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Stability of g-Frame Expansions

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Abstract. In this paper we investigate the stability of one-sided perturbation to g-frame expansions. We show that if Λ is a g-frame of a Hilbert space $\mathcal{H}$, $\Lambda_i^a = \Lambda_i + \Theta_i$ where $\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)$, and $\widetilde{f} = \sum_{i\in J} \Lambda_i^\star \widetilde{\Lambda}_i^a f$, $\widehat{f} = \sum_{i\in J} (\Lambda_i^a)^\star \widetilde{\Lambda_i} f$, then $\|\widehat{f} - f\| \leq \alpha\|f\|$ and $\|f - \widetilde{f}\| \leq \beta\|f\|$ for some α and β.

Keywords: g-Frames, g-Riesz bases, g-Orthonormal bases, Dual g-frames.

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1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [6]. Their work on frames was somewhat forgotten until 1986 when Daubechies, Grossmann and Meyer [5] brought it all back to life during their fundamental work on wavelets. Frames have many nice properties which make them very useful in many fields.Various generalizations of frames have been proposed; frame of subspaces [4], subfusion frames [1], frame in a 2-inner product space [2] and so on.

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Sun introduced g-frames in a complex Hilbert space and discussed their properties (see [16]). Recently, g-frames in Hilbert spaces have been studied intensively, for more details see [8, 9, 10, 11] and the references therein. G-frames in complex Hilbert spaces have some properties similar to those of frames, but not all the properties are similar (see [15]). In this paper we give some properties of g-frame and generalize some of the results from frame theory to g-frame.

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space and $\{\mathcal{H}_i\}_{i\in J}$ is a sequence of separable Hilbert spaces, where J is a subset of $\mathbb{Z}$, $\mathcal{L}(\mathcal{H},\mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_i$.

A frame for a complex Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i\in J}$ so that there are two positive constants A and B satisfying

$$A\|f\|^2 \leq \sum_{i\in J} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, f \in \mathcal{H}.$$

The constants A and B are called lower and upper frame bounds.

For each sequence $\{\mathcal{H}_i\}_{i\in J}$, we define the space $\sum_{i\in J}\bigoplus \mathcal{H}_i$ by

$$\sum_{i\in J}\bigoplus \mathcal{H}_i = \{\{f_i\}_{i\in J} : f_i \in \mathcal{H}_i,\ i \in J \text{ and } \sum_{i\in J}\|f_i\|^2 < \infty\}.$$

With the inner product defined by

$$\langle\{f_i\},\{g_i\}\rangle = \sum_{i\in J}\langle f_i, g_i\rangle,$$

it is clear that $\sum_{i\in J}\bigoplus \mathcal{H}_i$ is a Hilbert space.

A sequence $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in J\}$ is called a generalized frame, or simply a g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ if there exist two positive constants A_Λ and B_Λ such that, for all $f \in \mathcal{H}$,

$$A_\Lambda\|f\|^2 \leq \sum_{i\in J}\|\Lambda_i f\|^2 \leq B_\Lambda\|f\|^2.$$

The constants A_Λ and B_Λ are called the lower and upper g-frame bounds, respectively. The sequence $\widetilde{\Lambda} = \{\widetilde{\Lambda_i} \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in J\}$ is called a dual g-frame of Λ if it is g-frame and $f = \Sigma_{i\in J}\Lambda_i^\star\widetilde{\Lambda_i} f$ for all $f \in \mathcal{H}$.

Theorem 1.1. *([15]) Let Λ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$. The operator*

$$S : \mathcal{H} \to \mathcal{H},\ Sf = \textstyle\sum_{i\in J}\Lambda_i^\star\Lambda_i f,$$

is a positive invertible operator and every $f \in \mathcal{H}$ has an expansion

$$f = \textstyle\sum_{i\in J} S^{-1}\Lambda_i^\star\Lambda_i f = \sum_{i\in J}\Lambda_i^\star\Lambda_i S^{-1} f.$$

In particular, $\{\Lambda_i S^{-1} \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in J\}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ and is called canonical dual g-frame of Λ. The operator S is called the g-frame operator of Λ.

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Definition 1.2. Let Λ be a g-frame for $\mathcal{H}$. The synthesis operator for Λ is the operator

$$T_\Lambda : \sum_{i\in J}\bigoplus \mathcal{H}_i \longrightarrow \mathcal{H},$$

defined by $T_\Lambda(\{f_i\}_{i\in J}) = \sum_{i\in J} \Lambda_i^\star(f_i)$.

Proposition 1.3. *([13]) Λ is a g-frame for $\mathcal{H}$ with upper bound B_Λ if and only if the synthesis operator is well defined from $\sum_{i\in J}\bigoplus \mathcal{H}_i$ onto $\mathcal{H}$ and bounded with $\|T_\Lambda\| \leq \sqrt{B_\Lambda}$.*

We say that the g-frame Λ is g-complete, if $\{f : \Lambda_i f = 0, \text{for all } i \in J\} = \{0\}$ and g-orthonormal basis for $\mathcal{H}$, if

$$\langle \Lambda_i^\star g_i, \Lambda_j^\star g_j\rangle = \delta_{i,j}\langle g_i, g_j\rangle,\ i,j \in J,\ g_i \in \mathcal{H}_i,\ g_j \in \mathcal{H}_j,$$

and

$$\sum_{i\in J} \|\Lambda_i f\|^2 = \|f\|^2.$$

We say that Λ is a g- Riesz basis for $\mathcal{H}$, if it is g-complete and there exist constants $0 < A \leq B < \infty$, such that for any finite subset $I \subseteq J$ and $g_i \in \mathcal{H}_i$, $i \in I$,

$$A\sum_{i\in I} \|g_i\|^2 \leq \|\sum_{i\in I} \Lambda_i^\star g_i\|^2 \leq B\sum_{i\in I} \|g_i\|^2.$$

Proposition 1.4. *([3]) Let $F \in \mathcal{L}(\mathcal{H})$ and $G : \mathcal{H} \to \mathcal{H}$ be linear. If there exist two constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Gh - Fh\| \leq \lambda_1\|F(h)\| + \lambda_2\|G(h)\|,\ h \in \mathcal{H}$$

then $G \in \mathcal{L}(\mathcal{H})$, is invertible and

$$\frac{1-\lambda_2}{1+\lambda_1}\frac{1}{\|F\|}\|h\| \leq \|G^{-1}h\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|F^{-1}\|\|h\|,\ h \in \mathcal{H}.$$

Proposition 1.5. *([14]) Let $F : \mathcal{H} \to \mathcal{H}$ be invertible on $\mathcal{H}$. Suppose that $G : \mathcal{H} \to \mathcal{H}$ is a bounded operator and $\|Gh - Fh\| \leq \nu\|h\|$, for all $h \in \mathcal{H}$, where $\nu \in [0, \frac{1}{\|F^{-1}\|})$. Then*

i) *G is invertible on $\mathcal{H}$ and $G^{-1} = \sum_{k=0}^{\infty}[F^{-1}(F - G)]^k F^{-1}$;*
ii) *$\frac{1}{1+\nu\|F^{-1}\|}\frac{1}{\|F\|}\|h\| \leq \|G^{-1}h\| \leq \frac{1}{\frac{1}{\|F^{-1}\|}-\nu}\|h\|$, for all $h \in \mathcal{H}$.*

Proof. (i) is proved in [7]. For (ii) observe that

$$\|Gh - Fh\| \leq \nu\|F^{-1}\|\|F(h)\|$$

and apply Proposition 1.4 with $\lambda_1 = \nu\|F^{-1}\| < 1$ and $\lambda_2 = 0$. □

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2. Main Results

Theorem 2.1. *Let Λ be a g-frame of $\mathcal{H}$ with an upper bound B_Λ and $D, B' \geq 0$, $\Lambda_i^a = \Lambda_i + \Theta_i$ where $\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ satisfies*

$$\|\sum_{i\in J}\Theta_i^\star g_i\| \leq D\|\sum_{i\in J}\Lambda_i^\star g_i\| + \sqrt{B'}(\sum_{i\in J}\|g_i\|^2)^{\frac{1}{2}},$$

for every $g_i \in \mathcal{H}_i$, keeping Λ^a a g-frame. Assume further that $\widetilde{\Lambda}$ is a dual g-frame of Λ with an upper bound $B_{\widetilde{\Lambda}}$ and that $\widetilde{\Lambda}^a$ is a dual g-frame of Λ^a with an upper bound $B_{\widetilde{\Lambda}^a}$. For given $f \in \mathcal{H}$, if $\widetilde{f} = \sum_{i\in J}\Lambda_i^\star\widetilde{\Lambda}_i^a f$, $\widehat{f} = \sum_{i\in J}(\Lambda_i^a)^\star\widetilde{\Lambda_i} f$, $f^a = \sum_{i\in J}(\widetilde{\Lambda}_i^a)^\star\Lambda_i f$ and $f^b = \sum_{i\in J}(\widetilde{\Lambda}_i)^\star\Lambda_i^a f$, then

$$\|\widehat{f} - f\| \leq (D + \sqrt{B'B_{\widetilde{\Lambda}}})\|f\|,\ \|f - \widetilde{f}\| \leq (D\sqrt{B_{\widetilde{\Lambda}^a}B_\Lambda} + \sqrt{B'B_{\widetilde{\Lambda}^a}})\|f\|,$$

$$\|f - f^a\| \leq (D\sqrt{B_{\widetilde{\Lambda}^a}B_\Lambda} + \sqrt{B'B_{\widetilde{\Lambda}^a}})\|f\|,\ \|f^b - f\| \leq (D + \sqrt{B'B_{\widetilde{\Lambda}}})\|f\|.$$

Proof. Since $\widetilde{\Lambda}$ is a dual g-frame of Λ we have $f = \sum_{i\in J}\Lambda_i^\star\widetilde{\Lambda}_i f$. Hence

$$\widehat{f} - f = \sum_{i\in J}\Theta_i^\star\widetilde{\Lambda}_i f,$$

thus

$$\begin{aligned}\|\widehat{f} - f\| &= \|\sum_{i\in J}\Theta_i^\star\widetilde{\Lambda}_i f\| \leq D\|\sum_{i\in J}\Lambda_i^\star\widetilde{\Lambda}_i f\| + \sqrt{B'}(\sum_{i\in J}\|\widetilde{\Lambda}_i f\|^2)^{\frac{1}{2}}\\ &\leq D\|f\| + \sqrt{B'B_{\widetilde{\Lambda}}}\|f\| = (D + \sqrt{B'B_{\widetilde{\Lambda}}})\|f\|.\end{aligned}$$

If $f \in \mathcal{H}$, we have $f = \sum_{i\in J}(\Lambda_i^a)^\star\widetilde{\Lambda}_i^a f$, and hence

$$\begin{aligned}f - \widetilde{f} &= \sum_{i\in J}(\Lambda_i^a)^\star\widetilde{\Lambda}_i^a f - \sum_{i\in J}\Lambda_i^\star\widetilde{\Lambda}_i^a f\\ &= \sum_{i\in J}((\Lambda_i^a)^\star - \Lambda_i^\star)\widetilde{\Lambda}_i^a f = \sum_{i\in J}\Theta_i^\star\widetilde{\Lambda}_i^a f,\end{aligned}$$

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by Proposition 1.3 we know the synthesis operator T_Λ for Λ is bounded by $\sqrt{B_\Lambda}$. Thus

$$\begin{aligned}
\|f-\widetilde{f}\| = \|\sum_{i\in J}\Theta_i^\star\widetilde{\Lambda}_i^a f\| &\leq D\|\sum_{i\in J}\Lambda_i^\star\widetilde{\Lambda}_i^a f\| + \sqrt{B'}(\sum_{i\in J}\|\widetilde{\Lambda}_i^a f\|^2)^{\frac{1}{2}} \\
&\leq D\|T_\Lambda\{\widetilde{\Lambda}_i^a f\}_{i\in J}\| + \sqrt{B'}(\sum_{i\in J}\|\widetilde{\Lambda}_i^a f\|^2)^{\frac{1}{2}} \\
&\leq D\|T_\Lambda\|\|\{\widetilde{\Lambda}_i^a f\}_{i\in J}\| + \sqrt{B'}(\sum_{i\in J}\|\widetilde{\Lambda}_i^a f\|^2)^{\frac{1}{2}} \\
&\leq D\|T_\Lambda\|(\sum_{i\in J}\|\widetilde{\Lambda}_i^a f\|^2)^{\frac{1}{2}} + \sqrt{B'}(\sum_{i\in J}\|\widetilde{\Lambda}_i^a f\|^2)^{\frac{1}{2}} \\
&\leq D\sqrt{B_{\widetilde{\Lambda}^a}B_\Lambda}\|f\| + \sqrt{B'B_{\widetilde{\Lambda}^a}}\|f\| \\
&= (D\sqrt{B_{\widetilde{\Lambda}^a}B_\Lambda} + \sqrt{B'B_{\widetilde{\Lambda}^a}})\|f\|.
\end{aligned}$$

Now for $f, g \in \mathcal{H}$ we have

$$\begin{aligned}
f - f^a &= \sum_{i\in J}(\widetilde{\Lambda}_i^a)^\star\Lambda_i^a f - \sum_{i\in J}(\widetilde{\Lambda}_i^a)^\star\Lambda_i f \\
&= \sum_{i\in J}(\widetilde{\Lambda}_i^a)^\star(\Lambda_i^a - \Lambda_i)f = \sum_{i\in J}(\widetilde{\Lambda}_i^a)^\star\Theta_i f,
\end{aligned}$$

and hence

$$\begin{aligned}
\langle f - f^a, g\rangle &= \langle\sum_{i\in J}(\widetilde{\Lambda}_i^a)^\star\Theta_i f, g\rangle = \sum_{i\in J}\langle\Theta_i f, \widetilde{\Lambda}_i^a g\rangle \\
&= \sum_{i\in J}\langle f, \Theta_i^\star\widetilde{\Lambda}_i^a g\rangle = \langle f, \sum_{i\in J}\Theta_i^\star\widetilde{\Lambda}_i^a g\rangle \\
&= \langle f, g - \widetilde{g}\rangle.
\end{aligned}$$

Thus, for $g = f - f^a$ we have

$$\begin{aligned}
\|f - f^a\|^2 &= |\langle f, f - f^a - \widetilde{(f - f^a)}\rangle| \\
&\leq \|f\|\|f - f^a - \widetilde{(f - f^a)}\| \\
&\leq \|f\|(D\sqrt{B_{\widetilde{\Lambda}^a}B_\Lambda} + \sqrt{B'B_{\widetilde{\Lambda}^a}})\|f - f^a\|.
\end{aligned}$$

Hence

$$\|f - f^a\| \leq (D\sqrt{B_{\widetilde{\Lambda}^a}B_\Lambda} + \sqrt{B'B_{\widetilde{\Lambda}^a}})\|f\|.$$

With similar calculations as above we have

$$f^b - f = \sum_{i\in J}\widetilde{\Lambda}_i^\star\Theta_i f \text{ and } \langle f^b - f, g\rangle = \langle f, \widehat{g} - g\rangle.$$

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Thus, for $g = f^b - f$ we have

$$\begin{aligned}\|f^b - f\|^2 &= |\langle f, \widehat{f^b - f} - f^b - f)\rangle| \\ &\leq \|f\|\|(\widehat{f^b - f}) - f^b - f\| \\ &\leq \|f\|(D + \sqrt{B'B_{\widetilde{\Lambda}^a}})\|f^b - f\|.\end{aligned}$$

□

Lemma 2.2. *Let Λ be a g-frame of $\mathcal{H}$, and let $\Lambda_i^a = \Lambda_i + \Theta_i$ where $\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is a g-Riesz basis with an upper bound B' keeping Λ^a a g-frame. Assume further that $\widetilde{\Lambda}$ is a dual g-frame of Λ with an upper bound $B_{\widetilde{\Lambda}}$ and that $\widetilde{\Lambda}^a$ is a dual g-frame of Λ^a with an upper bound $B_{\widetilde{\Lambda}^a}$. If $\widetilde{f}$, $\widehat{f}$, f^a and f^b are as in Theorem 2.1, then*

$$\|f - \widetilde{f}\| \leq \sqrt{B_{\widetilde{\Lambda}^a}B'}\|f\|,\ \|\widehat{f} - f\| \leq \sqrt{B_{\widetilde{\Lambda}}B'}\|f\|$$

and

$$\|f - f^a\| \leq \sqrt{B_{\widetilde{\Lambda}^a}B'}\|f\|,\ \|f^b - f\| \leq \sqrt{B_{\widetilde{\Lambda}}B'}\|f\|.$$

Proof. Since $\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is a g-Riesz basis with upper bound B', the results follow from Theorem 2.1 with $D = 0$. □

By the above results and this fact that every g-orthonormal basis for $\mathcal{H}$ is a g-Riesz basis for $\mathcal{H}$ with bounds 1 (see [13]), it is easy to show that

$$\|f - \widetilde{f}\| \leq \sqrt{B_{\widetilde{\Lambda}^a}}\|f\| \text{ and } \|\widehat{f} - f\| \leq \sqrt{B_{\widetilde{\Lambda}}}\|f\|,$$

$$\|f - f^a\| \leq \sqrt{B_{\widetilde{\Lambda}^a}}\|f\| \text{ and } \|f^b - f\| \leq \sqrt{B_{\widetilde{\Lambda}}}\|f\|.$$

Proposition 2.3. *Let Λ be a g-frame of $\mathcal{H}$, with lower frame bound A and let $\Lambda_i^a = \Lambda_i + \Theta_i$ where $\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)$, $\|\Theta_i\| < \delta_i$ and $0 < (\sum_{i\in J}\delta_i^2)^{\frac{1}{2}} \leq \delta$ with $\sqrt{A} > \delta$. Then Λ^a is a g-frame. Assume further that $\widetilde{\Lambda}$ is a dual g-frame of Λ with upper bound $B_{\widetilde{\Lambda}}$ and $\widetilde{\Lambda}^a$ is a dual g-frame of Λ^a with upper bound $B_{\widetilde{\Lambda}^a}$. If $\widetilde{f}$, $\widehat{f}$, f^a and f^b are as in Theorem 2.1, then*

$$\|f - \widetilde{f}\| \leq \delta\sqrt{B_{\widetilde{\Lambda}^a}}\|f\| \text{ and } \|f - \widehat{f}\| \leq \delta\sqrt{B_{\widetilde{\Lambda}}}\|f\|$$

$$\|f - f^a\| \leq \delta\sqrt{B_{\widetilde{\Lambda}^a}}\|f\| \text{ and } \|f - f^b\| \leq \delta\sqrt{B_{\widetilde{\Lambda}}}\|f\|.$$

Proof. Since

$$\sum_{i\in J}\|(\Lambda_i^a - \Lambda_i)f\|^2 = \sum_{i\in J}\|\Theta_i f\|^2 \leq \sum \delta_i^2\|f\|^2 \leq \delta^2\|f\|^2,$$

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and $\sqrt{A} > \delta > 0$, by Theorem 3.1 of [16], Λ^a is a g-frame. If $g_i \in \mathcal{H}_i$, then

$$\begin{aligned} \|\sum_{i\in J}\Theta_i^\star g_i\| &\leq \sum_{i\in J}\|\Theta_i^\star\|\|g_i\| \\ &\leq (\sum_{i\in J}\delta_i^2)^{\frac{1}{2}}(\sum_{i\in J}\|g_i\|^2)^{\frac{1}{2}} \leq \delta(\sum_{i\in J}\|g_i\|^2)^{\frac{1}{2}}. \end{aligned}$$

The proof is complete by using Theorem 2.1. □

In [12], the authors investigated the stability of perturbation to frame expansions. We give similar results for the case of g-frame.

Theorem 2.4. *Let Λ be a g-frame of $\mathcal{H}$, and let $\Lambda_i^a = \Lambda_i + \Theta_i$ where $\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ and $\|\Theta_i\| < \delta$ and $\delta > 0$ is fixed so that Λ^a remains a g-frame. Assume further that $\widetilde{\Lambda}$ is a dual g-frame of Λ with an upper bound $B_{\widetilde{\Lambda}}$ and that $\widetilde{\Lambda}^a$ is a dual g-frame of Λ^a with upper bound $B_{\widetilde{\Lambda}^a}$. Let Θ be such that $\langle\Theta_i^\star g_i, \Theta_j^\star g_j\rangle = 0$ for $g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j$, and $|i-j| > K$ with some positive $K \geq 1$. For given $f \in \mathcal{H}$, we have*

$$\|f - \widetilde{f}\| \leq \delta(2K+1)^{\frac{1}{2}}\sqrt{B_{\widetilde{\Lambda}^a}}\|f\|$$

and

$$\|f - \widehat{f}\| \leq \delta(2K+1)^{\frac{1}{2}}\sqrt{B_{\widetilde{\Lambda}}}\|f\|.$$

Proof. If $f \in \mathcal{H}$, we have $f = \sum_{i\in J}(\Lambda_i^a)^\star\widetilde{\Lambda}_i^a f$, and so $f - \widetilde{f} = \sum_{i\in J}\Theta_i^\star\widetilde{\Lambda}_i^a f$. Since $\Theta = \{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is such that $\langle\Theta_i^\star g_i, \Theta_j^\star g_j\rangle = 0$ for $g_i \in \mathcal{H}_i$, $g_j \in \mathcal{H}_j$, $|i-j| > K$, given $i \in J$ we have

$$\begin{aligned} |\sum_{j\in J}\langle\Theta_i^\star\widetilde{\Lambda}_i^a f, \Theta_j^\star\widetilde{\Lambda}_j^a f\rangle| &= |\sum_{|i-j|\leq K}\langle\Theta_i^\star\widetilde{\Lambda}_i^a f, \Theta_j^\star\widetilde{\Lambda}_j^a f\rangle| \\ &\leq \sum_{|i-j|\leq K}|\langle\Theta_i^\star\widetilde{\Lambda}_i^a f, \Theta_j^\star\widetilde{\Lambda}_j^a f\rangle| \\ &\leq \sum_{|i-j|\leq K}\|\Theta_i^\star\widetilde{\Lambda}_i^a f\|\|\Theta_j^\star\widetilde{\Lambda}_j^a f\| \\ &\leq \sum_{|i-j|\leq K}\frac{\|\Theta_i^\star\widetilde{\Lambda}_i^a f\|}{2}^2 + \sum_{|i-j|\leq K}\frac{\|\Theta_j^\star\widetilde{\Lambda}_j^a f\|}{2}^2 \\ &\leq \sum_{|i-j|\leq K}\frac{\|\Theta_i^\star\|^2\|\widetilde{\Lambda}_i^a f\|}{2}^2 + \sum_{|i-j|\leq K}\frac{\|\Theta_j^\star\|^2\|\widetilde{\Lambda}_j^a f\|}{2}^2 \\ &\leq \frac{\delta^2}{2}\sum_{|i-j|\leq K}\|\widetilde{\Lambda}_i^a f\|^2 + \frac{\delta^2}{2}\sum_{|i-j|\leq K}\|\widetilde{\Lambda}_j^a f\|^2 \\ &\leq \frac{\delta^2}{2}(2K+1)\|\widetilde{\Lambda}_i^a f\|^2 + \frac{\delta^2}{2}\sum_{|i-j|\leq K}\|\widetilde{\Lambda}_j^a f\|^2. \end{aligned}$$

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Thus

$$\|f-\widetilde{f}\|^2 \leq \delta^2(2K+1)\sum_{i\in J}\|\widetilde{\Lambda}_i^a f\|^2 \leq \delta^2(2K+1)B_{\widetilde{\Lambda}^a}\|f\|^2,$$

and hence

$$\|f-\widetilde{f}\| \leq \delta(2K+1)^{\frac{1}{2}}\sqrt{B_{\widetilde{\Lambda}^a}}\|f\|.$$

If $f\in\mathcal{H}$, then we have $f=\sum_{i\in J}\Lambda_i^\star\widetilde{\Lambda}_i f$, and so $\widehat{f}-f=\sum_{i\in J}\Theta_i^\star\widetilde{\Lambda}_i f$. Thus

$$\|\widehat{f}-f\|^2 \leq \delta^2(2K+1)\sum_{i\in J}\|\widetilde{\Lambda}_i f\|^2 \leq \delta^2(2K+1)B_{\widetilde{\Lambda}}\|f\|^2,$$

and so

$$\|\widehat{f}-f\| \leq \delta(2K+1)^{\frac{1}{2}}\sqrt{B_{\widetilde{\Lambda}}}\|f\|.$$

□

Theorem 2.5. *Let Λ be a g-frame of $\mathcal{H}$, and let $\Lambda_i^a=\Lambda_i+\Theta_i$ where $\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i)$ and $\|\Theta_i\|<\delta$ and $\delta>0$ is fixed so that Λ^a remains a g-frame. Assume further that $\widetilde{\Lambda}$ is a dual g-frame of Λ with an upper bound $B_{\widetilde{\Lambda}}$ and that $\widetilde{\Lambda}^a$ is a dual g-frame of Λ^a with upper bound $B_{\widetilde{\Lambda}^a}$. Let Θ be such that $|\langle\Theta_i^\star g_i,\Theta_j^\star g_j\rangle|=\frac{\delta^2}{r^{|i-j|}}\langle g_i,g_j\rangle$ for $g_i\in\mathcal{H}_i, g_j\in\mathcal{H}_j$, and some $r>1$. Then there exist finite constants $C,C'>0$ such that*

$$\|f-\widetilde{f}\|\leq C\delta\|f\| \text{ and } \|f-\widehat{f}\|\leq C'\delta\|f\|.$$

Proof. For fixed $i\in J$, we have

$$\begin{aligned}|\sum_j\langle\Theta_i^\star\widetilde{\Lambda}_i^a f,\Theta_j^\star\widetilde{\Lambda}_j^a f\rangle| &\leq (\sum_{j=i}+\sum_{j>i}+\sum_{j<i})|\langle\Theta_i^\star\widetilde{\Lambda}_i^a f,\Theta_j^\star\widetilde{\Lambda}_j^a f\rangle|\\ &= I(i)+II(i)+III(i).\end{aligned}$$

Since $I(i)=|\langle\Theta_i^\star\widetilde{\Lambda}_i^a f,\Theta_i^\star\widetilde{\Lambda}_i^a f\rangle|=\|\Theta_i^\star\widetilde{\Lambda}_i^a f\|^2$,

$$\sum_{i\in J}I(i)=\sum_{i\in J}\|\Theta_i^\star\widetilde{\Lambda}_i^a f\|^2\leq\delta^2 B_{\widetilde{\Lambda}^a}\|f\|^2.$$

Also, since $II(i)=\sum_{j>i}|\langle\Theta_i^\star\widetilde{\Lambda}_i^a f,\Theta_j^\star\widetilde{\Lambda}_j^a f\rangle|\leq\sum_{j>i}\frac{\delta^2}{r^{j-i}}|\langle\widetilde{\Lambda}_i^a f,\widetilde{\Lambda}_j^a f\rangle|$, it follows that

$$\begin{aligned}\sum_{i\in J}II(i) &\leq \sum_i\sum_{j>i}\frac{\delta^2}{r^{j-i}}|\langle\widetilde{\Lambda}_i^a f,\widetilde{\Lambda}_j^a f\rangle|\leq\sum_i\sum_{j>i}\frac{\delta^2}{r^{j-i}}\|\widetilde{\Lambda}_i^a f\|\|\widetilde{\Lambda}_j^a f\|\\ &\leq \sum_i\sum_{j>i}\frac{\delta^2}{2r^{j-i}}(\|\widetilde{\Lambda}_i^a f\|^2+\|\widetilde{\Lambda}_j^a f\|^2)=\frac{\delta^2}{2}(R+S),\end{aligned}$$

with

$$R=\sum_i\sum_{j>i}\frac{1}{r^{j-i}}\|\widetilde{\Lambda}_i^a f\|^2\leq\frac{1}{r-1}\sum_i\|\widetilde{\Lambda}_i^a f\|^2\leq\frac{B_{\widetilde{\Lambda}^a}}{r-1}\|f\|^2$$

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and

$$S=\sum_i\sum_{j>i}\frac{1}{r^{j-i}}\|\widetilde{\Lambda}_j^a f\|^2=\sum_j\sum_{i<j}\frac{1}{r^{j-i}}\|\widetilde{\Lambda}_j^a f\|^2\leq\frac{B_{\widetilde{\Lambda}^a}}{r-1}\|f\|^2.$$

Therefore

$$\sum_{i\in J}II(i)\leq\frac{\delta^2}{2}(R+S)\leq\frac{B_{\widetilde{\Lambda}^a}\delta^2}{r-1}\|f\|^2.$$

For the last term, similarly we have

$$\sum_{i\in J}III(i)\leq\frac{B_{\widetilde{\Lambda}^a}\delta^2}{r-1}\|f\|^2.$$

Hence

$$\begin{aligned}\|f-\widetilde{f}\|^2 &= \sum_i\sum_j\langle\Theta_i^\star\widetilde{\Lambda}_i^a f,\Theta_j^\star\widetilde{\Lambda}_j^a f\rangle\leq(B_{\widetilde{\Lambda}^a}\delta^2+\frac{B_{\widetilde{\Lambda}^a}\delta^2}{r-1}+\frac{B_{\widetilde{\Lambda}^a}\delta^2}{r-1})\|f\|^2\\ &= B_{\widetilde{\Lambda}^a}\delta^2\frac{r+1}{r-1}\|f\|^2,\end{aligned}$$

and so

$$\|f-\widetilde{f}\|\leq\sqrt{B_{\widetilde{\Lambda}^a}}\sqrt{\frac{r+1}{r-1}}\delta\|f\|\ or\ \|f-\widetilde{f}\|\leq C\delta\|f\|.$$

Similarly we have $\|f-\widehat{f}\|\leq C'\delta\|f\|$ where $C'=\sqrt{B_{\widetilde{\Lambda}}}\sqrt{\frac{r+1}{r-1}}$. □

Theorem 2.6. *Let Λ be a g-frame of $\mathcal{H}$, and let $\Lambda_i^a=\Lambda_i+\Theta_i$ where $\Theta_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i)$ and $\|\Theta_i\|<\delta$ and $\delta>0$ is fixed so that Λ^a remains a g-frame. Assume further that $\widetilde{\Lambda}^a$ is a dual g-frame of Λ^a. Let Θ be such that $|\langle\Theta_i^\star g_i,\Theta_j^\star g_j\rangle|=\frac{\delta^2}{r^{|i-j|}}\langle g_i,g_j\rangle$ for $g_i\in\mathcal{H}_i,g_j\in\mathcal{H}_j$, and some $r>1$, or $\langle\Theta_i^\star g_i,\Theta_j^\star g_j\rangle=0$ for $|i-j|>K$ with some positive $K\geq1$. Then there exist finite constants $C,C'>0$ such that*

$$\|f-f^a\|\leq C\delta\|f\|\ and\ \|f-f^b\|\leq C'\delta\|f\|.$$

Proof. By Theorems 2.4, 2.5, there exists $C>0$ such that

$$\|g-\widetilde{g}\|\leq C\delta\|g\|\ \forall g\in\mathcal{H}.$$

Now, for $f,g\in\mathcal{H}$ we have

$$\langle f-f^a,g\rangle=\langle f,g-\widetilde{g}\rangle.$$

Thus, for $g=f-f^a$ we have

$$\|f-f^a\|^2 \leq \|f\|C\delta\|f-f^a\|,$$

and so

$$\|f-f^a\|\leq C\delta\|f\|.$$

Again, by Theorems 2.4, 2.5 there exists $C'>0$ such that $\|\widehat{g}-g\|\leq C'\delta\|g\|$, for all $g\in\mathcal{H}$. Now, imitating the first part of the proof, we deduce that $\|f^b-f\|\leq C'\delta\|f\|$.

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By the above Theorems and Proposition we conclude

$$\|\widehat{f} - f\| \leq C\|f\|,\ \|f - \widetilde{f}\| \leq C'\|f\|,$$

$$\|f - f^a\| \leq D\|f\|,\ \|f^b - f\| \leq D'\|f\|,$$

for some C, C', D, D'. By Proposition 1.5, if $C < 1$ then the operator $R : \mathcal{H} \to \mathcal{H}$ defined by $R(f) = \sum_{i\in J} \Lambda_i^\star \widetilde{\Lambda}_i^a f$ is a bounded invertible operators,

$$f = \sum_{i\in J} \Lambda_i^\star \widetilde{\Lambda}_i^a R^{-1} f,$$

$$\frac{1}{1+C}\|f\| \leq \|R^{-1}f\| \leq \frac{1}{1-C}\|f\|,$$

and if $C' < 1$ the operator $G : \mathcal{H} \to \mathcal{H}$ defined by $G(f) = \sum_{i\in J} (\Lambda_i^a)^\star \widetilde{\Lambda}_i f$ is a bounded invertible operator,

$$f = \sum_{i\in J} (\Lambda_i^a)^\star \widetilde{\Lambda}_i G^{-1} f$$

and

$$\frac{1}{1+C'}\|f\| \leq \|G^{-1}f\| \leq \frac{1}{1-C'}\|f\|,$$

and if $D < 1$ the operator $K : \mathcal{H} \to \mathcal{H}$ defined by $Kf = \sum_{i\in J} (\widetilde{\Lambda}_i^a)^\star \Lambda_i f$ is a bounded invertible operator,

$$f = \sum_{i\in J} (\widetilde{\Lambda}_i^a)^\star \Lambda_i K^{-1} f$$

and

$$\frac{1}{1+D}\|f\| \leq \|K^{-1}f\| \leq \frac{1}{1-D}\|f\|,$$

and if $D' < 1$ the operator $L : \mathcal{H} \to \mathcal{H}$ defined by $Lf = \sum_{i\in J} (\widetilde{\Lambda}_i)^\star \Lambda_i^a f$ is a bounded invertible operator,

$$f = \sum_{i\in J} (\widetilde{\Lambda}_i)^\star \Lambda_i^a L^{-1} f$$

and

$$\frac{1}{1+D'}\|f\| \leq \|K^{-1}f\| \leq \frac{1}{1-D'}\|f\|.$$

□

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