Iranian Journal of Mathematical Sciences and Informatics
Vol. 11, No. 2 (2016), pp 119-130
DOI: 10.7508/ijmsi.2016.02.009

Double Sequence Iteration for a Strongly Contractive Mapping in the Modular Space

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Abstract. In this paper, two double sequence iterations for a strongly ρ-contractive mapping in the modular space are introduced. It is proved that these sequence iterations are strongly convergent to a fixed point of a strongly ρ-contractive mapping. Finally, some examples are presented to show the novelty of the main result.

Keywords: Strongly ρ-contraction, Modular space, Double sequence, Strongly convergence.

2000 Mathematics subject classification: 47H10, 54H10.

1. Introduction

The notion of modular space, as a generalization of a metric space, was introduced by Nakano [13] in 1950 and redefined and generalized by Musielak and Orlicz in 1959 [12]. These spaces were developed following the successful theory of Orlicz spaces, which replaces the particular, integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties.
The existence of a fixed point (or a common fixed point) for a map has a broad set of application and because of these it is studied by many authors (such as

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Received 08 May 2014; Accepted 13 August 2016

[3] and [16]). On the other hand, fixed point theorems in the modular space, generalizing the classical Banach fixed point theorem in the metric space, have been studied extensively, for example in [1], [6], [7], [14], etc.

In 1974, Ishikawa [5] introduced a new iteration method and proved that it converges strongly to a fixed point of a Lipschitz pseudocontractive map defined in the Hilbert space. This result has been extended to continuous pseudocontractive maps defined on compact convex subsets of the Hilbert and Banach spaces. The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically k-strict pseudocontractive type mappings were studied by several authors (see [4], [8], [9] and [15]).

In 2002, Moore [10], introduced a double sequence iteration

$$x_{k,n+1} = (1-\alpha_n)x_{k,n} + \alpha_n T_k x_{k,n},$$

where $T_k x = (1-a_k)z + a_k Tx$ for all $x \in C$ where C is a bounded, closed, convex and nonempty subset of a real Hilbert H and also $\{\alpha_n\}_{n\geq}$ and $\{a_k\}_{k\geq 0}$ are sequences in $(0,1)$. He proved that the double sequence $\{x_{k,n}\}$ is strongly convergent to a fixed point of T in C.

Moradi et al. [11], introduced new double sequence iterations and proved that these sequences are strongly convergent to a fixed point of a ρ-quasi contraction mapping in the modular space.

In this paper, based on [11], we define two double sequence iterations for a strongly ρ-contractive mapping in the modular space. Then we show these sequences are strongly convergent to a fixed point of the strongly ρ-contractive mapping. Finally, some numerical examples are presented. In order to do this, the definition of the modular space is recalled (see [2], [6], [7] and [14]).

Definition 1.1. Let X be an arbitrary vector space over $K = (\mathcal{R}$ or $\mathcal{C})$.
$a)$ A functional $\rho : X \longrightarrow [0,\infty]$ is called modular if it satisfies the following conditions:
$i)$ $\rho(x) = 0 \ iff \ x = 0$.
$ii)$ $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$, for all $x \in X$.
$iii)$ $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, for all $x, y \in X$.
If $iii)$ is replaced by:
$iii)'$ $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for α, $\beta \geq 0$ with $\alpha + \beta = 1$, for all $x, y \in X$.
Then the modular ρ is called a convex modular.
$b)$ A modular ρ defines a corresponding modular space, i.e. the space X_ρ is given by:

$$X_\rho = \{x \in X \ |\rho(\alpha x) \to 0 \ as \ \alpha \to 0\}.$$

$c)$ If ρ is convex modular, the modular X_ρ can be equipped with a norm called the Luxemburg norm defined by:

$$\|x\|_\rho = \inf\{\alpha > 0; \quad \rho(\frac{x}{\alpha}) \leq 1\}.$$

Definition 1.2. Let X_ρ be a modular space.
a) A sequence $(x_n)_{n\in\mathcal{N}}$ in X_ρ is said to be:
i) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$.
ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.
b) X_ρ is ρ-complete if every ρ-Cauchy sequence is ρ-convergent.
c) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_{n\in\mathcal{N}} \subset B$ and $\rho(x_n - x) \to 0$ then $x \in B$.
d) A subset $B \subset X_\rho$ is called ρ-bounded if $\delta_\rho(B) = \sup \rho(x - y) < \infty$ for all $x, y \in B$, where $\delta_\rho(B)$ is called the ρ-diameter of B.
e) A function $f : X_\rho \to X_\rho$ is called ρ-continuous if $\rho(x_n - x) \to 0$, then $\rho(f(x_n) - f(x)) \to 0$.

Let C be a nonempty subset of a modular space X_ρ, $T : C \longrightarrow C$ be a mapping and $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T.

Lemma 1.3. [17] *Assume $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in real number such that

(I) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$.
(II) $\limsup_{n\longrightarrow\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\Sigma_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

Definition 1.4. Let X_ρ be a modular space and C be a nonempty subset of X_ρ. A mapping $T : C \to C$ is said to be strongly ρ-contraction if there exist $c, l, q \in \mathbb{R}^+$ with $c > l$ and $q \in (0, 1)$ such that,

$$\rho(c(Tx - Ty)) \leq q\rho(l(x - y)), \tag{1.1}$$

for all $x, y \in C$.

Theorem 1.5. [14] *Let X_ρ be a ρ-complete modular space where ρ is convex and satisfies the Δ_2-condition. Let C be a nonempty and ρ-closed subset of X_ρ and $T : C \to C$ be a strongly ρ-contractive mapping. Then there exists a unique fixed point of T in C.*

2. Strongly ρ-Contractive Mappings

In this section, we study the strongly convergence of the double sequences to a fixed point of a strongly ρ-contractive mapping in the modular space.

Definition 2.1. [10] Let X_ρ be a modular space and $\mathbb{N}$ denote the set of all natural numbers. We consider the function $f : \mathbb{N} \times \mathbb{N} \to X_\rho$ defined by $f(n, m) = x_{n,m} \in X_\rho$. A double sequence $\{x_{n,m}\}$ is said to be strongly ρ-convergence to z if for given any $\epsilon > 0$ there exist N, $M > 0$ such that

$\rho(x_{n,m} - z) < \epsilon$ for all $n \geq N$, $m \geq M$. If for all $n, r \geq N$ and $m, t \geq M$, we have that $\rho(x_{n,r} - x_{m,t}) < \epsilon$, then the double sequence is said to be ρ-Cauchy. Furthermore, if for each fixed n, $\rho(x_{n,m} - z_n) \longrightarrow 0$ as $m \to \infty$ and $\rho(z_n - z) \longrightarrow 0$ as $n \to \infty$, then $\rho(x_{n,m} - z) \longrightarrow 0$ as $n, m \to \infty$.

Theorem 2.2. *Let X_ρ be a ρ-complete modular space where ρ is convex and satisfies the Δ_2-condition. Let C be a ρ-closed and convex subset of X_ρ and $T : C \to C$ be a uniformly continuous and strongly ρ-contractive mapping. For an arbitrary but fixed $z \in C$, define $T_k : C \to C$ by $T_k x = (1 - \lambda_k)z + \lambda_k T x$, for all $x \in C$ such that $\frac{1}{2} < \lambda_k < 1$ for all $k \geq 0$. Let $\{\beta_{n,i}\}$, $i = 0, \cdots, m$ be real sequences in $(0, 1)$. Assume the following conditions are satisfied:*

(I): $\lim_{n \longrightarrow \infty} \beta_{n,m} = 0$.
(II): $0 < \liminf_{n \longrightarrow \infty} \beta_{n,i} \leq \limsup_{n \longrightarrow \infty} \beta_{n,i} < 1$, *for all* $i = 0, \cdots, m-1$.
(III): $\lim_{k \to \infty} \lambda_k = 1$.
(IV): *For all $x \in C$ and $c_0 \in \mathbb{R}^+$, $\rho(c_0(x - z)) \leq \nu < \infty$.*
(V): $F(T) \neq \emptyset$.

Then the double sequence $\{x_{k,n}\}_{k \geq 0, n \geq 0}$ generated from an arbitrary $x_{0,0} \in C$ by

$$\begin{cases} x^1_{k,n} = (1 - \beta_{n,0})x_{k,n} + \beta_{n,0} T_k x_{k,n}, \\ x^2_{k,n} = (1 - \beta_{n,1})x_{k,n} + \beta_{n,1} T_k x^1_{k,n}, \\ \vdots \\ x^m_{k,n} = (1 - \beta_{n,m-1})x_{k,n} + \beta_{n,m-1} T_k x^{m-1}_{k,n}, \\ x_{k,n+1} = (1 - \beta_{n,m})x_{k,n} + \beta_{n,m} T_k x^m_{k,n}, \end{cases} \tag{2.1}$$

is strongly convergent to a fixed point w of T in C and $\rho(Tx_{k,n} - x_{k,n}) \longrightarrow 0$.

Proof. For each $k \geq 0$ we have,

$$\begin{aligned} \rho(c(T_k x - T_k y)) &= \rho(c((1 - \lambda_k)z + \lambda_k T x - (1 - \lambda_k)z - \lambda_k T y)) \\ &= \rho(c\lambda_k(Tx - Ty)) \\ &\leq q\lambda_k \rho(l(x - y)). \end{aligned}$$

By Theorem 1.5, since $\lim_{k \to \infty} \lambda_k = 1$, for each $k \geq 0$, T_k has a unique fixed point w_k in C.

For each $k \geq 0$, we show $\rho(x_{k,n} - w_k) \longrightarrow 0$ as $n \longrightarrow \infty$. By (2.1),

$$\begin{aligned}
\rho(c(x_{k,n+1} - w_k)) \leq\ & (1-\beta_{n,m})\rho(c(x_{k,n} - w_k)) + \beta_{n,m}\rho(c(T_k x^m_{k,n} - w_k)) \\
\leq\ & (1-\beta_{n,m})\rho(c(x_{k,n} - w_k)) + \beta_{n,m} q\rho(l(x^m_{k,n} - w_k)) \\
\leq\ & (1-\beta_{n,m})\rho(c(x_{k,n} - w_k)) \\
& \quad +\beta_{n,m} q[(1-\beta_{n,m-1})\rho(c(x_{k,n} - w_k)) + \beta_{n,m-1}\rho(c(T_k x^{m-1}_{k,n} - w_k))] \\
\leq\ & (1-\beta_{n,m})\rho(c(x_{k,n} - w_k)) \\
& \quad +\beta_{n,m} q[(1-\beta_{n,m-1})\rho(c(x_{k,n} - w_k)) + \beta_{n,m-1} q\rho(l(x^{m-1}_{k,n} - w_k))] \\
\vdots\ & \\
\leq\ & (1-\beta_{n,m})\rho(c(x_{k,n} - w_k)) \\
& \quad +\beta_{n,m} q[(1-\beta_{n,m-1})\rho(c(x_{k,n} - w_k)) \\
& \quad +\beta_{n,m-1} q[(1-\beta_{n,m-2})\rho(c(x_{k,n} - w_k)) \\
& \quad +\beta_{n,m-2} q[\cdots[\cdots[(1-\beta_{n,1})\rho(l(x_{k,n} - w_k) \\
& \quad +\beta_{n,1} q[(1-\beta_{n,0})\rho(l(x_{k,n} - w_k) + \beta_{n,0} q\rho(l(x_{k,n} - w_k))]]\cdots],
\end{aligned} \tag{2.2}$$

therefore

$$\rho(c(x_{k,n+1} - w_k)) \leq \mu_n \rho(c(x_{k,n} - w_k)), \tag{2.3}$$

where

$$\begin{aligned}
\mu_n =\ & (1-\beta_{n,m}) + \beta_{n,m} q[(1-\beta_{n,m-1}) \\
& +\beta_{n,m-1} q[(1-\beta_{n,m-2}) + \beta_{n,m-2} q[\cdots[\cdots[(1-\beta_{n,1}) \\
& +\beta_{n,1} q[(1-\beta_{n,0}) + \beta_{n,0} q]]\cdots].
\end{aligned}$$

By Lemma 1.3, since $\lim_{n\to\infty} \beta_{n,m} = 0$, then $\rho(c(x_{k,n} - w_k)) \longrightarrow 0$ and by Δ_2-condition $\rho(x_{k,n} - w_k) \longrightarrow 0$ as $n \longrightarrow \infty$.
Since

$$w_k = (1-\lambda_k)z + \lambda_k T w_k,$$

then

$$\begin{aligned}
\rho(c(w_k - Tw_k)) =\ & \rho(c(w_k - \tfrac{1}{\lambda_k} w_k + \tfrac{1-\lambda_k}{\lambda_k} z)) \\
=\ & \rho(c(\tfrac{1-\lambda_k}{\lambda_k}(z - w_k))) \\
\leq\ & \tfrac{1-\lambda_k}{\lambda_k}\nu.
\end{aligned}$$

So $\lambda_k \longrightarrow 1$ shows $\rho(w_k - Tw_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore $\{w_k\}$ is an approximate fixed point sequence for T.
We show the sequence $\{w_k\}$ is ρ-Cauchy. For $0 < n \leq m$,

$$w_m - w_n = (\lambda_m - \lambda_n)(Tw_n - z) + \lambda_m(Tw_m - Tw_n),$$

therefore

$$\rho(l(w_m - w_n)) \leq (\lambda_m - \lambda_n)\rho(\beta l(Tw_n - z)) + \lambda_m q\rho(l(w_m - w_n)),$$

where $\beta \in \mathbb{R}^+$ is the conjugate of $\frac{c}{l}$. Then

$$\rho(l(w_m - w_n)) \leq \frac{\lambda_m - \lambda_n}{1 - q\lambda_m}\nu.$$

By $\lambda_k \longrightarrow 1$, $\{w_k\}$, is a ρ-Cauchy sequence. Since $\{w_k\} \in C$ and X_ρ is ρ-complete, then there exists $w \in C$ such that $\rho(w_k - w) \longrightarrow 0$ as $n \longrightarrow \infty$. Since T is a ρ-continuous mapping then $\rho(Tw_k - Tw) \longrightarrow 0$ as $n \longrightarrow \infty$. By

$$\rho(\frac{1}{3}(Tw - w)) \leq \rho(Tw - Tw_k) + \rho(Tw_k - w_k) + \rho(w_k - w),$$

we have $\rho(Tw - w) \longrightarrow 0$ as $n \longrightarrow \infty$ and w is a fixed point of T. Since

$$\rho(\frac{1}{3}(Tx_{k,n} - x_{k,n})) \leq \rho(Tx_{k,n} - Tw_k) + \rho(Tw_k - w_k) + \rho(w_k - x_{k,n}),$$

then $\rho(Tx_{k,n} - x_{k,n}) \longrightarrow 0$. □

We consider the following iteration sequence

$$\begin{cases} x_1 \in C \\ u_{k,n} = \frac{1}{n+1}\sum_{j=0}^{n} T_k^j x_{k,n}, \\ z_{k,n} = (1-\gamma_n)x_{k,n} + \gamma_n u_{k,n}, \\ y_{k,n} = (1-\beta_n)x_{k,n} + \beta_n T_k^n z_{k,n}, \\ x_{k,n+1} = (1-\alpha_n)x_{k,n} + \alpha_n T_k^n y_{k,n}, \end{cases} \tag{2.4}$$

where T is a strongly ρ-contraction and $T_k x = \lambda_k T x$ such that $\lambda_k \in (0,1)$. We prove the sequence $\{x_{k,n}\}$, generated by (2.4) is strongly convergent to $w \in F(T)$.

Theorem 2.3. *Let X_ρ be a ρ-complete modular space where ρ is convex and satisfies the Δ_2-condition. Let C be a ρ-closed and convex subset of X_ρ and $T : C \to C$ be a continuous linear strongly ρ-contractive mapping. Let $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$, $\{\gamma_n\}_{n\geq 0}$ and $\{\lambda_k\}_{k\geq 0}$ be real sequences in $(0,1)$. Assume the following conditions are satisfied:*

(I): $\sum_{n\geq 0} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$.
(II): $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.
(III): $0 < \liminf_{n\longrightarrow\infty} \gamma_n \leq \limsup_{n\longrightarrow\infty} \gamma_n < 1$.
(IV): $\frac{1}{2} < \lambda_k < 1$ *and* $\lim_{k\to\infty} \lambda_k = 1$.
(V): $F(T) \neq \emptyset$.

Then the double sequence $\{x_{k,n}\}_{k\geq 0, n\geq 0}$ generated by (2.4) is strongly convergent to a fixed point w of T in C.

Proof. For each $k \geq 0$, we have,

$$T_k^n x = \lambda_k^n T^n x,$$

also for $0 \leq j \leq n$,

$$\begin{aligned} \rho(c(u_{k,n} - w_k)) \leq & \quad \frac{1}{n+1}\sum_{j=0}^{n} \rho(c(T_k^j x_{k,n} - w_k)) \\ \leq & \quad \lambda_k^n q \rho(c(x_{k,n} - w_k)). \end{aligned}$$

By quality (2.4),

$$\rho(c(z_{k,n} - w_k)) \leq (1-\gamma_n)\rho(c(x_{k,n} - w_k)) + \gamma_n \lambda_k^n q \rho(c(x_{k,n} - w_k)).$$

Moreover

$$\begin{aligned}\rho(c(y_{k,n}-w_k)) \leq\ & (1-\beta_n)\rho(c(x_{k,n}-w_k))+\beta_n\rho(c(T_k^n z_{k,n}-w_k))\\ \leq\ & (1-\beta_n)\rho(c(x_{k,n}-w_k))\\ & +\beta_n\lambda_k^n q[(1-\gamma_n)\rho(c(x_{k,n}-w_k))+\gamma_n\lambda_k^n q\rho(c(x_{k,n}-w_k))],\end{aligned}$$

and

$$\begin{aligned}\rho(c(x_{k,n+1}-w_k)) \leq\ & (1-\alpha_n)\rho(c(x_{k,n}-w_k))+\alpha_n\rho(c(T_k^n y_{k,n}-w_k))\\ \leq\ & (1-\alpha_n)\rho(c(x_{k,n}-w_k))\\ & +\alpha_n\lambda_k^n q\{(1-\beta_n)\rho(c(x_{k,n}-w_k))\\ & +\beta_n\lambda_k^n q[(1-\gamma_n)\rho(c(x_{k,n}-w_k))\\ & +\gamma_n\lambda_k^n q\rho(c(x_{k,n}-w_k))]\},\end{aligned}$$

therefore

$$\rho(c(x_{k,n+1}-w_k)) \leq \mu_n\rho(c(x_{k,n}-w_k)),$$

where

$$\mu_n = [(1-\alpha_n)+\alpha_n\lambda_k^n q\{(1-\beta_n)+\beta_n\lambda_k^n q[(1-\gamma_n)+\gamma_n\lambda_k^n q]\}].$$

By Lemma 1.3, $\rho(c(x_{k,n}-w_k)) \longrightarrow 0$ and by Δ_2-condition $\rho(x_{k,n}-w_k) \longrightarrow 0$ as $n \longrightarrow \infty$.
Since $w_k = \lambda_k T w_k$ then $\rho(c(Tw_k - w_k)) \leq \frac{1-\lambda_k}{\lambda_k}\rho(c(w_k))$. So $\lambda_k \longrightarrow 1$ shows $\rho(w_k - Tw_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore $\{w_k\}$ is an approximate fixed point sequence for T. We have

$$w_m - w_n = (\frac{\lambda_m-\lambda_n}{\lambda_n})(w_n)+\lambda_m(Tw_m - Tw_n),$$

therefore

$$\rho(l(w_m-w_n)) \leq (\frac{\lambda_m-\lambda_n}{\lambda_n})\rho(\beta l(w_n))+\lambda_m q\rho(l(w_m-w_n)),$$

where $\beta \in \mathbb{R}^+$ is the conjugate of $\frac{c}{l}$. By $\lambda_k \longrightarrow 1$, $\{w_k\}$, is a ρ-Cauchy sequence and there exists a $w \in C$ such that $\rho(w_k - w) \longrightarrow 0$ as $n \longrightarrow \infty$. Since T is a ρ-continuous mapping then $\rho(Tw_k - Tw) \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore $\rho(Tw - w) \longrightarrow 0$ as $n \longrightarrow \infty$ and w is a fixed point of T. □

3. Examples

In this section, five examples will be presented as follows:

Example 3.1. Let $X_\rho = \mathbb{R}$ be the set of real numbers and $C = [0,\infty)$. Suppose that $\{x_{k,n}\}$ is the sequence defined by

$$\begin{cases} y_{k,n} = (1-\beta_n)x_{k,n}+\beta_n T_k x_{k,n},\\ x_{k,n+1} = (1-\alpha_n)x_{k,n}+\alpha_n T_k y_{k,n},\end{cases} \tag{3.1}$$

where $Tx = \frac{x}{10^5}$ and $z = 0$. Also $\alpha_n = \frac{1}{10n}$, $\beta_n = \frac{n}{4n+1}$ and $\lambda_k = \frac{k}{k+4}$. We have

$$\begin{cases} y_{k,n} = \frac{3n+1}{4n+1}x_{k,n}+\frac{n}{4n+1}\frac{k}{k+4}\frac{x_{k,n}}{10^5},\\ x_{k,n+1} = \frac{10n-1}{10n}x_{k,n}+\frac{1}{10n}\frac{k}{k+4}\frac{y_{k,n}}{10^5}.\end{cases}$$

If $x_{100,1} = \frac{1}{2}$, so

n	$x_{100,n}$	n	$x_{100,n}$
1	0.5	11	0.37001
2	0.45	12	0.36664
3	0.4275	13	0.36359
4	0.41325	14	0.36079
5	0.40291	15	0.35821
6	0.39486	16	0.35583
7	0.38828	17	0.3536
8	0.38273	18	0.35152
9	0.37794	19	0.34957
10	0.37374	20	0.34773

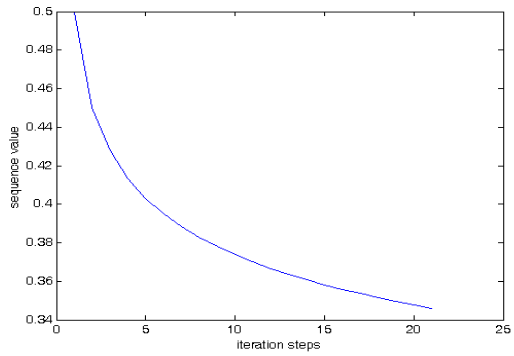

FIGURE 1. The iteration chart for $k = 100$.

EXAMPLE 3.2. Under the conditions of Example 3.1, suppose that $\{x_{k,n}\}$ is the sequence defined by

$$\begin{cases} y_{k,n} = \beta_n x_{k,n} + (1-\beta_n) T_k x_{k,n}, \\ x_{k,n+1} = \alpha_n x_{k,n} + (1-\alpha_n) T_k y_{k,n}, \end{cases} \tag{3.2}$$

we have

$$\begin{cases} y_{k,n} = \frac{n}{4n+1} x_{k,n} + \frac{3n+1}{4n+1} \frac{k}{k+4} \frac{x_{k,n}}{10^5}, \\ x_{k,n+1} = \frac{1}{10n} x_{k,n} + \frac{10n-1}{10n} \frac{k}{k+4} \frac{y_{k,n}}{10^5}. \end{cases}$$

If $x_{100,1} = \frac{1}{2}$, so

n	$x_{100,n}$	n	$x_{100,n}$
1	0.5	11	1.96×10^{-52}
2	0.05	12	4.57×10^{-58}
3	2.88×10^{-7}	13	1.06×10^{-63}
4	6.18×10^{-13}	14	2.5×10^{-69}
5	1.36×10^{-18}	15	5.86×10^{-75}
6	3.06×10^{-24}	16	1.37×10^{-80}
7	6.94×10^{-30}	17	3.24×10^{-86}
8	1.58×10^{-35}	18	7.63×10^{-92}
9	3.65×10^{-41}	19	1.79×10^{-97}
10	8.46×10^{-47}	20	4.24×10^{-103}

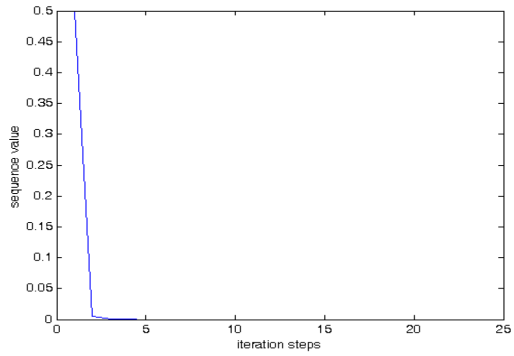

FIGURE 2. The iteration chart for $k = 100$.

Corollary 3.3. *Examples 3.1 and 3.2 show the rate of convergence of sequence $\{x_n\}$ generated by (3.2), is faster than the rate of convergence of sequence $\{x_n\}$ generated by (3.1).*

EXAMPLE 3.4. Let $X_\rho = \mathbb{R}$ be the set of real numbers and $C = [0, \infty)$. Suppose that $\{x_{k,n}\}$ is the sequence defined by (2.4), where $Tx = \frac{x}{10}$ and $z = 0$. Also $\alpha_n = \frac{1}{10n}$, $\beta_n = \frac{n}{5n+1}$, $\gamma_n = \frac{n}{7n+1}$ and $\lambda_k = \frac{k}{k+100}$. We have

$$\begin{cases} u_n = \frac{1}{n+1} \sum_{j=0}^{n} \frac{x_n}{10^j}, \\ z_{k,n} = \frac{6n+1}{7n+1} x_{k,n} + \frac{n}{7n+1} (\frac{k}{k+100})^n \frac{u_{k,n}}{10^n}, \\ y_{k,n} = \frac{4n+1}{5n+1} x_{k,n} + \frac{n}{5n+1} (\frac{k}{k+100})^n \frac{z_{k,n}}{10^n}, \\ x_{k,n+1} = \frac{10n-1}{10n} x_{k,n} + \frac{1}{10n} (\frac{k}{k+100})^n \frac{y_{k,n}}{10^n}. \end{cases}$$

If $x_{100,1} = 0.5$, so

n	$x_{100,n}$	n	$x_{100,n}$
1	0.5	11	0.37178
2	0.4521	12	0.3684
3	0.42954	13	0.36533
4	0.41522	14	0.36252
5	0.40484	15	0.35993
6	0.39674	16	0.35753
7	0.39013	17	0.35529
8	0.38456	18	0.3532
9	0.37975	19	0.35124
10	0.37553	20	0.34939

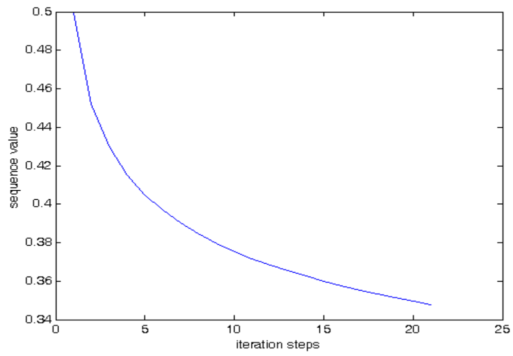

FIGURE 3. The iteration chart for $k = 100$.

EXAMPLE 3.5. Under the conditions of Example 3.4, we consider the following sequence $\{x_n\}$,

$$\begin{cases} x_1 \in C \\ u_{k,n} = \frac{1}{n+1}\sum_{j=0}^{n} T_k^j x_{k,n}, \\ z_{k,n} = \gamma_n x_{k,n} + (1-\gamma_n)u_{k,n}, \\ y_{k,n} = \beta_n x_{k,n} + (1-\beta_n)T_k^n z_{k,n}, \\ x_{k,n+1} = \alpha_n x_{k,n} + (1-\alpha_n)T_k^n y_{k,n}, \end{cases} \tag{3.3}$$

therefore, we have

$$\begin{cases} u_n = \frac{1}{n+1}\sum_{j=0}^{n} \frac{x_n}{10^j}, \\ z_{k,n} = \frac{n}{7n+1}x_{k,n} + \frac{6n+1}{7n+1}(\frac{k}{k+100})^n \frac{u_{k,n}}{10^n}, \\ y_{k,n} = \frac{n}{5n+1}x_{k,n} + \frac{4n+1}{5n+1}(\frac{k}{k+100})^n \frac{z_{k,n}}{10^n}, \\ x_{k,n+1} = \frac{1}{10n}x_{k,n} + \frac{10n-1}{10n}(\frac{k}{k+100})^n \frac{y_{k,n}}{10^n}. \end{cases}$$

If $x_{100,1} = 0.5$, so

n	$x_{100,n}$	n	$x_{100,n}$
1	0.5	11	1.49×10^{-17}
2	0.053	12	1.36×10^{-19}
3	2.7×10^{-3}	13	1.13×10^{-21}
4	9.06×10^{-5}	14	8.73×10^{-24}
5	2.26×10^{-6}	15	6.24×10^{-26}
6	4.53×10^{-8}	16	4.16×10^{-28}
7	7.55×10^{-10}	17	2.6×10^{-30}
8	1.07×10^{-11}	18	1.52×10^{-32}
9	1.34×10^{-13}	19	8.49×10^{-35}
10	1.49×10^{-15}	20	4.47×10^{-37}

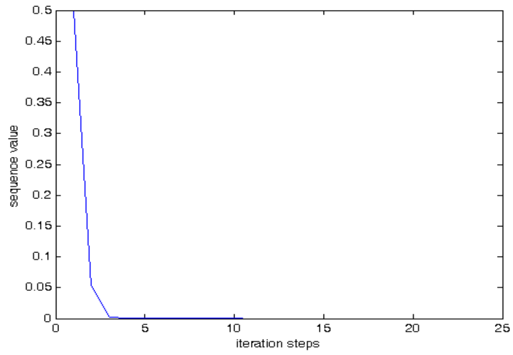

FIGURE 4. The iteration chart for $k = 100$.

Corollary 3.6. *Examples 3.4 and 3.5 show the rate of convergence of sequence $\{x_n\}$ generated by (3.3), is faster than the rate of convergence of sequence $\{x_n\}$ generated by (2.4).*

Acknowledgments

The authors would like to thank the referees for giving fruitful advices.

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