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The Representations and Positive Type Functions of Some Homogenous Spaces

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Abstract. For a homogeneous spaces G/H, we show that the convolution on $L^1(G/H)$ is the same as convolution on $L^1(K)$, where G is semidirect product of a closed subgroup H and a normal subgroup K of G. Also we prove that there exists a one to one correspondence between nondegenerat $*$-representations of $L^1(G/H)$ and representations of G/H. We propose a relation between cyclic representations of $L^1(G/H)$ and positive type functions on G/H. We prove that the Gelfand Raikov theorem for G/H holds if and only if H is normal.

Keywords: Homogenous space, Semidirect product, Convolution, Involution, Representation, Irreducible representation.

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1. Introduction

The ideas of representation theory are among the most important unifying concepts in current mathematical research, and they are relevant to several fields, from geometry and quantum mechanics to number theory, besides being an important subject by itself [9]. In this paper we study the convolution on $L^1(G/H)$ is the same as convolution on $L^1(K)$, when G is a semidirect product of closed subgroup H and normal subgroup K. Several various applications of semidirect product groups in physics have motivated us to study them.

Let G be a locally compact group and H be a closed subgroup of G. Consider G/H as a homogeneous space and μ as a relatively invariant measure on it (see [1, 10]).
The mapping $T : L^1(G) \to L^1(G/H)$ defined by

$$Tf(gH) = \int_H \frac{f(gh)}{\rho(gh)} dh, \tag{1.1}$$

is a surjective bounded linear operator with $\|T\| \leq 1$, in which ρ is a rho-function for (G, H). Moreover,

$$\int_{G/H} \int_H \frac{f(gh)}{\rho(gh)}\, dh\, d\mu(gH) = \int_G f(g)\, dg \tag{1.2}$$

for $f \in L^1(G)$ (for more details on homogeneous spaces one can consult with [10, 4, 2, 8]).

Let K and H be two locally compact groups and $h \mapsto \tau_h$ be a homomorphism of H into the group of automorphisms of K. Then the set $K \times H$ endowed with the operation

$$(k_1, h_1)(k_2, h_2) = (k_1 \tau_{h_1}(k_2), h_1 h_2)$$

and the Cartesian product topology on $K \times H$ is a locally compact group which is denoted by $K \times_\tau H$ [5]. The homogeneous space G/H has a relatively invariant Radon measure, which arises from the rho function $\rho : G \to (0, \infty)$ defined by $\rho(x) = \frac{\Delta_H(h)}{\Delta_G(h)}$, where $x \in G, h \in H$, and $x = kh$ for some $k \in K$. Moreover, there exists a left Haar measure dk on K for which

$$\int_G f(x)dx = \int_K \int_H f(kh)\delta(h)dhdk, \quad f \in L^1(G/H), \tag{1.3}$$

where $\delta : H \to (0, \infty)$ is a homomorphism such that $\delta(h) = \frac{\Delta_G(h)}{\Delta_H(h)}$ [7, 10].

In this paper we show that $L^1(G/H)$ is a Banach $*$-algebra and the convolution on $L^1(G/H)$ is the same as the convolution on $L^1(K)$, where K, H are two locally compact groups and $G = K \times_\tau H$. We prove that there exists a one to one correspondence between positive type functions on G/H and representations of $L^1(G/H)$. This paper is organized as follows:

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Section 2 is devoted to introducing a convolution and involution on $L^1(G/H)$, where $G = K \times_\tau H$, and K, H are two locally compact groups. Also, it is shown that $L^1(G/H)$ is a Banach $*$-algebra. In Section 3, a one to one correspondence between the representations of G/H and nondegenerat $*$-representations of $L^1(G/H)$ is given. Section 4 deals with introducing the positive type functions on G/H, where $G = K \times_\tau H$. A relation between positive type functions on G/H and positive type functions on K is stated. Moreover, it is proved that a generalized version of Gelfand-Raikove theorem for G/H holds if and only if H is normal.

2. The Banach Algebra $L^1(G/H)$

Throughout this paper, we assume that G is semidirect product of closed subgroup H and normal subgroup K. We give a reasonably generalized definition of convolution on homogeneous space G/H. It is worthwhile to note that if $G = K \times_\tau H$, then there exists a bijection between $C_C(G/H)$ and $C_C(K)$ which assigns to each $g \in C_C(K)$ a function $\gamma_g \in C_C(G/H)$ such that

$$\gamma_g(kH) = g(k), \quad k \in K, \tag{2.1}$$

(see [6, proposition1.5.3]). First, we define a convolution on $C_C(G/H)$ and then we extend it to $L^1(G/H)$.

Definition 2.1. Let G be a semidirect product of closed subgroup H and normal subgroup K. For each $\phi, \psi \in C_C(G/H)$, the convolution of ϕ, ψ is defined as follows:

$$* : C_C(G/H) \times C_C(G/H) \to C_C(G/H), \quad (\phi * \psi)(xH) = (f * g)(k), \tag{2.2}$$

where $x = kh, \phi = \gamma_f, \psi = \gamma_g, \quad f, g \in C_C(K)$.

By the uniqueness of the decomposition of x by h, k, it is easily shown that " $*$ " is well defined. To extend the convolution from $C_C(G/H)$ to $L^1(G/H)$ we require the following Lemma.

Lemma 2.2. *(i) For $\phi, \psi \in C_C(G/H)$, we have*

$$\int_{G/H} (\phi * \psi)(xH) d\mu(xH) = \int_K (f * g)(k) dk, \tag{2.3}$$

where $\phi = \gamma_f, \psi = \gamma_g, f, g \in C_C(K)$ (as in 2.1) and $x = kh$.
(ii) If $\phi, \psi \in C_C(G/H)$, then $\phi * \psi \in L^1(G/H)$ and

$$\|\phi * \psi\|_{L^1(G/H)} \leq \|\phi\|_{L^1(G/H)} \|\psi\|_{L^1(G/H)}.$$

Proof. Let $\phi, \psi \in C_C(G/H), \quad \phi = \gamma_f, \psi = \gamma_g, f, g \in C_C(K)$. Since the mapping T, defined as in (1.1) is surjective, there exists $\varphi \in L^1(G)$ such that $T(\varphi) = \gamma_f * \gamma_g$. Also, using (1.2) and (1.3), we have:

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$$\begin{aligned}\int_{G/H}(\phi*\psi)(xH)d\mu(xH) &= \int_{G/H}(\gamma_f*\gamma_g)(xH)d\mu(xH)\\ &= \int_{G/H}(T\varphi)(xH)d\mu(xH)\\ &= \int_G \varphi(x)dx\\ &= \int_K\int_H \varphi(kh)\delta(h)dhdk\\ &= \int_K\int_H \frac{\varphi(kh)}{\rho(kh)}dhdk\\ &= \int_K(T\varphi)(kH)dk\\ &= \int_K(\gamma_f*\gamma_g)(kH)dk\\ &= \int_K(f*g)(k)dk.\end{aligned}$$

By part (i), it can be shown

$$\|\phi*\psi\|_{L^1(G/H)} = \|f*g\|_{L^1(K)} \leq \|\phi\|_{L^1(G/H)}\|\psi\|_{L^1(G/H)}$$

. □

Now, the convolution extends by continuity to $L^1(G/H)$. One can easily show that for all $\phi,\phi_1,\phi_2,\psi,\psi_1,\psi_2 \in L^1(G/H)$ and $\alpha\in\mathbb{C}$:

$$(i)\ \ (\phi_1+\alpha\phi_2)*\psi = (\phi_1*\psi)+\alpha(\phi_2*\psi),$$

$$(ii)\ \ \phi*(\psi_1+\alpha\psi_2) = (\phi*\psi_1)+\alpha(\phi*\psi_2).$$

We conclude that $L^1(G/H)$ is a Banach algebra.
Now we define an involution on $L^1(G/H)$ as follows:

Definition 2.3. For all $\phi\in C_C(G/H)$, we define the mapping

$$^*: C_C(G/H)\to C_C(G/H),\quad \phi^* = \gamma_{g^*}, \tag{2.4}$$

where $\phi=\gamma_g, g\in C_C(K)$.

The following Lemma states that $L^1(G/H)$ is a Banach *-algebra.

Lemma 2.4. *The conjugate linear operator "*" in Definition 2.3 is an involution on $L^1(G/H)$.*

Proof. Let $\phi,\psi\in C_C(G/H),\quad \phi=\gamma_f,\psi=\gamma_g, f,g\in C_C(K)$. Then,

$$\begin{aligned}(\phi+\alpha\psi)^*(xH) &= (\gamma_f+\alpha\gamma_g)^*(kH)\\ &= (\gamma_{(f+\alpha g)^*})(kH)\\ &= (f+\alpha g)^*(k)\\ &= (f^*+\bar\alpha g^*)(k)\\ &= (\gamma_{f^*}+\bar\alpha\gamma_{g^*})(kH)\\ &= (\phi^*+\bar\alpha\psi^*)(xH).\end{aligned}$$

Also, similarly one can be shown $(\phi^*)^*=\phi, (\phi*\psi)^* = (\psi^**\phi^*)$ and $\|\phi^*\|_{L^1(G/H)} = \|\phi\|_{L^1(G/H)}$. The involution now can be extended to $L^1(G/H)$ by continuity. □

From Lemma 2.2, Lemma 2.4 imply the following theorem.

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Theorem 2.5. *Let H, K be two locally compact groups such that $G = K \times_\tau H$. Then $L^1(G/H)$ is a Banach $*$-algebra.*

The following proposition states that $L^1(G/H)$ is a Banach $*$-algebra with an approximate identity.

Proposition 2.6. *Let $\mathcal{U}$ be a neighborhood base at e in K. For $U \in \mathcal{U}$ choose $\psi_U = \gamma_{h_U}$ where $h_U \in C_C^+(K)$, such that $supp(h_U) \subseteq U, \breve{h_U} = h_U$ and $\int_K h_U(k)dk = 1$. Then for all $\phi \in L^1(G/H), \|\phi * \psi_U - \phi\|_{L^1(G/H)} \to 0$ and $\|\psi_U * \phi - \phi\|_{L^1(G/H)} \to 0$ as $U \to e$.*

Proof. Let $\mathcal{U}$ be a neighborhood base at e. For all $U \in \mathcal{U}$ there exists a function $h_U \in C_C^+(K)$, such that $supp(h_U) \subseteq U, \breve{h_U} = h_U$ and $\int_K h_U(k)dk = 1$. Then $\{h_U\}_{U\in\mathcal{U}}$ is an approximate identity for $L^1(K)$. So, $\|f * h_U - f\|_{L^1(K)} \to 0$ and $\|h_U * f - f\|_{L^1(K)} \to 0$, for all $f \in L^1(K)$ as $U \to e$. Now, let $\psi_U = \gamma_{h_U}$, then by (2.3),

$$\|\psi_U * \phi - \phi\|_{L^1(G/H)} = \|h_U * f - f\|_{L^1(K)} \to 0,$$

where $\phi \in L^1(G/H), f \in L^1(K)$, as $U \to e$. By a similar argument we can show that $\|\phi * \psi_U - \phi\|_{L^1(G/H)} \to 0$, as $U \to e$. □

3. The Representations of G/H and $L^1(G/H)$

For the reader's convenience, we recall from [1] some basic concepts in the theory of unitary representations of homogeneous spaces. A continuous unitary representation of G/H is a mapping σ from G/H into the group $U(\mathcal{H})$, of all unitary operators on some nonzero Hilbert space $\mathcal{H}$, for which $gH \mapsto < \sigma(gH)x, y >$ is continuous from G/H into $\mathbb{C}$, for each $x, y \in \mathcal{H}$ and

$$\sigma(g_1g_2H) = \sigma(g_1H)\sigma(g_2H), \quad \sigma(g^{-1}H) = \sigma(gH)^*,$$

for each $g, g_1, g_2 \in G$. In the sequel we always mean by a representation, a continuous unitary representation. This easily defines a representation π of G in which the subgroup H is considered to be contained in the kernel of π. Conversely, any representation π of G which is trivial on H induces a representation σ of G/H, by letting $\sigma(gH) = \pi(g)$. Moreover, a closed subspace M of $\mathcal{H}$ is said to be invariant with respect to σ if $\sigma(gH)M \subseteq M$, for all $g \in G$. A representation σ is said to be irreducible if the only closed invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$. Any representation σ of G/H determines a representation $\tilde{\sigma}$ of $L^1(G/H)$ as follows:

$$\tilde{\sigma}(\phi) = \int_{G/H} \phi(xH)\sigma(xH)d\mu(xH), \quad for \ \ \phi \in L^1(G/H). \tag{3.1}$$

That is, for any $u, v \in \mathcal{H}$,

$$< \tilde{\sigma}(\phi)u, v >= \int_{G/H} \phi(xH) < \sigma(xH)u, v > d\mu(xH). \tag{3.2}$$

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The linear operator $\tilde{\sigma}$ is bounded. Indeed,

$$\begin{aligned} |<\tilde{\sigma}(\phi)u,v>| &\leq \int_{G/H}|\phi(xH)|\ \ |<\sigma(xH)u,v>|d\mu(xH) \\ &\leq \|\phi\|_{L^1(G/H)}\|u\|\ \ \|v\|, \end{aligned}$$

which implies that $\|\tilde{\sigma}(\phi)\| \leq \|\phi\|_{L^1(G/H)}$.

In this section we show that there is a one to one corresponding between the representations of G/H and the nondegenerate $*$-representations of $L^1(G/H)$. A representation $\tilde{\sigma}$ of $L^1(G/H)$ is called nondegenerate if there is no $v \in \mathcal{H}$ such that $\|\tilde{\sigma}(\phi)v\| = 0$, for all $\phi \in L^1(G/H)$. Let σ be a representation of G/H. This representation defines a representation on K as follows:

$$\pi : K \to U(\mathcal{H}), \quad \pi(k) := \sigma(xH); \quad x = kh.$$

It can easily be shown that π is a representation of K.

The following theorem states that any representation of G/H determines a representation of $L^1(G/H)$.

Theorem 3.1. *Let σ be a representation of G/H. Then $\tilde{\sigma}$ defined as (3.1) is a nondegenerate $*$-representation of $L^1(G/H)$.*

Proof. Let $\phi \in C_C(G/H)$ such that $\phi = \gamma_f$, for $f \in C_C(K)$ and σ be a representation of G/H. The representation π of K as $\pi(k) = \sigma(gH), \quad g = kh \in G$ determines a representation of $L^1(K)$, denoted by $\tilde{\pi}$ ([4, Theorem 3.9]). By a similar argument as in Lemma 2.2, $\tilde{\sigma}(\phi) = \tilde{\pi}(f)$. It can be easily shown that $\tilde{\sigma}(\phi * \psi) = \tilde{\sigma}(\phi).\tilde{\sigma}(\psi)$ and $\tilde{\sigma}(\phi^*) = (\tilde{\sigma}(\phi))^*$. Since $\tilde{\pi}$ is nondegenerate, so is $\tilde{\sigma}$. It is easily shown that $\tilde{\sigma}$ is continuous on $C_C(G/H)$. Now by continuity we can extended $\tilde{\sigma}$ from $C_C(G/H)$ onto $L^1(G/H)$. □

Proposition 3.2. *Suppose that $\tilde{\sigma}$ is a nondegenerate $*$-representation of $L^1(G/H)$ on a Hilbert space $\mathcal{H}$. Then $\tilde{\sigma}$ arises from a unique representation σ of G/H on $\mathcal{H}$ such that*

$$<\tilde{\sigma}(\phi)u,v>=\int_{G/H}\phi(xH)<\sigma(xH)u,v>d\mu(xH),$$

for all $u, v \in \mathcal{H}$.

Proof. Let $\tilde{\sigma}$ be a nondegenerate $*$-representation of $L^1(G/H)$. This representation defines a $*$-representation $\tilde{\pi}$ of $C_C(K)$, such that $\tilde{\pi}(f) = \tilde{\sigma}(\gamma_f)$. By [4, Theorem 3.11] the representation $\tilde{\pi}$ arises from a unique representation π of K on $\mathcal{H}$ such that

$$<\tilde{\pi}(f)u,v>=\int_{G/H}f(k)<\pi(k)u,v>dk, \quad \textit{for all} \ \ u,v\in\mathcal{H}.$$

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Consider $\sigma : G/H \to U(\mathcal{H})$ such that $\sigma(xH) := \pi(k), \quad x = kh$, in which $\tilde{\sigma}$ arises from a unique representation σ of G/H such that

$$< \tilde{\sigma}(\phi)u, v >= \int_{G/H} \phi(xH) < \sigma(xH)u, v > d\mu(xH).$$

□

By [4, Theorem 3.12] and a similar argument as in the proof of Therem 3.1, one can easily show that the representation σ is irreducible if and only if $\tilde{\sigma}$ defined as (3.1) is irreducible.

4. Positive Type Functions on G/H

The representation theory is the most important concept in harmonic analysis. Out of any representation of a group, positive type functions on the group can be obtained. In this section we study the relation between representations and positive type functions of homogeneous spaces whose groups are semidirect products. Moreover, it is proved that a generalized version of Gelfand-Raikove theorem for G/H holds if and only if H is normal.

Now, we define positive type functions on G/H and investigate the relation between positive type functions on G/H and on the group K as introduced in Section1.

Definition 4.1. A function $\phi \in L^\infty(G/H)$ is said to be a positive type function if $\phi(\omega^* * \omega) \geq 0$, for all $\omega \in L^1(G/H)$, i.e. $\int_{G/H}(\omega^* * \omega)(xH)\phi(xH)d\mu(xH) \geq 0$, in which μ is a strongly quasi invariant measure on G/H. We denote the set of all continuous positive type functions in $L^\infty(G/H)$ by $P(G/H)$.

Proposition 4.2. *Let K, H be two locally compact groups and $G = K \times_\tau H$. Then there exists an isometric isomorphism between $L^\infty(G/H)$ and $L^\infty(K)$.*

Proof. Define two mappings α and β as follows:
$\alpha : K \to G/H, k \mapsto kH$, and $\beta : G/H \to K, xH \mapsto k, x = kh$. Due to uniqueness of the decomposition of $x \in G$ as $x = kh$, the mapping β is well defined. Obviously α, β are continuous. Moreover, for $\phi \in L^\infty(G/H), \varphi \in L^\infty(K)$, by [10, Theorem 3.3.29] we have $\|\phi o\alpha\|_{L^\infty(K)} = \|\phi\|_{L^\infty(G/H)}, \|\varphi\|_{L^\infty(K)} = \|\varphi o\beta\|_{L^\infty(G/H)}$. In particular, for $\varphi \in L^\infty(K), \varphi o\beta \in L^\infty(G/H)$ and for $\phi \in L^\infty(G/H), \phi o\alpha \in L^\infty(K)$. So the following mappings are well define and isometry. $\Gamma_1 : L^\infty(K) \to L^\infty(G/H), \quad \Gamma_1(\varphi)(xH) = \varphi o\beta(xH) = \varphi(k), x = kh$, and $\Gamma_2 : L^\infty(G/H) \to L^\infty(K), \quad \Gamma_2(\phi)(k) = \phi o\alpha(k) = \phi(kH)$, for $\varphi \in L^\infty(K), \phi \in L^\infty(G/H)$. Moreover, $\Gamma_1 o\Gamma_2 = id$, which complete the proof. □

Proposition 4.3. *Let K, H be two locally compact groups and $G = K \times_\tau H$. The function $\phi \in L^\infty(G/H)$ ($\varphi \in L^\infty(K)$) is positive type if and only if $\phi o\alpha \in L^\infty(K)$ ($\varphi o\beta (\in L^\infty(G/H))$ is positive type.*

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Proof. Let $\phi \in L^\infty(G/H)$ be positive type. Then Proposition 4.2 implies that $\phi o\alpha \in L^\infty(K)$. For all $\omega \in C_c(G/H)$, we have

$$\begin{aligned}
\int_{G/H}(\omega^* * \omega)(xH)\phi(xH)d\mu(xH) &= \int_{G/H}(\gamma_{f^*} * \gamma_f)(xH)\phi(xH)d\mu(xH)) \\
&= \int_{G/H}(T\psi)(xH)d\mu(xH) \\
&= \int_G \psi(x)dx \\
&= \int_K \int_H \psi(kh)\delta(h)dhdk \\
&= \int_K \int_H \frac{\varphi(kh)}{\rho(kh)}dhdk \\
&= \int_K (T\varphi)(kH)dk \\
&= \int_K (\gamma_{f^*} * \gamma_f)(kH)\phi(kH)dk \\
&= \int_K (f^* * f)(\phi o\alpha)(k)dk,
\end{aligned}$$

in which $\omega = \gamma_f; f \in C_c(K)$and $\psi \in L^1(G)$. By a standard density argument one can see that for $\omega \in L^1(G)$ is hold. The statement in parantheses is a direct consequnce of the present statement. □

Theorem 4.4. *Let K, H be two locally compact groups and $G = K \times_\tau H$. Then the following hold.*
(i) If σ is a representation of G/H, then the mapping ϕ on $L^1(G/H)$, defined by $\phi(xH) =< \sigma(xH)u, u >$, is a positive type function.
(ii) If $\phi \in L^\infty(G/H)$ be a positive type function, then there exists a unique (up to unitary equivalence) cyclic representation of G/H on a Hilbert space $\mathcal{H}_\sigma$ such that $\phi(xH) =< \sigma_\phi(xH)u, u >$, where $u \in \mathcal{H}_\sigma$ is a cyclic vector.

Proof. To show (i), let $\omega \in C_c(G/H)$ such that $\omega = \gamma_f, f \in C_C(K)$. By a similar argument as in proof of proposition4.3, we have

$$\int_{G/H}(\omega^* * \omega)(xH)\phi(xH)d\mu(xH) = \int_K (f^* * f)(k) < \pi(k)u, u > dk = \|\pi(f)u\|^2_{L^1(K)},$$

where $\pi(k) = \sigma(xH)$, $x = kh$, is a representation of K. By a standard density argument one can show that for $\omega \in L^1(G/H)$ holds. For (ii) Let $\phi \in L^\infty(G/H)$ be a positive type function. Then $\phi o\alpha \in L^\infty(K)$ is of positive type. By [4, Theorem 3.20] there exists a cyclic representation $\pi_{\phi o\alpha}$ of K on some Hilbert space $\mathcal{H}_{\pi_{\phi o\alpha}}$ such that $\phi o\alpha(k) =< \pi_{\phi o\alpha}(k)u, u >$, where $u \in \mathcal{H}_{\pi_{\phi o\alpha}}$ is a cyclic vector. Therefore, the representation σ_ϕ of G/H defined by

$$\sigma_\phi(xH) = \pi_{\phi o\alpha}(k), \tag{4.1}$$

is a cyclic representation, for $x = kh$. Also, we have

$$\phi(xH) = \phi o\alpha(k) =< \pi_{\phi o\alpha}(k)u, u >=< \sigma_\phi(xH)u, u >,$$

for $x = kh$. By [4, Theorem 3.23], the uniqueness is obvious. □

Now, it is an interesting question, if Gelfand Raikove theorem holds for G/H? We show that it is the case just when H is normal.

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Theorem 4.5. *(Generalized Gelfand Raikove theorem) The irreducible representations of G/H separate points on G/H if and only if H is normal.*

Proof. If H is normal, then G/H is group and Gelfand Raikove theorem is hold. For converse, if σ is an arbitrary irreducible representation of G/H which separates points, then $\sigma(hxH) = \sigma(hH)\sigma(xH) = \sigma(xH)$, in which $x \in G$ and $h \in H$. This follows that H is normal. □

Here we support our technical considerations developed by giving some examples.

Example 4.6. Let G be the semidirect product of two locally compact groups K and H respectively, i.e. $G = K \times_\tau H$. Assume that the representation σ of G/H is a trivial representation. Then $\phi = 1$ is a positive type function on G/H.

Example 4.7. Let G be the semidirect product of two locally compact groups K and H respectively, i.e. $G = K \times_\tau H$. Consider $\varphi = \delta_{e_K}$ which is positive type on K, where e_K is the identity element of group K, and the left regular representation ℓ of K on $l^2(K)$. There is a cyclic vector $\varepsilon = \delta_{e_K}$ for ℓ such that $\varphi(k) =< \ell(k)\varepsilon, \varepsilon >$. Indeed,

$$< \ell(k)\varepsilon, \varepsilon >=< \ell(k)\delta_e, \delta_e >= \sum_{s\in K} \ell(k)\delta_e(s)\delta_e(s) = \sum_{s\in K} \delta_e(k^{-1}s)\delta_e(s) = \delta_e(k).$$

Example 4.8. Let $G = \mathbb{R} \times_\tau \mathbb{R}^*$ be the affine group. The Euclidean group $\mathbb{R}$ is considered as the homogeneous space under the following action

$$G \times \mathbb{R} \to \mathbb{R}, ((b, a), x) \mapsto ax + b.$$

Let $H^2_+(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ defined dy

$$H^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}), supp\hat{f} \subseteq [0, \infty)\},$$

which is called Hardy space and it is a closed subspace of $L^2(\mathbb{R})$ (see [11]). Let $\pi : \mathbb{R} \to U(H^2_+(\mathbb{R})), \quad \pi(b)\varphi \mapsto T_b\varphi$, where $T_b\varphi(x) = \varphi(x - b)$, $x \in \mathbb{R}$ is the translation operator. Then π is irreducible. Let $\varphi = \chi_{[0,1]}$ is a cyclic vector for π and obviously $\psi(b) = \begin{cases} 0 & b < -1 \\ 1 + b & -1 < b < 0 \\ 1 - b & 0 < b < 1 \\ 0 & b > 1 \end{cases}$ is a positive type function on K. By Theorem 4.3, the function $\psi o\beta$ is positive type function on G/H, where β is as in proposition 4.2.

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