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(ϕ, ρ)-Representation of Γ-So-Rings

M. Siva Mala[a,*], K. Siva Prasad[b]

[a]Department of Mathematics, V.R. Siddhartha Engineering College, Kanuru, Vijayawada-520007, Andhra Pradesh, India.
[b]Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar-522510, Guntur(D.t), Andhra Pradesh, India.

E-mail: sivamala_aug9@yahoo.co.in
E-mail: siva235prasad@yahoo.co.in

Abstract. A Γ-so-ring is a structure possessing a natural partial ordering, an infinitary partial addition and a ternary multiplication, subject to a set of axioms. The partial functions under disjoint-domain sums and functional composition is a Γ-so-ring. In this paper we introduce the notions of subdirect product and (ϕ, ρ)-product of Γ-so-rings and study (ϕ, ρ)-representation of Γ-so-rings.

Keywords: Subdirectly irreducible Γ-so-ring, Subdirect product, (ϕ, ρ)-product of Γ_i-so-rings, (ϕ, ρ)-representation of a Γ-so-ring.

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1. Introduction

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming language semantics. The general cardinal algebras studied by Tarski in 1949, Housdorff topological commutative groups studied by Bourbaki in 1966, Σ-structures studied by Higgs in 1980, sum ordered partial monoids & sum ordered partial semirings studied by Arbib,

*Corresponding Author

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Manes, Benson[2],[3] and Streenstrup[6] are some of the algebraic structures of the above type.

M. Murali Krishna Rao[4] in 1995 introduced the notion of a Γ-semiring as a generalization of semirings and Γ-rings, and extended many fundamental results of semirings and Γ-rings to Γ-semirings. In [5] we introduced the notion of Γ-so-ring R and obtained a necessary and sufficient condition for the quotient R/θ to be a Γ/σ-so-ring, where (θ, σ) is a congruence relation on (R, Γ). As a continuation, in this paper we prove that a Γ-so-ring is a subdirect product of subdirectly irreducible Γ_i-so-rings and obtain (ϕ, ρ)-representation of Γ-so-rings.

2. Preliminaries

In this section we collect important definitions from the literature.

Definition 2.1[3]. A *partial monoid* is a pair (M, Σ) where M is a nonempty set and Σ is a partial addition defined on some, but not necessarily all families $(x_i : i \in I)$ in M subject to the following axioms:
(1) Unary Sum Axiom. If $(x_i : i \in I)$ is a one element family in M and $I = \{j\}$, then $\Sigma(x_i : i \in I)$ is defined and equals x_j.
(2) Partition-Associativity Axiom. If $(x_i : i \in I)$ is a family in M and $(I_j : j \in J)$ is a partition of I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J, $(\Sigma(x_i : i \in I_j) : j \in J)$ is summable, and $\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j) : j \in J)$.

Example 2.2[3]. Let D and E be two sets and let the set of all partial functions from D to E be denoted by $Pfn(D, E)$. A family $(x_i : i \in I)$ is summable if and only if for i, j in I, and $i \neq j$, $dom(x_i) \cap dom(x_j) = \emptyset$. If $(x_i : i \in I)$ is summable then for any d in D

$$d(\Sigma_i x_i) = \begin{cases} dx_i, \ if \ d \in dom(x_i) \ for \ some \ (necessarily \ unique) \ i \in I; \\ undefined, \ otherwise. \end{cases}$$

Then $(Pfn(D, E), \Sigma)$ is a partial monoid.

Definition 2.3[6]. Let (M, Σ) and (M', Σ') be partial monoids. Then (M', Σ') is said to be a *partial submonoid* of (M, Σ) if it satisfies the following:
(1) M' is a subset of M,
(2) $(x_i : i \in I)$ is a summable family in M' implies that $(x_i : i \in I)$ is summable family in M and $\Sigma'_i x_i = \Sigma_i x_i$.

Definition 2.4[5]. Let (R, Σ) and (Γ, Σ') be two partial monoids. Then R is said to be a *partial* Γ*-semiring* if there exists a mapping $R \times \Gamma \times R \longrightarrow R$

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(images to be denoted by $x\gamma y$ for $x, y \in R$ and $\gamma \in \Gamma$) satisfying the following axioms:
(1) $x\gamma(y\mu z) = (x\gamma y)\mu z$,
(2) a family $(x_i : i \in I)$ is summable in R implies $(x\gamma x_i : i \in I)$ is summable in R and $x\gamma[\Sigma(x_i : i \in I)] = \Sigma(x\gamma x_i : i \in I)$,
(3) a family $(x_i : i \in I)$ is summable in R implies $(x_i\gamma x : i \in I)$ is summable in R and $[\Sigma(x_i : i \in I)]\gamma x = \Sigma(x_i\gamma x : i \in I)$,
(4) a family $(\gamma_i : i \in I)$ is summable in Γ implies $(x\gamma_i y : i \in I)$ is summable in R and $x[\Sigma'(\gamma_i : i \in I)]y = \Sigma(x\gamma_i y : i \in I)$ for all $x, y, z, (x_i : i \in I)$ in R and μ, γ, $(\gamma_i : i \in I)$ in Γ.

Definition 2.5[5]. Let R be a partial Γ-semiring. Let A be a nonempty subset of R and Γ' be a nonempty subset of Γ. Then the pair (A, Γ') is said to be a *partial sub* Γ-*semiring* if
(i) A is a partial submonoid of R,
(ii) Γ' is a partial submonoid of Γ,
(iii) $A\Gamma' A \subseteq A$.

Definition 2.6[6]. The *sum ordering* $\leq$ on a partial monoid (M, Σ) is the binary relation such that $x \leq y$ if and only if there exists an h in M such that $y = x + h$ for $x, y \in M$.

Definition 2.7[6]. A *sum-ordered partial monoid* or *so-monoid*, in short, is a partial monoid in which the sum ordering is a partial ordering.

Definition 2.8[5]. A partial Γ-semiring R is said be a *sum-ordered partial* Γ-*semiring* (in short Γ-so-ring) if the partial monoids R and Γ are so-monoids.

The *support* of a family $(x_i : i \in I)$ in M is defined to be the subfamily $(x_i : i \in J)$ where $J = \{i \in I \mid x_i \neq 0\}$.

Example 2.9[5] Let $R = \Gamma := Z^- \bigcup \{0\}$, the set of all nonpositive integers. Then R and Γ are partial monoids with finite support addition. Now R is a partial Γ-semiring with usual multiplication of integers. Also R is a Γ-so-ring with the partial order "usual less than or equal to". However R is not a so-ring in the sense of [9]. Since $-2, -3 \in R$ and $(-2)(-3) = 6 \notin R$.

Example 2.10[5]. Let D, E be any two sets. Then $Pfn(D, E)$ and $Pfn(E, D)$ are partial monoids with the summations defined as in the Example 2.2. Consider the mapping $(f, \gamma, g) \mapsto f\gamma g$ of $Pfn(D, E) \times Pfn(E, D) \times Pfn(D, E)$ into $Pfn(D, E)$ where $d(f\gamma g) = (((df)\gamma)g)$, for any $d \in D$. Then $Pfn(D, E)$

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is a partial $Pfn(E, D)$-semiring.

In general $Pfn(D, E)$ is not $Pfn(E, D)$-semiring, since a family in the partial $Pfn(E, D)$-semiring $Pfn(D, E)$ need not be summable.

Example 2.11[5]. Let R be the partial monoid of all $m \times n$ matrices over the set of all non negative rational numbers and Γ be the partial monoid of all $n \times m$ matrices over the set of all non negative integers. For any $A, B \in R$ and $\alpha \in \Gamma$, consider the mapping $(A, \alpha, B) \mapsto A\alpha B$, usual matrix multiplication, of $R \times \Gamma \times R$ into R. Then R is a partial Γ-semiring.

Definition 2.12[3]. Let (M, Σ), (M', Σ') be two partial monoids. Then a function $\phi : M \to M'$ is said to be an *additive map* of (M, Σ) into (M', Σ') if $(x_i : i \in I)$ is a summable family in M implies $(x_i\phi : i \in I)$ is a summable family in M' and $(\Sigma_i x_i)\phi = \Sigma'_i(x_i\phi)$.

Definition 2.13[5]. Let R be a partial Γ-semiring and R' be a partial Γ'-semiring. Then a pair of mappings $f : R \longrightarrow R'$ and $g : \Gamma \longrightarrow \Gamma'$ is said to be a *homomorphism* from (R, Γ) into (R', Γ') if it satisfies the following:
(i) f is an additive map from R into R',
(ii) g is an additive map from Γ into Γ', and
(iii) $f(a\gamma b) = f(a)g(\gamma)f(b)$ for any $a, b \in R$ and $\gamma \in \Gamma$.

Definition 2.14[5]. Let R be a partial Γ-semiring, θ be a binary relation on R and σ be a binary relation on Γ. Then the pair (θ, σ) is called a *partial Γ-semiring congruence relation* on (R, Γ) if it satisfies the following:
(i) θ and σ are equivalence relations on R and Γ respectively,
(ii) θ (σ) is closed under the additive operation of the product partial monoid $R \times R$ $(\Gamma \times \Gamma)$. i.e., if $(a_i : i \in I)$ and $(b_i : i \in I)$ are summable families in R (Γ) such that $(a_i, b_i) \in \theta(\sigma)$ for all $i \in I$ then $\Sigma_i(a_i, b_i) \in \theta$ $(\Sigma'_i(a_i, b_i) \in \sigma)$,
(iii) $(a, b) \in \theta$, $(\alpha, \beta) \in \sigma$ and $(c, d) \in \theta$ then $(a\alpha c, b\beta d) \in \theta$.

Definition 2.15[1]. A partial monoid congruence relation θ on a partial monoid R is said to have the *diagonal property* if it satisfies the condition that for any $a, b \in R$, $a\theta(b + k)$ and $(a + h)\theta b$ for some $h, k \in R \Leftrightarrow a\theta b$.

Definition 2.16[5]. Let R be a partial Γ-semiring. Then a partial Γ-semiring congruence relation (θ, σ) on (R, Γ) is said to have the *diagonal property* if and only if θ and σ have the diagonal property.

Definition 2.17[5]. A partial Γ-semiring congruence relation (θ, σ) on a Γ-so-ring R is said to be a *congruence relation* on (R, Γ) if and only if (θ, σ) has the

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diagonal property.

Definition 2.18[5]. Let R be a partial Γ-semiring, θ be a partial Γ-semiring Γ-congruence relation on R. Then θ is said to be a Γ-*congruence relation* on R if and only if θ has the diagonal property.

Example 2.19. Consider the Γ-so-ring $R := Z^- \bigcup\{0\}$ as in the Example 2.9. Define $\theta = \sigma$ on R as $a\theta b$ if and only if $a = b$. Then (θ, σ) is a partial Γ-semiring congruence relation on (R, Γ) such that (θ, σ) satisfies the diagonal property. Therefore (θ, σ) is a congruence relation on (R, Γ). However θ is not a partial semiring congruence relation on R as given in [7] and (θ, σ) is not a congruence relation on (R, Γ) in the sense of [4]. Since R is not a so-ring as in [7] and Γ-semiring as in [4].

We denote the set of all congruence relations of a Γ-so-ring R by $Con(R, \Gamma)$ and the set of all Γ-congruence relations of R by $ConR$.

3. Structure Theorem

Definition 3.1. Let $\{R_i \mid i \in I\}$ be a family of Γ_i-so-rings. Take $R = \prod_{i\in I} R_i$ and $\Gamma = \prod_{i\in I}\Gamma_i$. Let R' and Γ' be subsets of R and Γ respectively. Then the pair (R', Γ') is said to be a *subdirect product* of (R_i, Γ_i), $i \in I$ if it satisfies the following:
(i) (R', Γ') is a sub Γ-so-ring of R,
(ii) all the projection mappings of (R, Γ) restricted to (R', Γ') are epimorphisms.

Definition 3.2. Let $\{R_i \mid i \in I\}$ be a family of Γ-so-rings. Then a subset R of $\prod_{i\in I} R_i$ is said to be a Γ-*subdirect product* of R_i, $i \in I$ if it satisfies the following:
(i) R is a Γ-sub so-ring of $\prod_{i\in I} R_i$,
(ii) all the projection mappings of $\prod_{i\in I} R_i$ restricted to R are epimorphisms.

Definition 3.3. A Γ-so-ring R is said to be *subdirectly irreducible* if and only if $\bigcap_{i\in I}(\theta_i, \sigma_i) = (0_R, 0_\Gamma)$ where $\{(\theta_i, \sigma_i) \mid i \in I\}$ is a family of congruence relations on (R, Γ) implies $(\theta_i, \sigma_i) = (0_R, 0_\Gamma)$ for some $i \in I$.

Definition 3.4. A Γ-so-ring R is said to be Γ-*subdirectly irreducible* if and only if $\bigcap_{i\in I}\theta_i = 0_R$ where $\{\theta_i \mid i \in I\}$ is a family of Γ-congruence relations on R

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implies $\theta_i = 0_R$ for some $i \in I$.

Lemma 3.5. *Let R be a Γ-so-ring and (θ, σ) be a congruence relation on (R, Γ). Then there is a one to one correspondence between congruence relations on (R, Γ) containing (θ, σ) and congruence relations on $(R/\theta, \Gamma/\sigma)$.*

Proof. Let (ϕ, ρ) be a congruence relation on (R, Γ) containing (θ, σ). Define a relation ϕ/θ on R/θ by $[a]^\theta(\phi/\theta)[b]^\theta$ if and only if $a\phi b$, where $[a]^\theta$ ($[b]^\theta$) denotes the equivalence class containing a (b) relative to θ and define a relation ρ/σ on Γ/σ by $[\alpha]^\sigma(\rho/\sigma)[\beta]^\sigma$ if and only if $\alpha\rho\beta$, where $[\alpha]^\sigma$ ($[\beta]^\sigma$) denotes the equivalence class containing α (β) relative to σ. Since (ϕ, ρ) is an equivalence relation on (R, Γ), it follows that $(\phi/\theta, \rho/\sigma)$ is also an equivalence relation on $(R/\theta, \Gamma/\sigma)$. To prove $(\phi/\theta, \rho/\sigma)$ is congruence, let $([a_i]^\theta : i \in I)$ and $([b_i]^\theta : i \in I)$ be summable families in R/θ such that $[a_i]^\theta(\phi/\theta)[b_i]^\theta$, $i \in I$. Then $(a_i : i \in I)$ and $(b_i : i \in I)$ are summable families in R such that $a_i\phi b_i$, $i \in I$. $\Rightarrow (\Sigma_i a_i)\phi(\Sigma_i b_i)$. $\Rightarrow [\Sigma_i a_i]^\theta(\phi/\theta)[\Sigma_i b_i]^\theta$ and hence $\overline{\Sigma}_i[a_i]^\theta(\phi/\theta)\overline{\Sigma}_i[b_i]^\theta$. Let $([\alpha_i]^\sigma : i \in I)$ and $([\beta_i]^\sigma : i \in I)$ be summable families in Γ/σ such that $[\alpha_i]^\sigma(\rho/\sigma)[\beta_i]^\sigma$, $i \in I$. Then $(\alpha_i : i \in I)$ and $(\beta_i : i \in I)$ are summable families in Γ such that $\alpha_i\rho\beta_i$, $i \in I$. $\Rightarrow (\Sigma_i\alpha_i)\rho(\Sigma_i\beta_i)$. $\Rightarrow [\Sigma_i\alpha_i]^\sigma(\rho/\sigma)[\Sigma_i\beta_i]^\sigma$ and hence $\overline{\Sigma}_i[\alpha_i]^\sigma(\rho/\sigma)\overline{\Sigma}_i[\beta_i]^\sigma$. Let $[a]^\theta(\phi/\theta)[b]^\theta$, $[\alpha]^\sigma(\rho/\sigma)[\beta]^\sigma$ and $[c]^\theta(\phi/\theta)[d]^\theta$ where $a, b, c, d \in R$ and $\alpha, \beta \in \Gamma$. Then $a\phi b$, $\alpha\rho\beta$ and $c\phi d$. $\Rightarrow (a\alpha c)\phi(b\beta d)$ and hence $([a]^\theta[\alpha]^\sigma[c]^\theta)(\phi/\theta)([b]^\theta[\beta]^\sigma[d]^\theta)$. Hence $(\phi/\theta, \rho/\sigma)$ is a partial Γ/σ-semiring congruence relation on R/θ. Note that $[a]^\theta(\phi/\theta)([b]^\theta + [h]^\theta)$ and $([a]^\theta + [k]^\theta)(\phi/\theta)[b]^\theta$ for some $h, k \in R \Leftrightarrow a\phi(b + h)$ and $(a + k)\phi b \Leftrightarrow a\phi b$ $\Leftrightarrow [a]^\theta(\phi/\theta)[b]^\theta$. Therefore ϕ/θ has the diagonal property. Also note that $[\alpha]^\sigma(\rho/\sigma)([\beta]^\sigma + [h]^\sigma)$ and $([\alpha]^\sigma + [k]^\sigma)(\rho/\sigma)[\beta]^\sigma$ for some $h, k \in \Gamma \Leftrightarrow \alpha\rho(\beta + h)$ and $(\alpha + k)\rho\beta \Leftrightarrow \alpha\rho\beta \Leftrightarrow [\alpha]^\sigma(\rho/\sigma)[\beta]^\sigma$. Therefore ρ/σ has the diagonal property. Hence $(\phi/\theta, \rho/\sigma)$ is a congruence relation on $(R/\theta, \Gamma/\sigma)$.

Let (ϕ', ρ') be a congruence relation on $(R/\theta, \Gamma/\sigma)$. Define a relation ϕ_θ on R by $a\phi_\theta b$ if and only if $[a]^\theta\phi'[b]^\theta$ and a relation ρ_σ on Γ by $\alpha\rho_\sigma\beta$ if and only if $[\alpha]^\sigma\rho'[\beta]^\sigma$. Since (ϕ', ρ') is an equivalence relation on $(R/\theta, \Gamma/\sigma)$, it follows that $(\phi_\theta, \rho_\sigma)$ is also an equivalence relation on (R, Γ). To prove (ϕ', ρ') is congruence, let $(a_i : i \in I)$ and $(b_i : i \in I)$ be summable families in R such that $a_i\phi_\theta b_i$, $i \in I$. Then $([a_i]^\theta : i \in I)$ and $([b_i]^\theta : i \in I)$ are summable families in R/θ such that $[a_i]^\theta\phi'[b_i]^\theta$, $i \in I$. $\Rightarrow \overline{\Sigma}_i[a_i]^\theta\phi'\overline{\Sigma}_i[b_i]^\theta$ and hence $(\Sigma_i a_i)\phi_\theta(\Sigma_i b_i)$. Let $(\alpha_i : i \in I)$ and $(\beta_i : i \in I)$ be summable families in Γ such that $\alpha_i\rho_\sigma\beta_i$, $i \in I$. Then $([\alpha_i]^\sigma : i \in I)$ and $([\beta_i]^\sigma : i \in I)$ are summable families in Γ/σ such that $[\alpha_i]^\sigma\rho'[\beta_i]^\sigma$, $i \in I$. $\Rightarrow \overline{\Sigma}_i[\alpha_i]^\sigma\rho'\overline{\Sigma}_i[\beta_i]^\sigma$ and hence $(\Sigma_i\alpha_i)\rho_\sigma(\Sigma_i\beta_i)$. Let $a\phi_\theta b$, $\alpha\rho_\sigma\beta$ and $c\phi_\theta d$ where $a, b, c, d \in R$ and $\alpha, \beta \in \Gamma$. Then $[a]^\theta\phi'[b]^\theta$, $[\alpha]^\sigma\rho'[\beta]^\sigma$ and $[c]^\theta\phi'[d]^\theta$. $\Rightarrow [a\alpha c]^\theta\phi'[b\beta d]^\theta$ and hence $(a\alpha c)\phi_\theta(b\beta d)$. Hence $(\phi_\theta, \rho_\sigma)$ is a partial Γ-semiring congruence relation on R. Note that $a\phi_\theta(b+h)$ and $(a+k)\phi_\theta b$ for some $h, k \in R \Leftrightarrow [a]^\theta\phi'([b]^\theta + [h]^\theta)$ and

$([a]^\theta + [k]^\theta)\phi'[b]^\theta \Leftrightarrow [a]^\theta \phi'[b]^\theta \Leftrightarrow a\phi_\theta b$. Therefore ϕ_θ has the diagonal property. Also note that $\alpha\rho_\sigma(\beta+h)$ and $(\alpha+k)\rho_\sigma\beta$ for some $h, k \in \Gamma \Leftrightarrow [\alpha]^\sigma \rho'([\beta]^\sigma + [h]^\sigma)$ and $([\alpha]^\sigma + [k]^\sigma)\rho'[\beta]^\sigma \Leftrightarrow [\alpha]^\sigma \rho'[\beta]^\sigma \Leftrightarrow \alpha\rho_\sigma\beta$. Therefore ρ_σ has the diagonal property. Let $a\theta b$. Then $[a]^\theta = [b]^\theta$. $\Rightarrow [a]^\theta \phi'[b]^\theta$. $\Rightarrow a\phi_\theta b$. Therefore $\theta \subseteq \phi_\theta$. Let $\alpha\sigma\beta$. Then $[\alpha]^\sigma = [\beta]^\sigma$. $\Rightarrow [\alpha]^\sigma \rho'[\beta]^\sigma$. $\Rightarrow \alpha\rho_\sigma\beta$. Therefore $\sigma \subseteq \rho_\sigma$. Hence $(\phi_\theta, \rho_\sigma)$ is a congruence relation of (R, Γ) containing (θ, σ). The above definitions readily gives the correspondence. Hence the lemma. □

Corollary 3.6. *Let R be a Γ-so-ring and θ be a Γ-congruence relation on R. Then there is a one to one correspondence between Γ-congruence relations on R containing θ and Γ-congruence relations on R/θ.*

Lemma 3.7. *Let R be a Γ-so-ring and $\{(\theta_i, \sigma_i) \mid i \in I\}$ be a family of congruence relations on (R, Γ) such that $\bigcap_{i\in I}(\theta_i, \sigma_i) = (0_R, 0_\Gamma)$. Then (R, Γ) is isomorphic to a subdirect product of $(R/\theta_i, \Gamma/\sigma_i)$, $i \in I$.*

Proof. Define $f : R \to \prod_{i\in I} R/\theta_i$ by $f(a) = ([a]^{\theta_i} : i \in I)$ $\forall a \in R$ and $g : \Gamma \to \prod_{i\in I} \Gamma/\sigma_i$ by $g(\alpha) = ([\alpha]^{\sigma_i} : i \in I)$ $\forall \alpha \in \Gamma$. First we prove that (f, g) is a monomorphism. Let $(x_j : j \in J)$ be a summable family in R. Then $f(\Sigma_j x_j) = ([\Sigma_j x_j]^{\theta_i} : i \in I) = (\overline{\Sigma_j}[x_j]^{\theta_i} : i \in I) = \overline{\Sigma_j} f(x_j)$. Let $(\alpha_j : j \in J)$ be a summable family in Γ. Then $g(\Sigma_j \alpha_j) = ([\Sigma_j \alpha_j]^{\sigma_i} : i \in I) = (\overline{\Sigma_j}[\alpha_j]^{\sigma_i} : i \in I) = \overline{\Sigma_j} g(\alpha_j)$. For any $x, y \in R$ and $\alpha \in \Gamma$, $f(x\alpha y) = ([x\alpha y]^{\theta_i} : i \in I) = ([x]^{\theta_i}[\alpha]^{\sigma_i}[y]^{\theta_i} : i \in I) = f(x)g(\alpha)f(y)$. Let $x, y \in R$ such that $f(x) = f(y)$. Then $([x]^{\theta_i} : i \in I) = ([y]^{\theta_i} : i \in I)$. $\Rightarrow [x]^{\theta_i} = [y]^{\theta_i}$ $\forall i \in I$. $\Rightarrow x(\bigcap_{i\in I}\theta_i)y$. $\Rightarrow x = y$. Let $\alpha, \beta \in \Gamma$ such that $g(\alpha) = g(\beta)$. Then $([\alpha]^{\sigma_i} : i \in I) = ([\beta]^{\sigma_i} : i \in I)$. $\Rightarrow [\alpha]^{\sigma_i} = [\beta]^{\sigma_i}$ $\forall i \in I$. $\Rightarrow \alpha(\bigcap_{i\in I}\sigma_i)\beta$. $\Rightarrow \alpha = \beta$. Hence (f, g) is a monomorphism from (R, Γ) into $\prod_{i\in I}(R/\theta_i, \Gamma/\sigma_i)$. Consequently (f, g) is an isomorphism from (R, Γ) onto $(f(R), g(\Gamma))$.

Now we prove that $(f(R), g(\Gamma))$ is a subdirect product of $(R/\theta_i, \Gamma/\sigma_i)$, $i \in I$. It can be noted that $(f(R), g(\Gamma))$ is a sub $\prod_{i\in I}\Gamma/\sigma_i$-so-ring of $\prod_{i\in I} R/\theta_i$. For any $i \in I$, $p_i(f(R)) = \{p_i(f(x)) \mid x \in R\} = \{p_i([x]^{\theta_i} : i \in I) \mid x \in R\} = \{[x]^{\theta_i} \mid x \in R\} = R/\theta_i$ and $p'_i(g(\Gamma)) = \{p'_i(g(\alpha)) \mid \alpha \in \Gamma\} = \{p'_i([\alpha]^{\sigma_i} : i \in I) \mid \alpha \in \Gamma\} = \{[\alpha]^{\sigma_i} \mid \alpha \in \Gamma\} = \Gamma/\sigma_i$. Hence $(f(R), g(\Gamma))$ is a subdirect product of $(R/\theta_i, \Gamma/\sigma_i)$, $i \in I$. Hence the lemma. □

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Corollary 3.8. *Let $\{\theta_i \mid i \in I\}$ be a family of Γ-congruence relations on a Γ-so-ring R such that $\bigcap_{i\in I}\theta_i = 0_R$. Then R is isomorphic to a Γ-subdirect product of R/θ_i, $i \in I$.*

Lemma 3.9. *Let R be a Γ-so-ring and $\{(\theta_i, \sigma_i) \mid i \in I\}$ be a simply ordered family of congruence relations on (R, Γ). Then $\bigvee_{i\in I}(\theta_i, \sigma_i) = \bigcup_{i\in I}(\theta_i, \sigma_i)$.*

Proof. Since the family $\{(\theta_i, \sigma_i) \mid i \in I\}$ is simply ordered, it is obvious. □

Corollary 3.10. Let $\{\theta_i \mid i \in I\}$ be a simply ordered family of Γ-congruence relations on a Γ-so-ring R. Then $\bigvee_{i\in I}\theta_i = \bigcup_{i\in I}\theta_i$.

Lemma 3.11. *Let R be a Γ-so-ring. If $a, b \in R$ with $a \neq b$ and $\alpha, \beta \in \Gamma$ with $\alpha \neq \beta$, then there is a congruence relation $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$ on (R, Γ) such that $(a, b) \notin \theta_{(a,b)}$, $(\alpha, \beta) \notin \sigma_{(\alpha,\beta)}$ and $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$ is maximal with respect to this property.*

Proof. For any $a, b \in R$ such that $a \neq b$ and $\alpha, \beta \in \Gamma$ such that $\alpha \neq \beta$, let $\mathcal{C} = \{(\phi, \rho) \in Con(R, \Gamma) \mid (a, b) \notin \phi, (\alpha, \beta) \notin \rho\}$. It is obvious that $(0_R, 0_\Gamma) \in \mathcal{C}$. So, $\mathcal{C}$ is nonempty. By Zorn's lemma, $\mathcal{C}$ has a maximal element, say $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$. Hence the lemma. □

Corollary 3.12. *Let R be a Γ-so-ring. If $a, b \in R$ with $a \neq b$, then there is a Γ-congruence relation $\theta_{(a,b)}$ on R such that $(a, b) \notin \theta_{(a,b)}$ and $\theta_{(a,b)}$ is maximal with respect to this property.*

Lemma 3.13. *Let R be a Γ-so-ring. Then R is subdirectly irreducible if and only if $Con(R, \Gamma)$ has one and only one atom which is contained in every congruence relation other than $(0_R, 0_\Gamma)$, the zero congruence relation on (R, Γ).*

Proof. It can be proved easily. □

Corollary 3.14. *Let R be a Γ-so-ring. Then R is Γ-subdirectly irreducible if and only if $ConR$ has one and only one atom which is contained in every Γ-congruence relation other than 0_R, the zero Γ-congruence relation on R.*

Theorem 3.15. *A Γ-so-ring R, where both R and Γ are nonzero, is a subdirect product of subdirectly irreducible Γ_i-so-rings R_i, $i \in I$.*

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Proof. Let R be a Γ-so-ring. Consider the family of congruences $\mathcal{C} = \{(\theta_{(a,b)}, \sigma_{(\alpha,\beta)}) \mid a, b \in R, \alpha, \beta \in \Gamma$ with $a \neq b$ and $\alpha \neq \beta\}$ as constructed in the Lemma 3.11. Suppose $(x, y) \in \bigcap_{a \neq b} \theta_{(a,b)}$ and $(\gamma, \mu) \in \bigcap_{\alpha \neq \beta} \sigma_{(\alpha,\beta)}$. Then $(x, y) \in \theta_{(a,b)}\ \forall a \neq b$ and $(\gamma, \mu) \in \sigma_{(\alpha,\beta)}\ \forall \alpha \neq \beta$. If $x \neq y$ and $\gamma \neq \mu$, then $(x, y) \in \theta_{(x,y)}$ and $(\gamma, \mu) \in \sigma_{(\gamma,\mu)}$, a contradiction. Hence $(\bigcap_{a \neq b} \theta_{(a,b)}, \bigcap_{\alpha \neq \beta} \sigma_{(\alpha,\beta)}) = (0_R, 0_\Gamma)$. By the Lemma 3.7, (R, Γ) is isomorphic to a subdirect product of $\Gamma/\sigma_{(\alpha,\beta)}$-so-rings $R/\theta_{(a,b)}, a \neq b, \alpha \neq \beta$.

Now we prove that $R/\theta_{(a,b)}, a \neq b$ is a subdirectly irreducible $\Gamma/\sigma_{(\alpha,\beta)}$-so-ring, $\alpha \neq \beta$. Let $([\theta_{(a,b)}), [\sigma_{(\alpha,\beta)}))$ denote the set of all congruence relations on (R, Γ) containing $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$. Let (ψ, ρ) be the smallest congruence relation such that $(a, b) \in \psi$ and $(\alpha, \beta) \in \rho$. Then $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)}) \subseteq (\psi, \rho) \vee (\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$. If $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)}) = (\psi, \rho) \vee (\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$, then $(\psi, \rho) \subseteq (\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$, and so $(a, b) \in \theta_{(a,b)}$ for $a \neq b$ and $(\alpha, \beta) \in \sigma_{(\alpha,\beta)}$ for $\alpha \neq \beta$, a contradiction. Hence $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)}) \subset (\psi, \rho) \vee (\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$. $\Rightarrow (\psi, \rho) \vee (\theta_{(a,b)}, \sigma_{(\alpha,\beta)}) \in ([\theta_{(a,b)}), [\sigma_{(\alpha,\beta)}))$ and $(\psi, \rho) \vee (\theta_{(a,b)}, \sigma_{(\alpha,\beta)}) \neq (\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$. Let (ϕ, τ) be another congruence relation in $([\theta_{(a,b)}), [\sigma_{(\alpha,\beta)}))$ other than $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$. Then $(a, b) \in \phi$ and $(\alpha, \beta) \in \tau$. $\Rightarrow (\psi, \rho) \subseteq (\phi, \tau)$. Hence $(\psi, \rho) \vee (\theta_{(a,b)}, \sigma_{(\alpha,\beta)}) \subseteq (\phi, \tau)$. Therefore $(\psi, \rho) \vee (\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$ is the only atom which is contained in every congruence relation in $([\theta_{(a,b)}), [\sigma_{(\alpha,\beta)}))$ other than $(\theta_{(a,b)}, \sigma_{(\alpha,\beta)})$. By the Lemma 3.5, there is a one to one correspondence between the congruence relations on $(R/\theta_{(a,b)}, \Gamma/\sigma_{(\alpha,\beta)})$, $a \neq b, \alpha \neq \beta$ and $([\theta_{(a,b)}), [\sigma_{(\alpha,\beta)}))$. $\Rightarrow \mathrm{Con}(R/\theta_{(a,b)}, \Gamma/\sigma_{(\alpha,\beta)})$ has one and only one atom which is contained in every congruence relation other than $\overline{(\theta_{(a,b)}, \sigma_{(\alpha,\beta)})}$ (the zero congruence on $(R/\theta_{(a,b)}, \Gamma/\sigma_{(\alpha,\beta)})$). By the Lemma 3.13, $R/\theta_{(a,b)}$, $a \neq b$ is a subdirectly irreducible $\Gamma/\sigma_{(\alpha,\beta)}$-so-ring, $\alpha \neq \beta$. Hence the theorem. □

Corollary 3.16. *A Γ-so-ring R, where R is nonzero, is a Γ-subdirect product of Γ-subdirectly irreducible Γ-so-rings R_i, $i \in I$.*

4. (ϕ, ρ)-REPRESENTATION OF Γ-SO-RINGS

Walendziak[9] introduced the notion of ϕ-representation of algebras and studied the necessary and sufficient condition for $< (A_i : i \in I), f >$ to be a ϕ-representation. We extend these to Γ-so-rings.

Definition 4.1. Let $\{R_i \mid i \in I\}$ be a family of Γ_i-so-rings, (R, Γ) be a subdirect product of (R_i, Γ_i), $i \in I$, and let $(\phi, \rho) \in Con(R, \Gamma)$. Then the pair (R, Γ) is said to be a (ϕ, ρ)-*product* of (R_i, Γ_i), $i \in I$ if it satisfies the following:
(i) for every $\overline{x} = (x_i : i \in I) \in R^I$, if $(x_i, x_j) \in \phi\ \forall i, j \in I$ then $< x_i(i) : i \in I > \in R$,

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(ii) for every $\overline{\alpha} = (\alpha_i : i \in I) \in \Gamma^I$, if $(\alpha_i, \alpha_j) \in \rho \ \forall i, j \in I$ then $< \alpha_i(i) : i \in I > \in \Gamma$.

Definition 4.2. Let $\{R_i \mid i \in I\}$ be a family of Γ-so-rings, R be a Γ-*subdirect product* of $R_i, i \in I$, and let $\phi \in ConR$. Then R is said to be a ϕ-product of R_i, $i \in I$ if it satisfies the condition that for every $\overline{x} = (x_i : i \in I) \in R^I$, if $(x_i, x_j) \in \phi \ \forall i, j \in I$ then $< x_i(i) : i \in I > \in R$.

Example 4.3. Take $R_1 = \{0, 1\}$. Define Σ on R_1 as

$$\Sigma_i x_i = \begin{cases} 1, \ if \ x_i = 0 \ \forall i \neq j \ for \ some \ j \\ 1, \ if \ x_h = 1, \ x_k = 1 \ for \ some \ h, k, \ x_i = 0 \ \forall i \neq h, k \\ undefined, \ otherwise. \end{cases}$$

Take $\Gamma_1 = \{0', 1'\}$. Define Σ' on Γ_1 as

$$\Sigma'_i \alpha_i = \begin{cases} 1', \ if \ \alpha_i = 0 \ \forall i \neq j \ for \ some \ j \\ undefined, \ otherwise. \end{cases}$$

Then R_1 and Γ_1 are partial monoids. Consider the mapping $(x, \alpha, y) \mapsto x\alpha y$ of $R_1 \times \Gamma_1 \times R_1$ into R_1 as follows:

$0'$	0	1
0	0	0
1	0	0

$1'$	0	1
0	0	0
1	0	1

Then R_1 is a Γ_1-so-ring. Take $R_2 = \{0, a, 1\}$. Define Σ on R_2 as

$$\Sigma_i x_i = \begin{cases} x_j, \ if \ x_i = 0 \ \forall i \neq j \ for \ some \ j \\ a, \ if \ x_h = x_k = a \ for \ some \ h, k, \ x_i = 0 \ \forall i \neq h, k \\ 1, \ if \ x_h = 1, \ x_k = a \ or \ 1, \ for \ some \ h, k, \ x_i = 0 \ \forall i \neq h, k \\ undefined, \ otherwise. \end{cases}$$

Then R_2 is a partial monoid. Take $\Gamma_2 := \Gamma_1$. Consider the mapping $(x, \alpha, y) \mapsto x\alpha y$ of $R_2 \times \Gamma_2 \times R_2$ into R_2 as follows:

$0'$	0	a	1
0	0	0	0
a	0	0	0
1	0	0	0

$1'$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

Then R_2 is a Γ_2-so-ring. Also $R_1 \times R_2$ is a $\Gamma_1 \times \Gamma_2$-so-ring. Take $R = \{< 0, 0 >, < 1, 0 >, < 1, a >, < 1, 1 >\}$, $\Gamma = \{< 0', 0' >, < 1', 1' >\}$, $\theta = \{(< 0, 0 >, < 0, 0 >), (< 1, 0 >, < 1, 0 >), (< 1, a >, < 1, a >), (< 1, 1 >, < 1, 1 >), (< 0, 0 >, < 1, 0 >), (< 1, 0 >, < 0, 0 >)\}$ and $\rho = \{(< 0', 0' >, < 0', 0' >), (< 1', 1' >, < 1', 1' >)\}$. Then (R, Γ) is a subdirect product of (R_i, Γ_i), $i = 1, 2$

and (θ, ρ) is a congruence relation on (R, Γ). Now for every $\overline{x} = (x_1, x_2) \in R^2$, if $(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2) \in \theta$ then $< x_1(1), x_2(2) > \in R$ and $\overline{\alpha} = (\alpha_1, \alpha_2) \in \Gamma^2$, if $(\alpha_1, \alpha_1), (\alpha_1, \alpha_2), (\alpha_2, \alpha_1), (\alpha_2, \alpha_2) \in \rho$ then $< \alpha_1(1), \alpha_2(2) > \in \Gamma$. Also $p_1(R) = R_1, p_2(R) = R_2$ and $p'_1(\Gamma) = \Gamma_1, p'_2(\Gamma) = \Gamma_2$. Hence (R, Γ) is a (θ, ρ)-product of (R_i, Γ_i), $i = 1, 2$.

Now take $\phi = \{(< 0, 0 >, < 0, 0 >), (< 1, 0 >, < 1, 0 >), (< 1, a >, < 1, a >), (< 1, 1 >, < 1, 1 >), (< 0, 0 >, < 1, 0 >), (< 1, 0 >, < 0, 0 >), (< 0, 0 >, < 1, a >), (< 1, a >, < 0, 0 >), (< 1, 0 >, < 1, a >), (< 1, a >, < 1, 0 >)\}$. Then (ϕ, ρ) is a congruence relation on (R, Γ). Take $\overline{x} = (< 0, 0 >, < 1, a >) \in R^2$. Then $(< 0, 0 >, < 0, 0 >), (< 0, 0 >, < 1, a >), (< 1, a >, < 0, 0 >), (< 1, a >, < 1, a >) \in \phi$. But $<< 0, 0 > (1), < 1, a > (2) > = < 0, a > \notin R$. Thus (R, Γ) is not a (ϕ, ρ)-product of (R_i, Γ_i), $i = 1, 2$.

Lemma 4.4. *Let R be a Γ-so-ring and $R_i (i \in I)$ be a family of Γ_i-so-rings. Then*
(i) (R, Γ) is a subdirect product of $(R_i, \Gamma_i), i \in I$ if and only if (R, Γ) is a $(0_R, 0_\Gamma)$-product of (R_i, Γ_i), $i \in I$.
(ii) (R, Γ) is a $(1_R, 1_\Gamma)$-product of (R_i, Γ_i), $i \in I$ if and only if $(R, \Gamma) = \prod\limits_{i \in I} (R_i, \Gamma_i)$.

Proof. (i) Suppose (R, Γ) is a $(0_R, 0_\Gamma)$-product of (R_i, Γ_i), $i \in I$. Then (R, Γ) is a subdirect product of (R_i, Γ_i), $i \in I$. Conversely, suppose that (R, Γ) is a subdirect product of Γ_i-so-rings $R_i, i \in I$. Let $\overline{x} := (x_i : i \in I) \in R^I \ni (x_i, x_j) \in 0_R \ \forall i, j \in I$. Then $< x_i(i) : i \in I >= x_i \in R$ (since $x_i = x_j, \ \forall i, j \in I$). Let $\overline{\alpha} := (\alpha_i : i \in I) \in \Gamma^I \ni (\alpha_i, \alpha_j) \in 0_\Gamma \ \forall i, j \in I$. Then $< \alpha_i(i) : i \in I >= \alpha_i \in \Gamma$ (since $\alpha_i = \alpha_j, \ \forall i, j \in I$). Hence (R, Γ) is a $(0_R, 0_\Gamma)$-product of (R_i, Γ_i), $i \in I$.
(ii) Suppose (R, Γ) is a $(1_R, 1_\Gamma)$-product of (R_i, Γ_i), $i \in I$. Let $x \in \prod\limits_{i \in I} R_i$. Then $x =< x_i : i \in I > \in \prod\limits_{i \in I} R_i$ where $x_i \in R_i \ \forall \ i \in I$. Since $p_i \mid_R : R \to R_i$ is a surjective homomorphism, $\exists \ a_i \in R \ni p_i \mid_R (a_i) = x_i \ \forall \ i \in I$. $\Rightarrow a_i(i) = x_i \ \forall \ i \in I$. Put $\overline{a} := (a_i : i \in I)$. Since $1_R = R^2$, $(a_i, a_j) \in 1_R \ \forall i, j \in I$. $\Rightarrow < a_i(i) : i \in I > \in R$. $\Rightarrow x =< x_i : i \in I > \in R$. Hence $R = \prod\limits_{i \in I} R_i$. Similarly, we can prove that $\Gamma = \prod\limits_{i \in I} \Gamma_i$.

Conversely suppose that $(R, \Gamma) = \prod\limits_{i \in I} (R_i, \Gamma_i)$. Then (R, Γ) is clearly a subdirect product of (R_i, Γ_i), $i \in I$. Let $\overline{x} := (x_i : i \in I) \in R^I \ni (x_i, x_j) \in 1_R \ \forall i, j \in I$. $\Rightarrow < x_i(i) : i \in I > \in \prod\limits_{i \in I} R_i = R$. Let $\overline{\alpha} := (\alpha_i : i \in I) \in \Gamma^I \ni$

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$(\alpha_i, \alpha_j) \in 1_\Gamma\ \forall i, j \in I.\ \Rightarrow <\alpha_i(i) : i \in I> \in \prod_{i\in I}\Gamma_i = \Gamma$. Hence (R, Γ) is a $(1_R, 1_\Gamma)$-product of (R_i, Γ_i), $i \in I$. □

Corollary 4.5. *Let R and $R_i (i \in I)$ be a family of Γ-so-rings. Then*
(i) R is a Γ-subdirect product of $R_i, i \in I$ if and only if R is a 0_R-product of $R_i, i \in I$.
(ii) R is a 1_R-product of $R_i, i \in I$ if and only if $R = \prod_{i\in I} R_i$.

Theorem 4.6. *Let $\{R_i \mid i \in I\}$ be a family of Γ_i-so-rings, let (R, Γ) be a sub $\prod_{i\in I}\Gamma_i$-so-ring of $\prod_{i\in I} R_i$ and let $(\phi, \rho) \in Con(R, \Gamma)$. For $i \in I$, let θ_i be the kernel of the projection at i, restricted to R and σ_i be the kernel of the projection at i, restricted to Γ. If (R, Γ) is a (ϕ, ρ)-product of (R_i, Γ_i), $i \in I$, then*
(i) $(0_R, 0_\Gamma) = \bigcap_{i\in I}(\theta_i, \sigma_i)$,
(ii) for every $\overline{x} = (x_i : i \in I) \in R^I$, if $(x_i, x_j) \in \phi\ \forall i, j \in I$, then $\exists\ x \in R \ni (x, x_i) \in \theta_i, \forall i \in I$,
(iii) for every $\overline{\alpha} = (\alpha_i : i \in I) \in \Gamma^I$, if $(\alpha_i, \alpha_j) \in \rho\ \forall i, j \in I$, then $\exists\ \alpha \in \Gamma \ni (\alpha, \alpha_i) \in \sigma_i, \forall i \in I$,
(iv) $(R/\theta_i, \Gamma/\sigma_i) \cong (R_i, \Gamma_i)\ \forall i \in I$.

Proof. (i) Let $(x, y) \in \bigcap_{i\in I}\theta_i$ where $x, y \in R$. Then $p_i \mid_R (x) = p_i \mid_R (y)\ \forall i \in I$. $\Rightarrow p_i(x) = p_i(y)\ \forall i \in I.\ \Rightarrow x(i) = y(i)\ \forall\ i \in I.\ \Rightarrow x = y$. Let $(\alpha, \beta) \in \bigcap_{i\in I}\sigma_i$ where $\alpha, \beta \in \Gamma$. Then $p'_i \mid_\Gamma (\alpha) = p'_i \mid_\Gamma (\beta)\ \forall i \in I.\ \Rightarrow p'_i(\alpha) = p'_i(\beta)\ \forall i \in I.\ \Rightarrow \alpha(i) = \beta(i)\ \forall\ i \in I.\ \Rightarrow \alpha = \beta$. Hence $\bigcap_{i\in I}(\theta_i, \sigma_i) = (0_R, 0_\Gamma)$.
(ii) Let $\overline{x} := (x_i : i \in I) \in R^I \ni (x_i, x_j) \in \phi\ \forall i, j \in I$. Put $x :=< x_i(i) : i \in I >$. So $x \in R, x_i \in R$ and $x(i) = x_i(i)\ \forall\ i \in I.\ \Rightarrow p_i \mid_R (x) = p_i \mid_R (x_i)\ \forall i \in I.$ $\Rightarrow (x, x_i) \in \theta_i\ \forall i \in I$. Hence $\exists\ x \in R \ni (x, x_i) \in \theta_i, \forall i \in I$.
(iii) Let $\overline{\alpha} := (\alpha_i : i \in I) \in \Gamma^I \ni (\alpha_i, \alpha_j) \in \rho\ \forall i, j \in I$. Put $\alpha :=< \alpha_i(i) : i \in I >$. So $\alpha \in \Gamma, \alpha_i \in \Gamma$ and $\alpha(i) = \alpha_i(i)\ \forall i \in I.\ \Rightarrow p'_i \mid_\Gamma (\alpha) = p'_i \mid_\Gamma (\alpha_i)\ \forall i \in I.$ $\Rightarrow (\alpha, \alpha_i) \in \sigma_i\ \forall i \in I$. Hence $\exists\ \alpha \in \Gamma \ni (\alpha, \alpha_i) \in \sigma_i, \forall i \in I$.
(iv). Define $f : R/\theta_i \to R_i$ by $[a]_{\theta_i} \mapsto a(i)$ and $g : \Gamma/\sigma_i \to \Gamma_i$ by $[\alpha]_{\sigma_i} \mapsto \alpha(i)$. Note that for any $[a]_{\theta_i}, [b]_{\theta_i} \in R/\theta_i$, $[a]_{\theta_i} = [b]_{\theta_i} \Leftrightarrow (a, b) \in \theta_i \Leftrightarrow p_i \mid_R (a) = p_i \mid_R (b) \Leftrightarrow a(i) = b(i) \Leftrightarrow f([a]_{\theta_i}) = f([b]_{\theta_i})$ and for any $[\alpha]_{\sigma_i}, [\beta]_{\sigma_i} \in \Gamma/\sigma_i$, $[\alpha]_{\sigma_i} = [\beta]_{\sigma_i} \Leftrightarrow (\alpha, \beta) \in \sigma_i \Leftrightarrow p'_i \mid_\Gamma (\alpha) = p'_i \mid_\Gamma (\beta) \Leftrightarrow \alpha(i) = \beta(i) \Leftrightarrow g([\alpha]_{\sigma_i}) = g([\beta]_{\sigma_i})$. Therefore (f, g) is well defined and one-one. For any $a_i \in R_i$ and $\alpha_i \in \Gamma_i$, $\exists\ a \in R$ and $\alpha \in \Gamma \ni a(i) = p_i \mid_R (a) = a_i$ and $\alpha(i) = p'_i \mid_\Gamma (\alpha) = \alpha_i$. Now $[a]_{\theta_i} \in R/\theta_i$, $[\alpha]_{\sigma_i} \in \Gamma/\sigma_i$ and $f([a]_{\theta_i}) = a(i) = a_i$, $g([\alpha]_{\sigma_i}) = \alpha(i) = \alpha_i$. Hence (f, g) is onto.

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To prove (f, g) is a homomorphism, let $([x_j]_{\theta_i} : j \in J)$ be a summable family in R/θ_i. Then $f(\Sigma_{j\in J}[x_j]_{\theta_i}) = f([\Sigma_j x_j]_{\theta_i}) = (\Sigma_j x_j)(i) = \Sigma_j f([x_j]_{\theta_i})$. Let $([\alpha_j]_{\sigma_i} : j \in J)$ be a summable family in Γ/σ_i. Then $g(\Sigma_{j\in J}[\alpha_j]_{\sigma_i}) = g([\Sigma_j \alpha_j]_{\sigma_i}) = (\Sigma_j \alpha_j)(i) = \Sigma_j g([\alpha_j]_{\sigma_i})$. Now for any $[a]_{\theta_i}, [b]_{\theta_i} \in R/\theta_i$ and $[\alpha]_{\sigma_i} \in \Gamma/\sigma_i$, $f([a]_{\theta_i}[\alpha]_{\sigma_i}[b]_{\theta_i}) = f([a\alpha b]_{\theta_i}) = (a\alpha b)(i) = a(i)\alpha(i)b(i)$
$= f([a]_{\theta_i})g([\alpha]_{\sigma_i})f([b]_{\theta_i})$ and hence (f, g) is a homomorphism from $(R/\theta_i, \Gamma/\sigma_i)$ onto (R_i, Γ_i), $i \in I$. Hence $(R/\theta_i, \Gamma/\sigma_i) \cong (R_i, \Gamma_i)$, $i \in I$. □

Corollary 4.7. *Let $\{R_i \mid i \in I\}$ be a family of Γ-so-rings, R be a Γ-sub so-ring of $\prod_{i\in I} R_i$ and let $\phi \in ConR$. For $i \in I$, let θ_i be the kernel of the projection at i, restricted to R. If R is a ϕ-product of $R_i, i \in I$, then*
(i) $0_R = \bigcap_{i\in I} \theta_i$,
(ii) for every $\overline{x} = (x_i : i \in I) \in R^I$, if $(x_i, x_j) \in \phi$ $\forall i, j \in I$, then $\exists$ $x \in R \ni (x, x_i) \in \theta_i, \forall i \in I$,
(iii) $R/\theta_i \cong R_i$ $\forall i \in I$.

Definition 4.8. Let R be a Γ-so-ring and let $(\phi, \rho) \in Con(R, \Gamma)$. For any family $\{(\theta_i, \sigma_i) \mid i \in I\}$ of congruence relations on (R, Γ), we write $(0_R, 0_\Gamma) = \Pi_{(\phi,\rho)}((\theta_i, \sigma_i) : i \in I)$ if and only if the conditions (i),(ii) and (iii) of above theorem are satisfied.

Definition 4.9. Let R be a Γ-so-ring and let $\phi \in ConR$. For any family $\{\theta_i \mid i \in I\}$ of Γ-congruence relations on R, we write $0_R = \Pi_\phi(\theta_i : i \in I)$ if and only if the conditions (i) and (ii) of above corollary are satisfied.

Remark 4.10. Let R be a Γ-so-ring and let $\{(\theta_i, \sigma_i) \mid i \in I\}$ be a family of congruence relations on (R, Γ). Then
(i) $(0_R, 0_\Gamma) = \Pi_{(0_R,0_\Gamma)}((\theta_i, \sigma_i) : i \in I)$ if and only if $(0_R, 0_\Gamma) = \bigcap_{i\in I}(\theta_i, \sigma_i)$,
(ii) $(0_R, 0_\Gamma) = \Pi_{(1_R,1_\Gamma)}((\theta_i, \sigma_i) : i \in I)$ if and only if $(0_R, 0_\Gamma) = \bigcap_{i\in I}(\theta_i, \sigma_i)$, for every $(x_i : i \in I) \in R^I$, $\exists$ $x \in R \ni (x, x_i) \in \theta_i$ and for every $(\alpha_i : i \in I) \in \Gamma^I$, $\exists$ $\alpha \in \Gamma \ni (\alpha, \alpha_i) \in \sigma_i$ $\forall i \in I$.

Remark 4.11. Let R be a Γ-so-ring and let $\{\theta_i \mid i \in I\}$ be a family of Γ-congruence relations on R. Then
(i) $0_R = \Pi_{0_R}(\theta_i : i \in I)$ if and only if $0_R = \bigcap_{i\in I} \theta_i$,
(ii) $0_R = \Pi_{1_R}(\theta_i : i \in I)$ if and only if $0_R = \bigcap_{i\in I} \theta_i$ and for every $(x_i : i \in I) \in R^I$,

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$\exists\ x \in R \ni (x, x_i) \in \theta_i\ \forall i \in I$.

Definition 4.12. Let R be a Γ-so-ring and R' be a Γ'-so-ring. Suppose (f, g) is an epimorphism from (R, Γ) onto (R', Γ'). For any congruence relation (ϕ, ρ) on (R, Γ), define $(f(\phi), g(\rho)) = (\{(f(x), f(y)) \mid (x, y) \in \phi\}, \{(g(\alpha), g(\beta)) \mid (\alpha, \beta) \in \rho\})$.

It can be easily observed that the relation $(f(\phi), g(\rho))$ is a congruence relation on (R', Γ').

Lemma 4.13. *Let R be a Γ-so-ring, R' be a Γ'-so-ring and (ϕ, ρ), $\{(\theta_i, \sigma_i) \mid i \in I\}$ be a family of congruence relations on (R, Γ). If (f, g) is an isomorphism from (R, Γ) onto (R', Γ'), then $(0_R, 0_\Gamma) = \Pi_{(\phi,\rho)}((\theta_i, \sigma_i) : i \in I)$ if and only if $(0_{R'}, 0_{\Gamma'}) = \Pi_{(f(\phi),g(\rho))}((f(\theta_i), g(\sigma_i)) : i \in I)$.*

Proof. Suppose $(0_R, 0_\Gamma) = \Pi_{(\phi,\rho)}((\theta_i, \sigma_i) : i \in I)$. Since (ϕ, ρ) and $\{(\theta_i, \sigma_i) \mid i \in I\}$ are congruence relations on (R, Γ), $(f(\phi), g(\rho))$ and $\{(f(\theta_i), g(\sigma_i)) \mid i \in I\}$ are also congruence relations on (R', Γ'). Let $(f(x), f(y)) \in \bigcap_{i\in I} f(\theta_i)$ and $(g(\alpha), g(\beta)) \in \bigcap_{i\in I} g(\sigma_i)$. Then $(x, y) \in \bigcap_{i\in I} \theta_i = 0_R$ and $(\alpha, \beta) \in \bigcap_{i\in I} \sigma_i = 0_\Gamma$. $\Rightarrow x = y$ and $\alpha = \beta$. $\Rightarrow f(x) = f(y)$ and $g(\alpha) = g(\beta)$. Therefore $\bigcap_{i\in I} (f(\theta_i), g(\sigma_i)) = (0_{R'}, 0_{\Gamma'})$. Let $\overline{y} := (y_i : i \in I) \in {R'}^I \ni (y_i, y_j) \in f(\phi)\ \forall i, j \in I$. Since f is onto, $\exists\ \overline{x} := (x_i : i \in I) \in R^I \ni f(x_i) = y_i\ \forall i \in I\ \&\ (x_i, x_j) \in \phi\ \forall i, j \in I$. $\Rightarrow \exists\ x \in R \ni (x, x_i) \in \theta_i\ \forall i \in I$. $\Rightarrow \exists\ y := f(x) \in R' \ni (y, y_i) \in f(\theta_i)\ \forall i \in I$. Let $\overline{\beta} := (\beta_i : i \in I) \in {\Gamma'}^I \ni (\beta_i, \beta_j) \in g(\rho)\ \forall i, j \in I$. Since g is onto, $\exists\ \overline{\alpha} := (\alpha_i : i \in I) \in \Gamma^I \ni g(\alpha_i) = \beta_i\ \forall i \in I\ \&\ (\alpha_i, \alpha_j) \in \rho\ \forall i, j \in I$. $\Rightarrow \exists\ \alpha \in \Gamma \ni (\alpha, \alpha_i) \in \sigma_i\ \forall i \in I$. $\Rightarrow \exists\ \beta := g(\alpha) \in \Gamma' \ni (\beta, \beta_i) \in g(\sigma_i)\ \forall i \in I$. Hence $(0_{R'}, 0_{\Gamma'}) = \Pi_{(f(\phi),g(\rho))}((f(\theta_i), g(\sigma_i)) : i \in I)$.

Suppose $(0_{R'}, 0_{\Gamma'}) = \Pi_{(f(\phi),g(\rho))}((f(\theta_i), g(\sigma_i)) : i \in I)$. Let $(x, y) \in \bigcap_{i\in I} \theta_i$ and $(\alpha, \beta) \in \bigcap_{i\in I} \sigma_i$. Then $(f(x), f(y)) \in \bigcap_{i\in I} f(\theta_i) = 0_{R'}$ and $(g(\alpha), g(\beta)) \in \bigcap_{i\in I} g(\sigma_i) = 0_{\Gamma'}$. $\Rightarrow f(x) = f(y)$ and $g(\alpha) = g(\beta)$. Since f and g are one-one, $x = y$ and $\alpha = \beta$. Hence $\bigcap_{i\in I} (\theta_i, \sigma_i) = (0_R, 0_\Gamma)$. Let $\overline{x} := (x_i : i \in I) \in R^I \ni (x_i, x_j) \in \phi\ \forall i, j \in I$. Then $\overline{y} := (f(x_i) : i \in I) \in {R'}^I \ni (f(x_i), f(x_j)) \in f(\phi)\ \forall i, j \in I$. Since f is onto, $\exists\ y \in R' \ni (y, f(x_i)) \in f(\theta_i)\ \forall i \in I$. $\Rightarrow \exists\ x \in R \ni f(x) = y\ \&\ (x, x_i) \in \theta_i\ \forall i \in I$. Let $\overline{\alpha} := (\alpha_i : i \in I) \in \Gamma^I \ni (\alpha_i, \alpha_j) \in \rho\ \forall i, j \in I$. Then $\overline{\beta} := (g(\alpha_i) : i \in I) \in {\Gamma'}^I \ni (g(\alpha_i), g(\alpha_j)) \in g(\sigma_i)\ \forall i, j \in I$. Since g is onto, $\exists\ \beta \in \Gamma' \ni (\beta, g(\alpha_i)) \in g(\sigma_i)\ \forall i \in I$. $\Rightarrow \exists\ \alpha \in$

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$\Gamma \ni g(\alpha) = \beta$ & $(\alpha, \alpha_i) \in \rho\ \forall i \in I$. Hence $(0_R, 0_\Gamma) = \Pi_{(\phi,\rho)}((\theta_i, \sigma_i) : i \in I)$. $\square$

Corollary 4.14. *Let R and R' be Γ-so-rings and $\phi, \theta_i (i \in I)$ be a family Γ-congruence relations on R. If f is a Γ-isomorphism from R onto R', then $0_R = \Pi_\phi(\theta_i : i \in I)$ if and only if $0_{R'} = \Pi_{f(\phi)}(f(\theta_i) : i \in I)$.*

Theorem 4.15. *Let R be a Γ-so-ring, $(\phi, \rho) \in Con(R, \Gamma)$ and $\{(\theta_i, \sigma_i) \mid i \in I\}$ be a family of congruence relations on (R, Γ) such that $(0_R, 0_\Gamma) = \Pi_{(\phi,\rho)}((\theta_i, \sigma_i) : i \in I)$. If the mappings $f : R \to \prod_{i\in I} R/\theta_i$ and $g : \Gamma \to \prod_{i \in I} \Gamma/\sigma_i$ are defined by $f(x) = ([x]_{\theta_i} : i \in I)$ and $g(\alpha) = ([\alpha]_{\sigma_i} : i \in I)$ respectively, then $(f(R), g(\Gamma))$ is a $(f(\phi), g(\rho))$-product of $(R/\theta_i, \Gamma/\sigma_i)$, $i \in I$.*

Proof. By the proof of the Lemma 3.7, (f, g) is a monomorphism from (R, Γ) into $\prod_{i\in I}(R/\theta_i, \Gamma/\sigma_i)$ and $(f(R), g(\Gamma))$ is a subdirect product of $(R/\theta_i, \Gamma/\sigma_i)$, $i \in I$. Let $(y_i : i \in I) \in f(R)^I \ni (y_i, y_j) \in f(\phi)\ \forall i, j \in I$. Then $\exists\ x_i \in R\ \forall i \in I \ni f(x_i) = y_i$ & $(x_i, x_j) \in \phi\ \forall i, j \in I$. $\Rightarrow \exists\ x \in R \ni (x, x_i) \in \theta_i\ \forall i \in I$. $\Rightarrow [x]_{\theta_i} = [x_i]_{\theta_i}\ \forall i \in I$. $\Rightarrow f(x)(i) = f(x_i)(i) = y_i(i)\ \forall i \in I$ and hence $(y_i(i) : i \in I) = f(x) \in f(R)$. Let $(\beta_i : i \in I) \in g(\Gamma)^I \ni (\beta_i, \beta_j) \in g(\rho)\ \forall i, j \in I$. Then $\exists\ \alpha_i \in \Gamma\ \forall i \in I \ni g(\alpha_i) = \beta_i$ & $(\alpha_i, \alpha_j) \in \rho\ \forall i, j \in I$. $\Rightarrow \exists\ \alpha \in \Gamma \ni (\alpha, \alpha_i) \in \sigma_i\ \forall i \in I$. $\Rightarrow [\alpha]_{\sigma_i} = [\alpha_i]_{\sigma_i}\ \forall i \in I$. $\Rightarrow g(\alpha)(i) = g(\alpha_i)(i) = \beta_i(i)\ \forall i \in I$ and hence $(\beta_i(i) : i \in I) = g(\alpha) \in g(\Gamma)$. Hence $(f(R), g(\Gamma))$ is a $(f(\phi), g(\rho))$-product of $(R/\theta_i, \Gamma/\sigma_i)$, $i \in I$. $\square$

Corollary 4.16. *Let R be a Γ-so-ring, $\phi \in ConR$ and $\{\theta_i \mid i \in I\}$ be a family of Γ-congruence relations on R such that $0_R = \Pi_\phi(\theta_i : i \in I)$. If the mapping $f : R \to \prod_{i\in I} R/\theta_i$ is defined by $f(x) = ([x]_{\theta_i} : i \in I)\ \forall x \in R$, then $f(R)$ is a $f(\phi)$-product of $R/\theta_i, i \in I$.*

Definition 4.17. Let R be a Γ-so-ring, $\{R_i \mid i \in I\}$ be a family of Γ_i-so-rings and $(\phi, \rho) \in Con(R, \Gamma)$. If (f, g) is a monomorphism of (R, Γ) into $\prod_{i\in I}(R_i, \Gamma_i)$ and if $(f(R), g(\Gamma))$ is a $(f(\phi), g(\rho))$-product of (R_i, Γ_i), $i \in I$, then the ordered pair $< ((R_i, \Gamma_i) : i \in I), (f, g) >$ is called a (ϕ, ρ)-*representation* of (R, Γ).

Definition 4.18. Let R, $R_i(i \in I)$ be a family of Γ-so-rings and $\phi \in ConR$. If f is a Γ-monomorphism of R into $\prod_{i\in I} R_i$ and if $f(R)$ is a $f(\phi)$-product of $R_i, i \in I$, then the ordered pair $< (R_i : i \in I), f >$ is called a ϕ-*representation* of R.

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Remark 4.19. Let R be a Γ-so-ring and $\{R_i \mid i \in I\}$ be a family of Γ_i-so-rings. Suppose (f, g) is a monomorphism from (R, Γ) into $\prod\limits_{i\in I}(R_i, \Gamma_i)$. Then
(i) $< ((R_i, \Gamma_i) : i \in I), (f, g) >$ is a $(0_R, 0_\Gamma)$-representation of (R, Γ) if and only if $(f(R), g(\Gamma))$ is a subdirect product of (R_i, Γ_i), $i \in I$,
(ii) $< ((R_i, \Gamma_i) : i \in I), (f, g) >$ is a $(1_R, 1_\Gamma)$-representation of (R, Γ) if and only if $(f(R), g(\Gamma))$ is the direct product of (R_i, Γ_i), $i \in I$.

Remark 4.20. Let R and $R_i (i \in I)$ be a family of Γ-so-rings. Suppose f is a Γ-monomorphism from R into $\prod\limits_{i\in I} R_i$. Then

(i) $< (R_i : i \in I), f >$ is a 0_R-representation of R if and only if $f(R)$ is a Γ-subdirect product of $R_i, i \in I$,

(ii) $< (R_i : i \in I), f >$ is a 1_R-representation of R if and only if $f(R)$ is the Γ-direct product of $R_i, i \in I$.

Theorem 4.21. *Let R be a Γ-so-ring, $(\phi, \rho) \in Con(R, \Gamma)$ and $\{(\theta_i, \sigma_i) \mid i \in I\}$ be a family of congruence relations on (R, Γ). If the mappings $f : R \to \prod\limits_{i\in I} R/\theta_i$ and $g : \Gamma \to \prod\limits_{i\in I} \Gamma/\sigma_i$ are defined by $f(x) = ([x]_{\theta_i} : i \in I)$ and $g(\alpha) = ([\alpha]_{\sigma_i} : i \in I)$ respectively, then $< ((R/\theta_i, \Gamma/\sigma_i) : i \in I), (f, g) >$ is a (ϕ, ρ)-representation of $(R, \Gamma) \Leftrightarrow (0_R, 0_\Gamma) = \Pi_{(\phi,\rho)}((\theta_i, \sigma_i) : i \in I)$.*

Proof. Suppose $< ((R/\theta_i, \Gamma/\sigma_i) : i \in I), (f, g) >$ is a (ϕ, ρ)-representation of (R, Γ). Then $(f(R), g(\Gamma))$ is a $(f(\phi), g(\rho))$-product of $(R/\theta_i, \Gamma/\sigma_i)$, $i \in I$. Note that $(f(x), f(y)) \in f(\theta_i) \Leftrightarrow (x, y) \in \theta_i \Leftrightarrow [x]_{\theta_i} = [y]_{\theta_i} \Leftrightarrow p_i \mid_{f(R)} (f(x)) = p_i \mid_{f(R)} (f(y)) \Leftrightarrow (f(x), f(y)) \in ker(p_i \mid_{f(R)})$. Also note that $(g(\alpha), g(\beta)) \in g(\sigma_i) \Leftrightarrow (\alpha, \beta) \in \sigma_i \Leftrightarrow [\alpha]_{\sigma_i} = [\beta]_{\sigma_i} \Leftrightarrow p'_i \mid_{g(\Gamma)} (g(\alpha)) = p'_i \mid_{g(\Gamma)} (g(\beta)) \Leftrightarrow (g(\alpha), g(\beta)) \in ker(p'_i \mid_{g(\Gamma)})$. Therefore $(f(\theta_i), g(\sigma_i)), i \in I$ are the kernels of the projections at i restricted to $(f(R), g(\Gamma))$. Then by the Theorem 4.6, $(0_{f(R)}, 0_{g(\Gamma)}) = \bigcap((f(\theta_i), g(\sigma_i)) : i \in I)$. Hence $(0_{f(R)}, 0_{g(\Gamma)}) = \prod_{(f(\phi),g(\rho))}((f(\theta_i), g(\sigma_i)) : i \in I)$. By the Lemma 4.13, $(0_R, 0_\Gamma) = \Pi_{(\phi,\rho)}((\theta_i, \sigma_i) : i \in I)$. The converse part is trivial in view of the Theorem 4.15. Hence the theorem. □

Corollary 4.22. *Let R be a Γ-so-ring, ϕ, $\theta_i (i \in I) \in ConR$. Define $f : R \to \prod R/\theta_i$ by $x \mapsto ([x]_{\theta_i} : i \in I)$. Then $< (R/\theta_i : i \in I), f >$ is a ϕ-representation of R if and only if $0_R = \prod_\phi(\theta_i : i \in I)$.*

Remark 4.23. (i) A family $\{(\theta_i, \sigma_i) \mid i \in I\}$ of congruence relations on a Γ-so-ring R gives a subdirect representation if and only if $\bigcap\limits_{i\in I}(\theta_i, \sigma_i) = (0_R, 0_\Gamma)$.

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(ii) A family $\{(\theta_i, \sigma_i) \mid i \in I\}$ of congruence relations on a Γ-so-ring R constitutes a direct representation if and only if $(0_R, 0_\Gamma) = \prod_{(1_R, 1_\Gamma)}((\theta_i, \sigma_i) : i \in I)$.

Remark 4.24. (i) A family $\{\theta_i \mid i \in I\}$ of Γ-congruence relations on a Γ-so-ring R gives a Γ-subdirect representation if and only if $\bigcap_{i \in I} \theta_i = 0_R$.

(ii) A family $\{\theta_i \mid i \in I\}$ of Γ-congruence relations on a Γ-so-ring R constitutes a Γ-direct representation if and only if $0_R = \prod_{1_R}(\theta_i : i \in I)$.

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References

1. G. V. S. Acharyulu, *A Study of Sum-Ordered Partial Semirings*, Doctoral thesis, Andhra University, 1992.
2. M. A. Arbib, E. G. Manes, Partially Additive Categories and Flow-diagram Semantics, *Journal of Algebra*, **62**, (1980), 203-227.
3. A. Iampan, Characterizing Ordered Bi-ideals in Ordered Gamma-semigroups, *Iranian Journal of Mathematical Sciences and Informatics*, **4** (1),(2009), 17-25.
4. A. Iampan, Some Properties of Ideal Extensions in Ternary semigroups, *Iranian Journal of Mathematical Sciences and Informatics*, **8** (1), (2013), 67-74.
5. E. G. Manes, D.B. Benson, The Inverse Semigroup of a Sum-Ordered Semiring, *Semigroup Forum*, **31**, (1985), 129-152.
6. M. Murali Krishna Rao, *Γ-Semirings*, Doctoral thesis, Andhra University, 1995.
7. M. Siva Mala, K. Siva Prasad, Partial Γ-Semirings, *Southeast Asian Bulletin of Mathematics*, **38** (6),(2014), 873-885.
8. M. E. Streenstrup, *Sum-Ordered Partial Semirings*, Doctoral thesis, Graduate school of the University of Massachusetts, Feb 1985 (Department of Computer and Information Science).
9. P. V. Srininvasa Rao, ϕ- Representation of So-Rings, *Intenational Journal of Computational Cognition*, **7** (2),(2009), 65-70.
10. P. V. Srinivasa Rao, *Ideal Theory of Sum-Ordered Partial Semirings*, Doctoral thesis, Acharya Nagarjuna University, 2011.
11. A. Walendziak, A Common Generalization of Subdirect and Direct product of Algebras, *Annales Societatis Mathematicae Polanae, Series I, Commentationes Mathemathicae*, **17**, (1992).

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