $3/7=\{z,3\in z*7\}=\{3z,z\in H\}$ and $13/5=\{z,13\in z*5\}=\{13z,z\in H\}$. So, $39\in 3/7\bigcap 13/5$ but $3*5=\{3,5\}$ and $13*7=\{13,7\}$ have no element in common i.e. $3*5\bigcap 13*7=\emptyset$. This implies that the associated EL^2–hyperstructure $(H,*)$ is not a transposition hypergroup.

Theorem 3.8. *Suppose $(H,\circ,\leq)$ be a quasi ordered hypergroup in which the trichotomy (or comparability) law holds i.e. for any pair $(a,b)\in H^2$, we have $a\leq b$ or $b\leq a$. Then, the associated EL^2–hypergroup $(H,*)$ is a transposition hypergroup.*

Proof. Suppose $a/b\bigcap c/d\neq\emptyset$. Since, the relation $\leq$ is total for the elements $(a,b,c,d)\in H^4$, we have the following cases:

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(1) $a \leq b$ and $d \leq c$
(2) $a \leq b$ and $c \leq d$
(3) $b \leq a$ and $c \leq d$
(4) $b \leq a$ and $d \leq c$.

In each case, we prove that $a * d \bigcap b * c \neq \emptyset$.

(1) Since $a \leq b$ and $d \leq c$, by monotone condition, we can conclude that $a \circ d \leq b \circ d \leq b \circ c$. Now, for an arbitrary element $y_0 \in a \circ d$ there exists $q_0 \in b \circ c \subseteq b * c$ such that $y_0 \leq q_0$. But, $y_0 \leq q_0$ implies that $q_0 \in [y_0)_{\leq} \subseteq \bigcup_{y \in a \circ d} [y)_{\leq} = a * d$. So, $q_0 \in a * d \bigcap b * c \neq \emptyset$.
(2) Since $a \leq b$ and $c \leq d$, we have $a \circ d \leq b \circ d$ and $b \circ c \leq b \circ d$. For an arbitrary element $t \in b \circ d$ there are $m_1 \in b \circ c \subseteq b * c$ and $n_1 \in a \circ d \subseteq a * d$ such that $m_1 \leq t$ and $n_1 \leq t$. Due to the totality of $\leq$, there are two possibilities: $m_1 \leq n_1$ or $n_1 \leq m_1$. First, suppose $m_1 \leq n_1$ which means that $n_1 \in [m_1)_{\leq} \subseteq \bigcup_{m \in b \circ c} [m)_{\leq} = b * c$. Hence, $n_1 \in a * d \bigcap b * c$. In the case $n_1 \leq m_1$, a similar argument can show the statement.
(3) The relations $b \leq a$ and $c \leq d$ imply that $b \circ c \leq b \circ d \leq a \circ d$. Now, for $q_0 \in b \circ c$, there exists $y_0 \in a \circ d \subseteq a * d$ with the property $q_0 \leq y_0$. Thus, $y_0 \in [q_0)_{\leq} \subseteq \bigcup_{q \in b \circ c} [q)_{\leq} = b * c$. Therefore, $y_0 \in a * d \bigcap b * c \neq \emptyset$.
(4) The proof is similar to the proof of part (2).

□

Example 10. Suppose $(H = \mathbb{R}, \circ, \leq)$, where for all $x \in H$, $x \circ x = x$ and for all x, y such that $x \neq y$, $x \circ y$ is the open interval between x and y and $\leq$ is the ordinary $\leq$ relation. It is easy to check that $(H, \circ, \leq)$ is a totally ordered hypergroup. Also, $[x)_{\leq} = [x, \infty)$ and $x * y = \bigcup_{m \in x \circ y} [m)_{\leq} = [x, \infty)$ or $[y, \infty)$ depending on $x \leq y$ or $y \leq x$ respectively. also, $x/y = \{z \in \mathbb{R} | x \in z * y\} = \mathbb{R}$. Now, for all $(a, b, c, d) \in H^4$ the relation $a/b \bigcap c/d = \mathbb{R} \neq \emptyset$, implies that $a * d \bigcap b * c = [t, \infty) \neq \emptyset$ where $t = min\{a, b, c, d\}$.

Theorem 3.9. *Let $(H, \circ, \leq)$ be a commutative (partially) quasi ordered hypergroup. Then its EL^2–hypergroup $(H, *)$ is commutative.*

Proof. Let $a, b \in H$. There is $a \circ b = b \circ a$. Then

$$a * b = \bigcup_{m \in a \circ b} [m)_{\leq} = \bigcup_{m \in b \circ a} [m)_{\leq} = b * a$$

□

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4. Subhyperstructures of Associated *El*–Hyperstructures

In this section, we focus on the relation between subhyperstructures of $(H, \circ, \leq)$ and $(H, *)$. First of all, we answer the natural question that, is there any relation among sub(semi)hypergroups of $(H, \circ, \leq)$ and $(H, *)$? More exactly is it true that $(K, *)$ is a sub(semi)hypergroup of $(H, *)$, wherever $(K, \circ)$ is a sub(semi)hypergroup of $(H, \circ)$? The following definition is motivated by reasoning in [23], where a detailed discussion of problems connected with the notion of subhyperstructures was performed. In Lemma 4.2 and Theorem 4.3, we generalized results of [23], some of which were included also in [22].

Definition 4.1. Let $(H, \circ, \leq)$ be a (partially) quasi ordered (semi)hypergroup and G be a non empty subset of H. If for all $g \in G$ we have $[g)_{\leq} \subseteq G$, we call G an *upper end* of H. If there exists an element $g \in G$ such that there exists $x \in H \setminus G$ such that $x \in [g)_{\leq}$, we say G is not an upper end of H because of the element x.

Example 11. Consider $(\mathbb{Z}, \circ, |)$ where $|$ is the ordinary divisibility and the hyperoperation $\circ$ is defined as follows:

$a \circ b =< a, b >$, the ideal generated by a and b, for all $a, b \in \mathbb{Z}$,

Then, it is not difficult to check that $(\mathbb{Z}, \circ, |)$ is a partially ordered hypergroup and $I =< 2 >$ is an upper end of $(\mathbb{Z}, \circ)$.

Now we prove a lemma similar to the one proved in [23] for EL–hyperstructures constructed from single-valued structures.

Lemma 4.2. *Let $(H, *)$ be the associated EL^2–semihypergroup of the (partially) quasi ordered (semi)hypergroup $(H, \circ, \leq)$. Let u be the scalar identity of $(H, \circ)$ and $G \subseteq H$. Further, suppose that $(G, \circ)$ is a subhypergroupoid of $(H, \circ)$ i.e $G \circ G \subseteq G$. Then,*

1. *if G is an upper end of H, then $(G, *)$ is a subhypergroupoid of $(H, *)$;*
2. *if G is not an upper end of H and $u \in G$, then $(G, *)$ is not a subhypergroupoid of $(H, *)$;*
3. *the statement in part (2) is valid in the case that u does not exist (or $u \notin G$) yet for some $a, b \in G$, there exists $c \in a \circ b$, where $c \in G$ is such that there exists an element x_i because of which G is not an upper end of H such that $c \leq x_i$;*
4. *the couple $(G, *)$ is a subhypergroupoid of $(H, *)$ provided that the following conditions are hold simultaneously:*
   (a) u does not exist or $u \notin G$.
   (b) G is not an upper end of H because of elements x_i, $i \in I$.
   (c) For every $a, b, c \in G$ there holds $c \in a \circ b$ and all triples are such that for no x_i there holds $c \leq x_i$.

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Proof. (1) Suppose a, b are an arbitrary pair in G. We have $a * b = \bigcup_{m \in a \circ b} [m)_{\leq}$. Now, as G is an upper end of H, for each $m \in a \circ b$ there holds $[m)_{\leq} \subseteq G$. So, $a * b \subseteq G$ which implies that $G * G \subseteq G$.

(2) Since G is not an upper end of H, there exists an element $g_1 \in G$ with the property $[g_1)_{\leq} \nsubseteq G$. Now, for $u \in G$ we see $u * g_1 = \bigcup_{m \in u \circ g_1} [m)_{\leq} = [g_1)_{\leq} \nsubseteq G$, which means that $G * G \nsubseteq G$.

(3) It is easy to check. Indeed, there exists an element $c \in a \circ b$ for which there, by definition, holds $[c)_{\leq} \nsubseteq G$. Hence, $a * b = \bigcup_{m \in a \circ b} [m)_{\leq} \nsubseteq G$ which implies that $G * G \nsubseteq G$. Elements $a, b, c \in G$ are those defined in part (3).

(4) For any pair $a, b \in G$, we have $a * b = \bigcup_{m \in a \circ b} [m)_{\leq}$ where for each $m \in a \circ b$ there holds $[m)_{\leq} \subseteq G$. Hence $G * G \subseteq G$.

□

The following Theorem is similar to the on proved in [23] for EL–hyperstructures constructed from single-valued structures.

Theorem 4.3. *Let $(H, *)$ be the associated EL^2–(semi)hypergroup of the (partially) quasi ordered (semi)hypergroup $(H, \circ, \leq)$ and G is an upper end of H. If $(G, \circ)$ is a subhypergroup of $(H, \circ)$, then $(G, *)$ is a subhypergroup of $(H, *)$*

Proof. Since hyperoperation $*$ is associative in H, so for all $a, b, c \in G \subseteq H$ we have $a * (b * c) = (a * b) * c$. To complete the proof, we must show that $a * G = G = G * a$ for any $a \in G$. Since $(G, \circ)$ is a subhypergroup of $(H, \circ)$, we can conclude that $a \circ G = G$ and by Proposition 3.3, $a \circ G \subseteq a * G$, so $G \subseteq a * G$. In order to prove $\supseteq$, suppose an arbitrary element $b \in G$. Then $a \circ b \subseteq G$ and for all $m \in a \circ b$ there holds $[m)_{\leq} \subseteq G$ (since G is an upper end of H). Hence $a * b = \bigcup_{m \in a \circ b} [m)_{\leq} \subseteq G$. Therefore $a * G \subseteq G$. Similarly, it can be proved that $G * a = G$. □

In the next theorem, we study the hyperideals of $(H, \circ, \leq)$ and $(H, *)$.

Theorem 4.4. *Let $(H, \circ, \leq)$ be a (partially) quasi semihypergroup and $(H, *)$ is the associated EL^2–semihypergroup.*

(1) *If I is a right (left) ideal (hyperideal) and, in addition, an upper end of $(H, \circ, \leq)$, then I is a right (left) hyperideal of $(H, *)$.*

(2) *Every right (left) hyperideal of $(H, *)$ is a right (left) hyperideal (not necessarily an ideal) of $(H, \circ, \leq)$.*

Proof. (1) It is enough to show that $I * H \subseteq I$ ($H * I \subseteq I$). We show the first one.

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Let $x \in I$ and $y \in H$. Then $x \circ y \subseteq I$. But I is an upper end of H, so $[m)_{\leq} \subseteq I$ for every $m \in x \circ y$. Consequently, $x * y = \bigcup_{m \in x \circ y} [m)_{\leq} \subseteq I$.

Finally, we have

$$I * H = \bigcup_{x \in I, y \in H} x * y \subseteq I.$$

(2) It is straightforward since $I \circ H \subseteq I * H \subseteq I$.

□

Example 12. Consider $(\mathbb{Z}, \circ, |)$ and $I =< 2 >$ defined in Example 11. Clearly I is a two sided ideal of $(\mathbb{Z}, \circ, |)$. So by Theorem 4.4, I is a two sided hyperideal of $(\mathbb{Z}, *)$.

Corollary 4.5. *Let I be a minimal (hyper) ideal of a (partially) quasi ordered semihypergroup $(H, \circ, \leq)$ which is also an upper end of H. Then I is a minimal hyperideal of $(H, *)$.*

Proof. By Theorem 4.4, I is a hyperideal of $(H, *)$. Suppose $J \subseteq I$ for some non-trivial hyperideal J of $(H, *)$. Now, by the second part of Theorem 4.4, we conclude that J is a hyperideal of $(H, \circ)$, which implies that $I = J$. □

Corollary 4.6. *Let P be a prime (hyper) ideal of a (partially) quasi ordered semihypergroup $(H, \circ, \leq)$ which is also an upper end of H. Then, P is a prime hyperideal of $(H, *)$.*

Proof. Suppose I and J are two nonempty hyperideals of $(H, *)$ and $I * J \subseteq P$. By Theorem 4.4, I and J are hyperideal of $(H, \circ)$. On the other hand, $I \circ J \subseteq I * J \subseteq P$ which implies that $I \subseteq P$ or $J \subseteq P$. □

5. Important Elements

In this section, we consider the important elements like identities, scalar identities and inverses, if they exist, in $(H, \circ, \leq)$ and $(H, *)$ and the relation between them. M. Novak in [20, 22] showed that an *El*–hypergroup derived from a quasi ordered group does not have a scalar identity. So, in commutative case, it can not be a canonical hypergroup and further a Krasner hyperring. In the following, we evaluate these results in EL^2–hypergroups derived from a hypergroup, i.e. EL–hypergroups derived from hyperstructures.

Theorem 5.1. *Let $(H, *)$ be the associated EL^2–hypergroup of (partially) quasi ordered hypergroup $(H, \circ, \leq)$ and e is an (a) identity (partial identity) in $(H, \circ, \leq)$. Then e is an (a) identity (partial identity) in $(H, *)$.*

Proof. We prove the first statement. By the hypothesis, $x \in x \circ e \cap e \circ x$ for all $x \in H$. But by Proposition 3.3, $x \circ e \subseteq x * e$ and $e \circ x \subseteq e * x$. Hence

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$x \in x * e \cap e * x$.
The second proof is similar. □

Corollary 5.2. *Let $(H, *)$ and $(H, \circ, \leq)$ be as in the Theorem 5.1 and e be a scalar identity in $(H, \circ, \leq)$. Then e is an identity in $(H, *)$.*

Proof. Straightforward verification. □

In the next theorem, it is shown that a scalar identity in $(H, \circ, \leq)$ can not be a scalar identity in $(H, *)$.

Theorem 5.3. *Let $(H, *)$ be the associated EL^2–hypergroup of (partially) quasi ordered hypergroup $(H, \circ, \leq)$ and e be scalar identity in $(H, \circ, \leq)$. Then e is not a scalar identity in $(H, *)$ whenever the relation " $\leq$ " is non-trivial.*

Proof. Suppose, by contradiction, $a = a * e = e * a$ for all $a \in H$. Then $a = \bigcup_{m \in a \circ e} [m]_{\leq}$. But $a \circ e = a$ and consequently $a = [a]_{\leq}$ for all $a \in H$. This means that the relation $\leq$ is trivial, a contradiction. □

Corollary 5.4. *Let $(H, \circ, \leq)$ be a non-trivial (partially) quasi ordered canonical hypergroup. Then $(H, *)$, the associated EL^2–hypergroup, can never be a canonical hypergroup and further a hyperring.*

Naturally, it comes to the reader's mind that what happens if $(H, *)$, itself, has some scalar identities And further, is there exists any relation between scalar identities of $(H, *)$ and scalar identities of $(H, \circ, \leq)$?

Theorem 5.5. *Let $(H, \circ, \leq)$ be (partially) quasi ordered hypergroup with scalar identity u and $(H, *)$ be the associated EL^2–hypergroup with scalar identity e. Then $e = u$*

Proof. Since u is a scalar identity in $(H, \circ, \leq)$ for $e \in H$, there holds $e = e \circ u = u \circ e$. Also e is a scalar identity in $(H, *)$. Hence, for $u \in H$ we have $e = e * u = u * e$. Now, $e = e \circ u \subseteq e * u = u$. □

As in classic algebra, the issues of inverse elements and identities in hyperstructure theory are relatively closed to each other. Let us now concentrate on the concept of inverse elements of the EL^2–hyperstructures introduced in this paper.

Theorem 5.6. *Let $(H, \circ, \leq)$ be (partially) quasi ordered hypergroup endowed with scalar identity u and $(H, *)$ be its EL^2–hypergroup. In addition, suppose the element $a \in H$ has an inverse in $(H, \circ, \leq)$ denoted by a^{-1}. Then a^{-1} is an inverse of a in $(H, *)$.*

Proof. Since a^{-1} is the inverse of a in $(H, \circ, \leq)$ we have $u \in a \circ a^{-1} \cap a^{-1} \circ a$. By Theorem 5.1, u is an identity in $(H, *)$. Also, by Proposition 3.3, we have, $a \circ a^{-1} \subseteq a * a^{-1}$ and $a^{-1} \circ a \subseteq a^{-1} * a$. Therefore $u \in a * a^{-1} \cap a^{-1} * a$, which means that a^{-1} is an inverse of a in $(H, *)$. □

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Corollary 5.7. *Let $(H, \circ, \leq)$ be (partially) quasi ordered hypergroup which is also regular. Then the associated EL^2–hypergroup $(H, *)$ is regular.*

Proof. There is at least one identity, named u, in $(H, \circ, \leq)$ and each element has at least one inverse in $(H, \circ, \leq)$. Now, by Theorem 5.1, u is an identity in $(H, *)$ and by Theorem 5.6 inverse elements in $(H, \circ, \leq)$ can be regarded as inverse elements in $(H, *)$. Therefore, $(H, *)$ is a regular hypergroup. □

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