

A Generalized Singular Value Inequality for Heinz Means

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ABSTRACT. In this paper we will generalize a singular value inequality that was proved before. In particular we obtain an inequality for numerical radius as follows:

$$2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \leq \omega(tA + (1-t)B),$$

where, A and B are positive semidefinite matrices, $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$.

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1. INTRODUCTION

Let \mathbb{M}_n be the algebra of all $n \times n$ complex matrices. A norm $\|\cdot\|$ on \mathbb{M}_n is said to be unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in \mathbb{M}_n$ and all unitary $U, V \in \mathbb{M}_n$. Special examples of such norms are the "Ky Fan norms"

$$\|A\|_{(k)} = \sum_{j=1}^k s_j(A), \quad 1 \leq k \leq n.$$

Note that the operator norm, in this notation, is $\|A\| = \|A\|_{(1)} = s_1(A)$; see [4] and [9] for more information.

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If $\|A\|_{(k)} \leq \|B\|_{(k)}$ for $1 \leq k \leq n$, then $\|A\| \leq \|B\|$ for all unitary invariant norms. This is called the "Fan dominance theorem." If A is a Hermitian element of \mathbb{M}_n , then we arrange its eigenvalues in decreasing order as $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$. If A is arbitrary, then its singular values are enumerated as $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$. These are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$. If A and B are Hermitian matrices, and $A - B$ is positive semidefinite, then we say that $B \leq A$.

Weyl's monotonicity theorem [4, p. 63] says that $B \leq A$ implies $\lambda_j(A) \leq \lambda_j(B)$, for all $j = 1, \dots, n$. Let f be a real valued function on an interval I . Then f is said to be matrix monotone if $A, B \in \mathbb{M}_n$ are Hermitian matrices with all their eigenvalues in I and $A \geq B$, then $f(A) \geq f(B)$ and also, f is said to be matrix convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B), \quad 0 \leq t \leq 1$$

and matrix concave if

$$f(tA + (1-t)B) \geq tf(A) + (1-t)f(B), \quad 0 \leq t \leq 1.$$

In response to a conjecture by Zhan [13], Audenaert [2] has proved that if $A, B \in \mathbb{M}_n$ are positive semidefinite, then the inequality

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leq s_j(A + B), \quad 1 \leq j \leq n$$

holds, for all $0 \leq \nu \leq 1$. In this paper we generalize this inequality as follows: If $A, B \in \mathbb{M}_n$ are positive semidefinite matrices, then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq s_j(tA + (1-t)B).$$

For more details about inequalities and their generalizations with their history of origin, the reader may refer to [1, 5, 6, 11, 12, 13].

2. MAIN RESULTS

Lemma 2.1. [14] If $X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is positive, then $2s_j(C) \leq s_j(X)$ for all $1 \leq j \leq n$.

Theorem 2.2. Let f be a matrix monotone function on $[0, \infty)$ and A and B be positive semidefinite matrices. Then

$$tAf(A) + (1-t)Bf(B) \geq (tA + (1-t)B)^{1/2}(tf(A) + (1-t)f(B))(tA + (1-t)B)^{1/2} \quad (2.1)$$

for all $0 \leq t \leq 1$.

Proof. The function f is also matrix concave, and $g(x) = xf(x)$ is matrix convex. (See [4]). The matrix convexity of g implies the inequality

$$(tA + (1-t)B)f(tA + (1-t)B) \leq tAf(A) + (1-t)Bf(B), \quad 0 \leq t \leq 1. \quad (2.2)$$

Since the matrix $tA + (1-t)B$ is positive semidefinite, in view of the spectral decomposition theorem, it is easy to see that for all $0 \leq t \leq 1$,

$$(tA + (1-t)B)f(tA + (1-t)B) = (tA + (1-t)B)^{1/2} f(tA + (1-t)B) (tA + (1-t)B)^{1/2}. \quad (2.3)$$

Also, the matrix concavity of f implies that

$$tf(A) + (1-t)f(B) \leq f(tA + (1-t)B), \quad 0 \leq t \leq 1. \quad (2.4)$$

Combining the relations (2.2), (2.3) and (2.4), we get (2.1). \square

Theorem 2.3. Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$

$$2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu) \leq s_j(tA + (1-t)B). \quad (2.5)$$

Proof. The proof depends on the fact that the matrices XY and YX have the same eigenvalues. Let $f(x) = x^r, 0 \leq r \leq 1$. This function is matrix monotone on $[0, \infty)$. Hence from (2.1) and Weyl's monotonicity theorem we have

$$\lambda_j(tA^{r+1} + (1-t)B^{r+1}) \geq \lambda_j((tA + (1-t)B)(tA^r + (1-t)B^r)). \quad (2.6)$$

Except for trivial zeroes the eigenvalues of $(tA + (1-t)B)(tA^r + (1-t)B^r)$ are the same as those of the matrix

$$\begin{aligned} & \begin{bmatrix} tA + (1-t)B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & 0 \\ \sqrt{1-t}B^{r/2} & 0 \end{bmatrix} \\ & \text{and in turn, these are the same as the eigenvalues of} \\ & \begin{bmatrix} \sqrt{t}A^{r/2} & 0 \\ \sqrt{1-t}B^{r/2} & 0 \end{bmatrix} \begin{bmatrix} tA + (1-t)B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{t}A^{r/2} & \sqrt{1-t}B^{r/2} \\ 0 & 0 \end{bmatrix} \\ & = \begin{bmatrix} tA^{r/2}(tA + (1-t)B)A^{r/2} & \sqrt{t(1-t)}A^{r/2}(tA + (1-t)B)B^{r/2} \\ \sqrt{t(1-t)}B^{r/2}(tA + (1-t)B)A^{r/2} & (1-t)B^{r/2}(tA + (1-t)B)B^{r/2} \end{bmatrix}. \end{aligned}$$

So, Lemma 2.1 and inequality (2.6) together give

$$\begin{aligned} \lambda_j(tA^{r+1} + (1-t)B^{r+1}) & \geq 2\sqrt{t(1-t)}s_j(A^{r/2}(tA + (1-t)B)B^{r/2}) \\ & = 2\sqrt{t(1-t)}s_j(tA^{1+\frac{r}{2}}B^{r/2} + (1-t)A^{r/2}B^{1+\frac{r}{2}}). \end{aligned}$$

Replacing A and B by $A^{1/r+1}$ and $B^{1/r+1}$, respectively, we get from this

$$s_j(tA + (1-t)B) \geq 2\sqrt{t(1-t)}s_j(tA^{\frac{r+2}{2r+2}}B^{\frac{r}{2r+2}} + (1-t)A^{\frac{r}{2r+2}}B^{\frac{2+r}{2r+2}}), \quad 0 \leq r, t \leq 1.$$

Now, if we put $\nu = \frac{r+2}{2r+2}$, then trivially, we get

$$s_j(tA + (1-t)B) \geq 2\sqrt{t(1-t)}s_j(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu}B^\nu),$$

for all $0 \leq t \leq 1$ and $\frac{1}{2} \leq \nu \leq \frac{3}{2}$ and we have proved (2.5) for this special range.

Symmetry, if we put $\nu = \frac{r}{2r+2}$, then it is easy to see that the inequality (2.5) holds for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{1}{2}$. Hence the proof is complete. \square

If in Theorem 2.3, we put $t = \frac{1}{2}$, then we have the following corollary, which obtained by Audenaert in [2] and by Bhatia and Kittaneh in [6].

Corollary 2.4. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq \nu \leq 1$*

$$s_j(A^\nu B^{1-\nu} + A^{1-\nu} B^\nu) \leq s_j(A + B).$$

Corollary 2.5. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$*

$$2\sqrt{t(1-t)} \left\| tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu \right\| \leq \left\| tA + (1-t)B \right\|.$$

For $A \in \mathbb{M}_n$, the numerical radius of A is defined and denoted by

$$\omega(A) = \max\{|x^* Ax| : x \in \mathbb{C}^n, x^* x = 1\}.$$

The quantity $\omega(A)$ is useful in studying perturbations, convergence, stability, approximation problems, iterative method, etc. For more information see [3, 7]. It is known that $\omega(\cdot)$ is a vector norm on \mathbb{M}_n , but is not unitarily invariant. We recall the following results about the numerical radius of matrices which can be found in [8] (see also [10, Chapter 1]).

Lemma 2.6. *Let $A \in \mathbb{M}_n$ and $\omega(\cdot)$ be the numerical radius. Then the following assertions are true:*

- (i) $\omega(U^*AU) = \omega(A)$, where U is unitary;
- (ii) $\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|$;
- (iii) $\omega(A) = \|A\|$ if (but not only if) A is normal.

Utilizing Lemma 2.6 (parts (ii) and (iii)) and by Corollary 2.5 we obtain the following corollary.

Corollary 2.7. *Let $A, B \in \mathbb{M}_n$ be positive semidefinite matrices. Then for all $0 \leq t \leq 1$ and $0 \leq \nu \leq \frac{3}{2}$*

$$2\sqrt{t(1-t)}\omega(tA^\nu B^{1-\nu} + (1-t)A^{1-\nu} B^\nu) \leq \omega(tA + (1-t)B).$$

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