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Filters and the Weakly Almost Periodic Compactification of a Semitopological Semigroup

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Abstract. Let S be a semitopological semigroup. The $wap-$ compactification of semigroup S, is a compact semitopological semigroup with certain universal properties relative to the original semigroup, and the $Lmc-$ compactification of semigroup S is a universal semigroup compactification of S, which are denoted by S^{wap} and S^{Lmc} respectively. In this paper, an internal construction of the $wap-$compactification of a semitopological semigroup is constructed as a space of $z-$filters. Also we obtain the cardinality of S^{wap} and show that if S^{wap} is the one point compactification then $(S^{Lmc} - S) * S^{Lmc}$ is dense in $S^{Lmc} - S$.

Keywords: Semigroup compactification, *Lmc*-compactification, *wap*-compactification, z-filter.

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1. Introduction

A semigroup S which is also a Hausdorff topological space is called a semitopological semigroup, if for each $s \in S$, $\lambda_s : S \to S$ and $r_s : S \to S$ are continuous, where for each $x \in S$, $\lambda_s(x) = sx$ and $r_s(x) = xs$. Notice that if

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just r_s, for each $s \in S$, is continuous, S is called a right topological semigroup. Throughout this paper S is a semitopological semigroup. Generally, the Stone-$\check{C}$ech compactification βS of space S is defined in two approaches:
i) as the spectrum of $\mathcal{CB}(S)$, the C^*-algebra of bounded complex valued continuous functions on S,(see [2]), or
ii) the collection of all z-ultrafilters on S, (see [5]).

Generally βS is not a semigroup. A necessary and sufficient condition for βS to be a semigroup naturally is that $\mathcal{CB}(S)$ should be an $m-$admissible algebra. If S is realized as a discrete semigroup, then $\mathcal{CB}(S)$ will be an $m-$admissible algebra and as a result, βS will be a semigroup. This semigroup, as the collection of all ultrafilters on S, has a known operation attributed to Glazer. Capability and competence of ultrafilter approach are illustrated clearly in [4], [5], [7] and [14]. It is well known that ultrafilters play a prominent role in the Stone-$\check{C}$ech compactification of discrete semigroups, see [3], [7] and [14], and as an application of filters in the Logic see [8]. It is natural to study the semigroup compactification of a semitopological semigroup as a collection of z-filters, see [12] for more details. By getting help from what has been done in [12], some scientists have been tried to introduce some new subjects about semigroup compactification through z-filters, that was not applicable without these objects until now. See [9],[11] and [13].

This paper by getting ideas from what filters do in the Stone-$\check{C}$ech compactification, has been prepared. Indeed, ideas are taken from [3] and it has been used in [10].
In Section 1, the semigroup compactification has been introduced briefly and m-admissible subalgebras of $Lmc(S)$ and $wap(S)$ have been explained, then semigroup compactification is rebuilt as a collection of z-filters, and some Theorems and Definitions of [10] are presented.

In Section 2, regarding to [10], the wap-compactification as a quotient space of the Lmc- compactification is described, and we obtain the cardinal of the wap-compactification under some conditions and present some examples in this area.

2. Preliminary

A semigroup compactification of S is a pair (ψ, X), where X is a compact, Hausdorff, right topological semigroup and $\psi : S \to X$ is continuous homomorphism with dense image such that for all $s \in S$, the mapping $x \mapsto \psi(s)x : X \to X$ is continuous. The last property say that $\psi[S]$ is in the topological center of X [2, Definition 3.1.1]. Let $\mathcal{F}$ be a C^*-subalgebra of $\mathcal{CB}(S)$ containing the constant functions. Then the set of all multiplicative means of $\mathcal{F}$, called the spectrum of $\mathcal{F}$, denoted by $S^{\mathcal{F}}$, equipped with the Gelfand topology, is a compact Hausdorff topological space. Given $\mathcal{F}$ is left translation invariant if $L_s f = f \circ \lambda_s \in \mathcal{F}$ for all $s \in S$ and $f \in \mathcal{F}$. Then $\mathcal{F}$

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is called m-admissible too if the function $s \mapsto (T_\mu f(s)) = \mu(L_s f)$ is in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$. If so, $S^{\mathcal{F}}$ under the multiplication $\mu\nu = \mu \circ T_\nu$ for $\mu, \nu \in S^{\mathcal{F}}$, equipped with the Gelfand topology, makes semigroup compactification $(\varepsilon, S^{\mathcal{F}})$ of S, called the $\mathcal{F}$-compactification, where $\varepsilon : S \to S^{\mathcal{F}}$ is the evaluation mapping. Also $\varepsilon^* : \mathcal{C}(S^{\mathcal{F}}) \to \mathcal{F}$ is an isometric isomorphism and $\widehat{f} = (\varepsilon^*)^{-1}(f) \in \mathcal{C}(S^{\mathcal{F}})$ for $f \in \mathcal{F}$ is given by $\widehat{f}(\mu) = \mu(f)$ for all $\mu \in S^{\mathcal{F}}$. For more details see [2, Section 2].
A function $f \in \mathcal{CB}(S)$ is left multiplicative continuous if and only if $\mathbf{T}_\mu f \in \mathcal{CB}(S)$ for all $\mu \in \beta S = S^{\mathcal{CB}(S)}$, then

$$Lmc(S) = \bigcap \{\mathbf{T}_\mu^{-1}(\mathcal{CB}(S)) : \mu \in \beta S\},$$

and $Lmc(S)$ is the largest m-admissible subalgebra of $\mathcal{CB}(S)$. Then (S^{Lmc}, ε) is the universal compactification of S, see [2, Definition 4.5.1 and Theorem 4.5.2].

A function $f \in \mathcal{CB}(S)$ is said to be weakly almost periodic if $R_S f = \{f \circ r_s : s \in S\}$ is relatively compact (i.e., $\sigma(\mathcal{CB}(S), \mathcal{CB}(S)^*)$) in $\mathcal{CB}(S)$. The set of all weakly almost periodic functions on S is denoted by $wap(S)$,(see 4.2.1 in [2].)

Theorem 2.1. *Let S be a semitopological semigroup and $f \in \mathcal{CB}(S)$. The following statements are equivalent:*
(i) $f \in wap(S)$.
(ii) $lim_{m\to\infty} lim_{n\to\infty} f(s_m t_n) = lim_{n\to\infty} lim_{m\to\infty} f(s_m t_n)$ whenever $\{s_m\}$ and $\{t_n\}$ are sequences in S such that all the limits exist.

Proof. See Theorem 4.2.3 in [2]. □

Theorem 2.2. *Let S be a semitoplogical semigroup. There is a compact semigroup S^{wap} and a continuous homomorphism $\varepsilon : S \to S^{wap}$ such that*
i) S^{wap} is a semitoplogical semigroup,
ii) $\varepsilon(S)$ is dense in S^{wap},
iii) the pair (S^{wap}, ε) is maximal with respect to these properties in the sense that ϕ is a continuous homomorphism from S to a compact semigroup T, and (T, ϕ) satisfies (i) and (ii) with ϕ replacing ε and T replacing S^{wap}, then there is a continuous homomorphism η from S^{wap} onto T such that $\eta \circ \varepsilon = \phi$. Moreover, a function $f \in \mathcal{CB}(S)$ extends to S^{wap} if and only if f is weak almost periodic.

Proof. See Theorem 2.5 in [3]. □

M. Amini and A. R. Medghalchi have shown that the spectrum of the Fourier-Stieltjes algebra $B(S)$ is a compact semitopological semigroup when S is a unital foundation topological $*$-semigroup whose representations separate points of S, see [1]. In fact, the spectrum of $B(S)$ is a factor of the wap-compactification of S. For the definition of factor see [2].

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Lemma 2.3. *(i) Let S be a semitopological semigroup, and let $f, g \in wap(S)$ be such that $Range(f) \subseteq \mathbb{R}$ and $Range(g) \subseteq \mathbb{R}$. Define h by*

$$h(s) = max\{f(s), g(s)\}$$

for each $s \in S$. Then $h \in wap(S)$.
(ii) If S is a compact semitopological semigroup, then $wap(S) = \mathcal{CB}(S)$.

Proof. See Lemma 2.8 in [3]. □

Now we quote some prerequisite material from [12] for the description of (S^{Lmc}, ε) in terms of $z-$filters and some Definitions and Theorems of [10] is explained that we need. For $f \in Lmc(S)$, $Z(f) = f^{-1}(\{0\})$ is called zero set for all $f \in Lmc(S)$ and we denote the collection of all zero sets by $Z(Lmc(S))$. For an extensive account of ultrafilters, the readers may refer to [4], [5] and [7].

Definition 2.4. $\mathcal{A} \subseteq Z(Lmc(S))$ is called a $z-$filter in $Lmc(S)$ if
(i) $\emptyset \notin \mathcal{A}$ and $S \in \mathcal{A}$,
(ii) if $A, B \in \mathcal{A}$, then $A \bigcap B \in \mathcal{A}$, and
(iii) if $A \in \mathcal{A}$, $B \in Z(Lmc(S))$ and $A \subseteq B$ then $B \in \mathcal{A}$.

A $z-$filter is a $z-$ultrafilter if it is not contained properly in any other $z-$filters. The collection of all ultrafilters in $Lmc(S)$ is denoted by $zu(S)$. If $p, q \in zu(S)$, then the following statements hold.
1) If $B \in Z(Lmc(S))$ and for all $A \in p$, $A \cap B \neq \emptyset$ then $B \in p$,
2) if $A, B \in Z(Lmc(S))$ such that $A \cup B \in p$ then $A \in p$ or $B \in p$,
3) let p and q be distinct $z-$ultrafilters, then there exist $A \in p$ and $B \in q$ such that $A \cap B = \emptyset$.(See Lemma 2.3 in [12]).

For $x \in S$, we define $\widehat{x} = \{Z(f) : f \in Lmc(S),\ f(x) = 0\}$. Let $Z(h) \in Z(Lmc(S))$ and $Z(h) \notin \widehat{x}$. Then $M = \{\widehat{f} \in \mathcal{C}(S^{Lmc}) : f(x) = 0\}$ is a maximal ideal in $\mathcal{C}(S^{Lmc})$ and so $\widehat{h} \notin M$. Thus there exists $\widehat{f} \in M$ such that $Z(\widehat{f}) \cap Z(\widehat{h}) = \emptyset$. Thus $\widehat{x} \cup \{Z(h)\}$ has not the finite intersection property, and so $\widehat{x}$ is a $z-$ultrafilter.

The space $zu(S)$ is equipped with a topology that base is $\{(\widehat{A})^c : A \in Z(Lmc(S))\}$, where $\widehat{A} = \{p \in zu(S) : A \in p\}$, is a compact space which is not Hausdorff in general. By Lemma 2.7. in [12], for each $p \in zu(S)$ there exists $\mu \in S^{Lmc}$ such that $\bigcap_{A \in p} \overline{\varepsilon(A)} = \{\mu\}$, and also for each $\mu \in S^{Lmc}$ there exists $p \in zu(S)$ such that $\bigcap_{A \in p} \overline{\varepsilon(A)} = \{\mu\}$. We say $p \in zu(S)$ converges to $\mu \in S^{Lmc}$ if $\mu \in \bigcap_{A \in p} \overline{\varepsilon(A)}$.
Now, we define the relation $\sim$ on $zu(S)$ such that $p \sim q$ if and only if

$$\bigcap_{A \in p} \overline{\varepsilon(A)} = \bigcap_{B \in q} \overline{\varepsilon(B)}.$$

It is obvious that $\sim$ is an equivalence relation on $zu(S)$ and $[[p]]$ denotes the equivalence class of $p \in zu(S)$. So for each $p \in zu(S)$ there is a unique $\mu \in S^{Lmc}$

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such that

$$[[p]] = \{p \in zu(S) : \bigcap_{A\in p} \overline{\varepsilon(A)} = \{\mu\}\},$$

in fact, $[[p]]$ is the collection of all $z-$ultrafilters in $Lmc(S)$ that converge to $\mu \in S^{Lmc}$. Let $\frac{zu(S)}{\sim}$ be the corresponding quotient space with the quotient map $\pi : zu(S) \to \frac{zu(S)}{\sim}$. For every $p \in zu(S)$, define $\widetilde{p}$ by $\widetilde{p} = \bigcap[[p]]$, put $\widetilde{A} = \{\widetilde{p} : A \in p\}$ for $A \in Z(Lmc(S))$ and $\mathcal{R} = \{\widetilde{p} : p \in zu(S)\}$. It is obvious that $\{(\widetilde{A})^c : A \in Z(Lmc(S))\}$ is a basis for a topology on $\mathcal{R}$, $\mathcal{R}$ is a Hausdorff and compact space and also S^{Lmc} and $\mathcal{R}$ are homeomorphic (see [12]). So we have $\mathcal{R} = \{\mathcal{A}^\mu : \mu \in S^{Lmc}\}$, where $\mathcal{A}^\mu = \widetilde{p}$ and $\mathcal{A}^{\varepsilon(x)} = \widehat{x}$ for $x \in S$. For all $x, y \in S$, we define

$$\mathcal{A}^{\varepsilon(x)} * \mathcal{A}^{\epsilon(y)} = \{Z(f) \in Z(Lmc(S)) : \ Z(\mathbf{T}_{\varepsilon(y)} f) \in \mathcal{A}^{\varepsilon(x)}\}.$$

It has been shown that $\mathcal{A}^{\varepsilon(x)} * \mathcal{A}^{\varepsilon(y)} = \mathcal{A}^{\varepsilon(xy)}$, for each $x, y \in S$, see [12]. Now, let $\{x_\alpha\}$ and $\{y_\beta\}$ be two nets in S, such that $lim_\alpha \varepsilon(x_\alpha) = \mu$ and $lim_\beta \varepsilon(y_\beta) = \nu$, for $\mu, \nu \in S^{Lmc}$. We define

$$\mathcal{A}^\mu * \mathcal{A}^\nu = lim_\alpha(lim_\beta(\mathcal{A}^{\varepsilon(x_\alpha)} * \mathcal{A}^{\varepsilon(y_\beta)})).$$

This Definition is well-defined and $(\mathcal{R}, e)$ is a compact right topological semigroup, where $e : S \to \mathcal{R}$ is defined by $e[x] = \widehat{x}$. Also the mapping $\varphi : S^{Lmc} \to \mathcal{R}$ defined by $\varphi(\mu) = \widetilde{p}$, where $\bigcap_{A\in p} \overline{\varepsilon(A)} = \{\mu\}$, is an isomorphism (see [12]).

The operation "$\cdot$" on S extends uniquely to $(\mathcal{R}, *)$. Thus $(\mathcal{R}, e)$ is a semigroup compactification of $(S, \cdot)$, that $e : S \to \mathcal{R}$ is an evaluation map. Also $e[S]$ is a subset of the topological center of $\mathcal{R}$ and $cl_{\mathcal{R}}(e[S]) = \mathcal{R}$. Hence S^{Lmc} and $\mathcal{R}$ are topologically isomorphic and so $S^{Lmc} \simeq \mathcal{R}$. For more details see [12].

A $z-$filter $\mathcal{A}$ on $Lmc(S)$ is called a pure $z-$filter if for some z-ultrafilter p, $\mathcal{A} \subseteq p$ implies that $\mathcal{A} \subseteq \widetilde{p}$. For a $z-$filter $\mathcal{A} \subseteq Z(Lmc(S))$, we define
(i) $\overline{\mathcal{A}} = \{\widetilde{p} \in S^{Lmc} : \text{There exists a } z - \text{ultrafilter } p \text{ such that } \mathcal{A} \subseteq p\}$,
(ii) $\mathcal{A}^\circ = \{A \in \mathcal{A} : \overline{\mathcal{A}} \subseteq (\overline{\varepsilon(A)})^\circ\}$.

Definition 2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two $z-$filters on $Lmc(S)$ and $A \in Z(Lmc(S))$. We say $A \in \mathcal{A} + \mathcal{B}$ if and only if for each $F \in Z(Lmc(S))$, $\Omega_{\mathcal{B}}(A) \subset F$ implies $F \in \mathcal{A}$, where $\Omega_{\mathcal{B}}(A) = \{x \in S : \lambda_x^{-1}(A) \in \mathcal{A}\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be $z-$filters in $Lmc(S)$ then $\mathcal{A} + \mathcal{B}$ is a $z-$filter, and we define $\mathcal{A} \odot \mathcal{B} = \bigcap \overline{\mathcal{A} + \mathcal{B}}$. So $\mathcal{A} \odot \mathcal{B}$ is a pure $z-$filter generated by $\overline{\mathcal{A} + \mathcal{B}}$.

In this paper, $\mathbb{Q}$ denotes rational numbers with natural topology. Also we replace $\overline{\varepsilon(A)}$ with $\overline{A}$ for simplicity.
Now we describe some Definitions and Theorems of [10] that applied in next sections.

Definition 2.6. Let Γ be a subset of pure $z-$filters in $Lmc(S)$. We say $f : \Gamma \to \bigcup \Gamma$ is a topological choice function if for each $\mathcal{L} \in \Gamma$, $\overline{\mathcal{L}} \subseteq (\overline{f(\mathcal{L})})^\circ$.

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Theorem 2.7. *Let Γ be a set of pure $z-$filters in $Lmc(S)$. Statements (a) and (a') are equivalent, statements (b) and (b') are equivalent and statement (c) is equivalent to the conjunction of statements (a) and (b).*
(a) Given any topological choice function f for Γ, there is a finite subfamily $\mathcal{F}$ of Γ such that $\overline{S} = S^{Lmc} = \bigcup_{\mathcal{L}\in\mathcal{F}}(\overline{f(\mathcal{L})})^\circ$.
(a') For each $\widetilde{p} \in S^{Lmc}$, there is some $\mathcal{L} \in \Gamma$ such that $\mathcal{L} \subseteq \widetilde{p}$.
(b) Given distinct $\mathcal{L}$ and $\mathcal{K}$ in Γ, there exists $B \in \mathcal{K}^\circ$ such that for every $A \in Z(Lmc(S))$ if $\overline{S} - (\overline{B})^\circ \subseteq (\overline{A})^\circ$ then $A \in \mathcal{L}$.
(b') For each $\widetilde{p} \in S^{Lmc}$, there is at most one $\mathcal{L} \in \Gamma$ such that $\mathcal{L} \subseteq \widetilde{p}$.
(c) There is an equivalence relation R on S^{Lmc} such that each equivalence class is closed in S^{Lmc} and $\Gamma = \{ \bigcap[\widetilde{p}]_R : \widetilde{p} \in S^{Lmc}\}$.

Proof. See Theorem 3.2 in [10]. □

Let $\Gamma \subseteq Z(Lmc(S))$ be a set of pure $z-$filters. We define

$$A^* = \{\mathcal{L} \in \Gamma : \overline{\mathcal{L}} \bigcap \overline{A} \neq \emptyset\},$$

for each $A \in Z(Lmc(S)$. Then $\{ (A^*)^c : A \in Z(Lmc(S))\}$ is a basis for the topology τ on Γ and τ is called quotient topology on Γ generated by $\{ (A^*)^c : A \in Z(Lmc(S))\}$. Let R be an equivalence relation on S^{Lmc} and let $\Gamma = \{\cap[\widetilde{p}]_R : \widetilde{p} \in S^{Lmc}\}$. Then for every $A \in Z(Lmc(S))$, we have $A^* = \{\cap[\widetilde{p}]_R : \widetilde{p} \in \overline{A}\}$. It has been shown that, if S^{Lmc}/R is a Hausdorff space then with the quotient topology on Γ, Γ and S^{Lmc}/R are homeomorphic (i.e. $\Gamma \approx S^{Lmc}/R$).(See Theorem 3.5 in [10].)

Theorem 2.8. *Let Γ be a set of pure $z-$filters in $Lmc(S)$ with the quotient topology. There is an equivalence relation R on S^{Lmc} such that S^{Lmc}/R is Hausdorff, $\Gamma = \{ \cap[p]_R : p \in S^{Lmc}\}$, and $\Gamma \approx S^{Lmc}/R$ if and only if both of the following statements hold:*
(a) Given any topological choice function f for Γ, there is a finite subfamily $\mathcal{F}$ of Γ such that $\overline{S} = \bigcup_{\mathcal{L}\in\mathcal{F}}(\overline{f(\mathcal{L})})^\circ$,
(b) Given distinct $\mathcal{L}$ and $\mathcal{K}$ in Γ, there exist $A \in \mathcal{L}^\circ$ and $B \in \mathcal{K}^\circ$ such that whenever $\mathcal{C} \in \Gamma$, for each $H, T \in Z(Lmc(S))$ either $\overline{S} - (\overline{A})^\circ \subseteq (\overline{H})^\circ$ then $H \in \mathcal{C}$ or $\overline{S} - (\overline{B})^\circ \subseteq (\overline{T})^\circ$ then $T \in \mathcal{C}$.

Proof. See Theorem 3.9 in [10]. □

Remark 2.9. i) Γ is called a quotient of S^{Lmc} if and only if Γ is a set of pure $z-$filters in $Lmc(S)$ with the quotient topology satisfying statements (a) and (b) of Theorem 1.8. Let Γ be a quotient of S^{Lmc}, then by Theorem 1.7, for each $\widetilde{p} \in S^{Lmc}$ there is a unique $\mathcal{L} \in \Gamma$ such that $\mathcal{L} \subseteq \widetilde{p}$. Then $\gamma : S^{Lmc} \to \Gamma$ by $\gamma(\widetilde{p}) \subseteq \widetilde{p}$ is a quotient map. We define $e : S \to \Gamma$ by $e(s) = \gamma(\widehat{s})$,(see Corollary 3.12 of [10].)

ii) Let Γ be a quotient of S^{Lmc}. If for each $\mathcal{A}$ and $\mathcal{B}$ in Γ there is some $\mathcal{C} \in \Gamma$ with $\mathcal{C} \subseteq \mathcal{A} \odot \mathcal{B}$, we define $\dot{+}$ on Γ by $\mathcal{A}\dot{+}\mathcal{B} \in \Gamma$ and $\mathcal{A}\dot{+}\mathcal{B} \subseteq \mathcal{A} \odot \mathcal{B}$. Then e is

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a continuous homomorphism from S to Γ, Γ is a right topological semigroup, $e[S]$ is dense in Γ, and the function $\lambda_{e(s)}$ is continuous for every $s \in S$,(see Theorem 3.15 in [10].)

Theorem 2.10. *Let Γ be a set of pure $z-$filters with the quotient topology. The following statements are equivalent:*
(a) There exists a continuous function $h : S \longrightarrow \Gamma$ such that Γ is a Hausdorff compact space, $h[S]$ is dense in Γ.
(i) for each $s \in S$, $s \in \bigcap h(s)$,
(ii) for distinct $\mathcal{L}$ and $\mathcal{K}$ in Γ, there exists $B \in \mathcal{K}^\circ$ such that for each $A \in Z(Lmc(S))$ if $\overline{S} - (\overline{B})^\circ \subseteq (\overline{A})^\circ$ then $A \in \mathcal{L}$.
(b) Γ is quotient of S^{Lmc}.
(c) Γ is quotient of S^{Lmc}, Γ is a Hausdorff compact space and $e[S]$ is dense in Γ.

Proof. See Theorem 3.16 in [10]. □

3. Weakly Almost Periodic compactification as a Quotient Space

In this section, S^{wap} is described as a space of pure $z-$filters. We obtain, in Theorem 2.13, a description of S^{wap} as a space of pure filters which is internal to the set of pure filters. In this section assume $\widehat{f} = (\varepsilon^*)^{-1}(f) \in \mathcal{C}(S^{Lmc})$ for $f \in Lmc(S)$.

Lemma 3.1. *Let Γ be a quotient of S^{Lmc}, T be a compact Hausdorff space, and let $f : S \to T$ be a continuous function with a continuous extension $\widetilde{f} : S^{Lmc} \to T$. Statements (a) and (d) are equivalent. If $\{\hat{s} : s \in S\} \subseteq \Gamma$, then all four statements are equivalent:*
(a) f has a continuous extension to Γ,(that is, there exists $g : \Gamma \to T$ such that $g \circ e = f$).
(b) For each $\mathcal{L} \in \Gamma$, $\bigcap\{cl_T(f[A]) : \overline{\mathcal{L}} - \overline{A^c} \neq \emptyset, A \in Z(Lmc(S))\} \neq \emptyset$.
(c) For each $\mathcal{L} \in \Gamma$, $card(\bigcap\{cl_T f[A] : \overline{\mathcal{L}} - \overline{A^c} \neq \emptyset, A \in Z(Lmc(S))\}) = 1$.
(d) For each $\mathcal{L} \in \Gamma$, $\widetilde{f}$ is constant on $\overline{\mathcal{L}}$.

Proof. To see that (a) implies (d), note that $g \circ \gamma$ is a continuous extension of f and hence $g \circ \gamma = \widetilde{f}$.
To see that (d) implies (a), define g by $g(\gamma(\widetilde{p})) = \widetilde{f}(\widetilde{p})$. Since $\widetilde{f}$ is constant on $\overline{\mathcal{L}}$ for each $\mathcal{L} \in \Gamma$, g is well defined . Therefore $g \circ \gamma = \widetilde{f}$, where γ is a quotient map. This implies g is continuous.
Now assume that $\{\hat{s} : s \in S\} \subseteq \Gamma$ so that for all $s \in S$, $e(s) = \hat{s}$. That (c) implies (b) is trivial.
To see that (a) implies (c), we show that

$$\bigcap\{cl_T f(A) : \overline{\mathcal{L}} - \overline{A^c} \neq \emptyset, A \in Z(Lmc(S))\} = \{g(\mathcal{L})\}.$$

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Since the following diagram commutes,

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle\varepsilon}\downarrow & & \uparrow{\scriptstyle g} \\ S^{Lmc} & \xrightarrow{\gamma} & \Gamma, \end{array}$$

(*i.e.* $f = g \circ \gamma \circ \varepsilon$).
Hence for each $A \in Z(Lmc(S))$, we have

$$\begin{aligned} cl_{_T} f(A) &= cl_T g \circ \gamma \circ \varepsilon(A) = g \circ \gamma(\overline{\varepsilon(A)}) \\ &= g \circ \gamma(\overline{A}) = g(\gamma(\overline{A})). \end{aligned}$$

Also if $\overline{\mathcal{L}} - \overline{A^c} \neq \emptyset$, then there exists $\widetilde{p} \in \overline{\mathcal{L}} \cap \overline{A}$, (it is obvious that $\overline{A}^c \subseteq A^c \subseteq \overline{A^c}$ hence if $\widetilde{p} \notin \overline{A^c}$ then $\widetilde{p} \in \overline{A}$). So

$$cl_T f(A) = g(\gamma(\overline{A})) = \{g(\gamma(\widetilde{r})) : \widetilde{r} \in \overline{A}\}.$$

Since $\widetilde{p} \in \overline{A} \bigcap \overline{\mathcal{L}}$ so $g(\mathcal{L}) = g(\gamma(\widetilde{p})) \in cl_T f(A)$. Also for each $\mathcal{K} \in \Gamma$, if $\mathcal{K} \neq \mathcal{L}$ then there exists $A \in Z(Lmc(S))$ such that $\overline{A} \bigcap \overline{\mathcal{K}} = \emptyset$, and this conclude that $g(\mathcal{K}) \notin cl_T f(A)$. Hence

$$\bigcap \{cl_T f(A) : \overline{\mathcal{L}} - \overline{A^c} \neq \emptyset\} = \{g(\mathcal{L})\}.$$

To see that (b) implies (a). Choose for each $\mathcal{L} \in \Gamma$, some $g(\mathcal{L}) \in \bigcap \{cl_T f(A) : \overline{\mathcal{L}} - \overline{A^c} \neq \emptyset\}$. It is obvious that $g \circ e = f$. To see that g is continuous, we show that $\widetilde{f}$ is constant on $\overline{\mathcal{L}}$, for each $\mathcal{L} \in \Gamma$. Let $\widetilde{p} \in \overline{\mathcal{L}}$ and $A \in Z(Lmc(S))$, if $\widetilde{p} \in \overline{A}$ then $\overline{\mathcal{L}} - \overline{A^c} \neq \emptyset$ and so $g(\mathcal{L}) \in cl_T f(A)$. Since $cl_T f(A) = \widetilde{f}(\overline{A})$ so $\bigcap_{\widetilde{p} \in \overline{A}} \widetilde{f}(\overline{A}) = \{\widetilde{f}(\widetilde{p})\}$. This conclude that $g(\mathcal{L}) = \widetilde{f}(\widetilde{p})$ and $g \circ \gamma = \widetilde{f}$. γ is a quotient map. Therefore $\widetilde{f}$ is continuous hence g is continuous. □

Lemma 3.2. *Let Γ be a quotient of S^{Lmc} such that $\{\hat{s} : s \in S\} \subseteq \Gamma$, $(T, +)$ be a compact Hausdorff right topological semigroup, and f be a continuous homomorphism from S to T such that $l_{f(x)}$ is continuous for each $x \in S$. Let $\bigcap \{cl_T f(A) : \overline{\mathcal{L}} - \overline{A^c} \neq \emptyset,\ A \in Z(Lmc(S))\} \neq \emptyset$ for each $\mathcal{L} \in \Gamma$. Let $\mathcal{L} \dot{+} \mathcal{K} \in \Gamma$ for each $\mathcal{L}, \mathcal{K} \in \Gamma$. Then there is a continuous homomorphism $g : \Gamma \to T$ such that $g \circ e = f$.*

Proof. Since $\overline{f(S)} = H$ is a compact right topological semigroup, so (H, f) is a semigroup compactification of S. By Lemma 2.1, $g : \Gamma \to T$ exists, g is continuous and $g \circ e = f$. If $\widetilde{f} : S^{Lmc} \to T$ is extension of $f : S \to T$, then $\widetilde{f}$ is a continuous homomorphism and the below diagram commutes,

$$\begin{array}{ccc} S^{Lmc} & \xrightarrow{\gamma} & \Gamma \\ & {\scriptstyle\widetilde{f}}\searrow \quad \swarrow{\scriptstyle g} & \\ & T & \end{array}$$

(i.e. $g \circ \gamma = \widetilde{f}$).

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Hence for each $\mathcal{L}, \mathcal{K} \in \Gamma$, there exist $\widetilde{p}, \widetilde{q} \in S^{Lmc}$ such that $\gamma(\widetilde{p}) = \mathcal{L}$ and $\gamma(\widetilde{q}) = \mathcal{K}$. Now we have:

$$\begin{aligned} g(\mathcal{L}\dot{+}\mathcal{K}) &= g(\gamma(\widetilde{p})\dot{+}\gamma(\widetilde{q})) \\ &= g(\gamma(\widetilde{p} * \widetilde{q})) \qquad \gamma \text{ is a homomorphism} \\ &= \widetilde{f}(\widetilde{p} * \widetilde{q}) \\ &= \widetilde{f}(\widetilde{p}) + \widetilde{f}(\widetilde{q}) \qquad \widetilde{f} \text{ is a homomorphism} \\ &= g \circ \gamma(\widetilde{p}) + g \circ \gamma(\widetilde{q}) \\ &= \widetilde{f}(\widetilde{p}) + \widetilde{f}(\widetilde{q}) \\ &= g(\mathcal{L}) + g(\mathcal{K}), \end{aligned}$$

and so g is a homomorphism. □

We use the set $\mathbb{Q}^+$ of positive rational in the definition, but any other dense subsets of the positive reals would do.

Definition 3.3. (*a*) φ is a topological $W - nest$ on S if and only if
(1) $\varphi : \mathbb{Q}^+ \to Z(Lmc(S))$,
(2) $S \in Range(\varphi)$ and $\emptyset \notin Range(\varphi)$,
(3) if $r, s \in \mathbb{Q}^+$ and $r < s$ then $(\overline{\varphi(r)})^\circ \subseteq \overline{\varphi(r)} \subseteq (\overline{\varphi(s)})^\circ$, and
(4) there do not exist r and υ in $\mathbb{Q}^+$ and sequences $< t_n >_{n=1}^{\infty}$ and $< s_n >_{n=1}^{\infty}$ in S such that $r < \upsilon$ and whenever $k < n$, $t_n s_k \in (\overline{\varphi(r)})^\circ$ and $t_k s_n \notin \overline{\varphi(\upsilon)}$.
(*b*) $\mathcal{W}(S)$=$\{\varphi : \varphi$ is a topological $W - nest$ in S$\}$.

Lemma 3.4. *(a) If $f \in wap(S)$, $a \in cl_{\mathbb{C}} f(S)$, and for $r \in Q^+$,*

$$\varphi(r) = \{s \in S :| f(s) - a |\leq r\},$$

then $\varphi \in \mathcal{W}(S)$. Also if $\widehat{f}(\widetilde{p}) = a$ for some $\widetilde{p} \in S^{Lmc}$, then $\widetilde{p} \in (\overline{\varphi(r)})^\circ$ for each $r \in \mathbb{Q}$.
(b) Let $\varphi \in \mathcal{W}(S)$, then $f(s) = inf\{r \in \mathbb{Q}^+ : s \in \varphi(r)\}$ is a continuous function.
(c) If $\varphi \in \mathcal{W}(S)$ and f is defined on S by $f(s) = inf\{r \in \mathbb{Q}^+ : s \in \varphi(r)\}$, then $f \in wap(S)$. Further, if $\widetilde{p} \in S^{Lmc}$ and $\widetilde{p} \in \bigcap_{r \in \mathbb{Q}^+} \overline{\varphi(r)}$ then $\widehat{f}(\widetilde{p}) = 0$.
(d) Let $f \in wap(S)$, $a \in cl_{\mathbb{C}} f(S)$ and $r > 0$. Then

$$B = \{x \in S : |f(x) - a| \leq r\} \in Z(wap(S)).$$

Also let $\widehat{f}(\widetilde{p}) = 0$ for some $\widetilde{p} \in S^{Lmc}$. Then $\widetilde{p} \in (\overline{f^{-1}([0, r])})^\circ$ for every $r > 0$.

Proof. (a) Condition (3) is immediate, (2) holds since f is bounded. To see that (1), it is obvious that for each $s \in S$, $g(s) = min(|\ f(s) - a\ | - r, 0)$ is continuous, and

$$Z(g) = \{s \in S : g(s) = 0\} = \{s \in S :| f(s) - a |\leq r\}.$$

Therefore $\varphi(r) = Z(g) \in Z(Lmc(S))$. To see that (4) holds, suppose that we have such $r, \upsilon, < t_n >_{n=1}^{\infty}$ and $< s_n >_{n=1}^{\infty}$. Then the sequences $< s_n >_{n=1}^{\infty}$ and

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$< t_n >_{n=1}^{\infty}$ so that $lim_{n\to\infty}lim_{k\to\infty}f(t_ns_k)$ and $lim_{k\to\infty}lim_{n\to\infty}f(t_ns_k)$ exist. Then $lim_{n\to\infty}lim_{k\to\infty} \mid f(t_ns_k)-a \mid \geq \upsilon$ while $lim_{k\to\infty}lim_{n\to\infty} \mid f(t_ns_k)-a \mid \leq r$, a contradiction.
If there exists $r \in \mathbb{Q}$ such that $\widetilde{p} \notin (\overline{\varphi(r)})^{\circ}$, then for every $s \in \mathbb{Q} \cap (0,r)$ we have $\widetilde{p} \notin (\overline{\varphi(s)})$ and this is a contradiction.

(b) Let there exists a net $\{s_\alpha\}$ in S such that $lim_\alpha s_\alpha = s$ in S and $lim_\alpha f(s_\alpha) \neq f(s)$. By thinning, there exists $\epsilon_\circ > 0$ such that

$$(f(s)-\epsilon_\circ, f(s)+\epsilon_\circ)\bigcap\{f(s_\alpha) : \alpha > \beta\} = \emptyset$$

for some β. Pick $r_1, r_2 \in Q^+$ such that

$$(f(s)-\epsilon_\circ < r_1 < f(s) < r_2 < f(s)+\epsilon_\circ),$$

so $\varphi(r_1) \subseteq \varphi(r_2)$, $\varphi(r_1) \neq \varphi(r_2)$ and $U = \varepsilon^{-1}((\overline{\varphi(r_1)})^c \bigcap (\overline{\varphi(r_2)})^{\circ})$ is a non empty open set in S. Since $s \in U$, so there exists η such that for each $\alpha > \eta$, $f(s_\alpha) \in U$ and so $r_1 < f(s_\alpha) < r_2$ and this is a contradiction.

(c) By (b), f is defined every where and is bounded and continuous. Suppose $f \notin wap(S)$, so pick sequences $< t_n >_{n=1}^{\infty}$ and $< s_n >_{n=1}^{\infty}$ such that

$$lim_{n\to\infty}lim_{k\to\infty}f(t_ns_k) = x$$

and $lim_{k\to\infty}lim_{n\to\infty}f(t_ns_k) = y$ with $x > y$. Let $\epsilon = \frac{x-y}{3}$. By thinning the sequences we may assume that whenever $k < n$, $f(t_ns_k) < y+\epsilon$ and $f(t_ks_n) > x-\epsilon$. Pick $r < \upsilon$ in Q^+ such that $y+\epsilon < r < \upsilon < x-\epsilon$. Then for $k < n$, $t_ns_k \in \varphi(r)$ and $t_ns_k \notin \varphi(\upsilon)$. Hence $\varphi \notin \mathcal{W}(S)$, and this is a contradiction. The last conclusion holds since each neighborhood of $\widetilde{p}$ includes points $\hat{s}$ with $f(s)$ arbitrarily close to 0.

(d) It is obvious. □

Definition 3.5. Define an equivalence relation on S^{Lmc} by $\widetilde{p} \equiv \widetilde{q}$ if and only if $\widehat{f}(\widetilde{p}) = \widehat{f}(\widetilde{q})$ whenever $f \in wap(S)$.

We observe that $\equiv$ is trivially an equivalence relation on S^{Lmc} and let $[\widetilde{p}]$ be the equivalence class of $\widetilde{p} \in S^{Lmc}$. We shall be interested in the $z-$filters $\bigcap[\widetilde{p}]$, where $\widetilde{p} \in S^{Lmc}$. Consequently, we want to know when $A \in \bigcap[\widetilde{p}]$ in terms of $\widetilde{p}$.

Lemma 3.6. *Let $\widetilde{p} \in S^{Lmc}$ and let $A \in Z(Lmc(S))$. The following statements are equivalent.*
(a) $[\widetilde{p}] \subseteq (\overline{A})^{\circ}$.
(b) There exist $f \in wap(S)$ and $\delta > 0$ such that $\{s \in S :\mid f(s)-\widehat{f}(\widetilde{p}) \mid \leq \delta\} \subseteq A$.
(c) There exist $f \in wap(S)$ and $\delta > 0$ such that $Range(f) \subseteq [0,1], \widehat{f}(\widetilde{p}) = 0$, and $f^{-1}([0,\delta]) \subseteq A$.

Proof. That (c) implies (b) is trivial. To see that (b) implies (a), let $\widetilde{q} \in [\widetilde{p}]$. Then $\widehat{f}(\widetilde{p}) = \widehat{f}(\widetilde{q})$ so $B = \{s \in S :\mid f(s) - \widehat{f}(\widetilde{p}) \mid \leq \delta\} \subseteq A$. Hence $[\widetilde{p}] \subseteq (\overline{A})^{\circ}$.

To see that (a) implies (c), suppose that the conclusion fails and assume

$$\mathcal{G} = \{f \in wap(S) : Range(f) \subseteq [0,1] \text{ and } \widehat{f}(\widetilde{p}) = 0\}.$$

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For each $f \in \mathcal{G}$ and each $\delta > 0$, let $B(f,\delta) = f^{-1}([0,\delta]) - \overline{A}$ and let

$$\mathcal{L} = \{B(f,\delta) : f \in \mathcal{G} \text{ and } \delta > 0\}.$$

Given f and g in $\mathcal{G}, \delta > 0$ and $\gamma > 0$, let $\mu = min\{\delta,\gamma\}$ and define h by $h(s) = max\{f(s), g(s)\}$. Then $Range(h) \subseteq [0,1]$ and by Lemma 1.3, $h \in wap(S)$. To see that $\widehat{h}(\widetilde{p}) = 0$, suppose that $\{s_\alpha\}$ be a net in S such that $\{s_\alpha\}$ converges to $\widetilde{p}$ in S^{Lmc}, since $h(s) = \frac{|f(s)-g(s)|}{2} + \frac{f(s)-g(s)}{2}$ so

$$\begin{aligned}
\widehat{h}(\widetilde{p}) &= lim_\alpha h(s_\alpha) \\
&= lim_\alpha \frac{\mid f(s_\alpha) - g(s_\alpha) \mid}{2} + \frac{f(s_\alpha) - g(s_\alpha)}{2} \\
&= \frac{\mid \widehat{f}(\widetilde{p}) - \widehat{g}(\widetilde{p}) \mid}{2} + \frac{\widehat{f}(\widetilde{p}) - \widehat{g}(\widetilde{p})}{2} \\
&= 0.
\end{aligned}$$

Thus $h \in \mathcal{G}$ and $B(h,\mu) \subseteq B(f,\delta) \bigcap B(g,\gamma)$. Therefore $\mathcal{L}$ has the finite intersection property. Pick $\widetilde{r} \in S^{Lmc}$ such that $\widetilde{r} \in \bigcap_{f\in\mathcal{G},\,\delta>0} \overline{B(f,\delta)}$. Then $\widetilde{r} \notin (\overline{A})^\circ$ and so $\widetilde{r} \notin [\widetilde{p}]$. Pick $g \in wap(S)$ such that $\widehat{g}(\widetilde{p}) \neq \widehat{g}(\widetilde{r})$. Let

$$b = sup\{\mid \widehat{g}(\widetilde{q}) - \widehat{g}(\widetilde{p}) \mid : \widetilde{q} \in S^{Lmc}\},$$

and pick $n \in \mathbb{N}$ such that $n > b$. Define φ as in Lemma 2.4 with $a = \widehat{g}(\widetilde{p})$. Then $\varphi \in \mathcal{W}(S)$ and $\varphi(n) = S$. Define $\mu(t)$ for $t \in Q^+$ by $\mu(t) = \varphi(nt)$. Then trivially $\mu \in \mathcal{W}(S)$. Define f on S by

$$\begin{aligned}
f(s) &= inf\{t \in Q^+ : s \in \mu(t)\} \\
&= inf\{t \in Q^+ : s \in \varphi(nt)\} \\
&= inf\{t \in Q^+ :\mid g(s) - a \mid \leq nt\} \\
&= inf\{t \in Q^+ :\mid \frac{1}{n}g(s) - \frac{1}{n}a \mid \leq t\} \\
&= \mid \frac{1}{n}g(s) - \frac{1}{n}a \mid .
\end{aligned}$$

By Lemma 2.4, $f \in wap(S)$. Since $\mu(1) = S$, $Range(f) \subseteq [0,1]$. Given $t \in Q^+$, $\mu(t) = \{s \in S :\mid g(s) - \widehat{g}(\widetilde{p}) \mid \leq nt\}$ so $\widetilde{p} \in \overline{\mu(t)}$. Thus $\widehat{f}(\widetilde{p}) = 0$. Also

$$\widehat{f}(\widetilde{r}) = \mid \widehat{g}(\widetilde{r}) - \widehat{g}(\widetilde{p}) \mid /n > 0.$$

Thus $\widetilde{r} \notin \overline{B(f, \widehat{f}(\widetilde{r})/2)}$ is a contradiction. □

Theorem 3.7. *Let Γ be a set of pure z−filters in $Lmc(S)$. Statement (a) below is equivalent to the conjunction of statements (b),(c) and (d) and implies that Γ is a Hausdorff compact space and $e[S]$ is dense in Γ.*
(a) $\Gamma = \{\bigcap[p] : p \in S^{Lmc}\}$.
(b) For $\varphi \in \mathcal{W}(S)$ and $\mathcal{L} \in \Gamma$, if $\mathcal{L} \bigcup Range(\varphi)$ has the finite intersection property, then $Rang(\varphi) \subseteq \mathcal{L}$.
(c) Given distinct $\mathcal{L}$ and $\mathcal{K}$ in Γ, there exists $\varphi \in \mathcal{W}(S)$ such that $Range(\varphi) \subseteq \mathcal{L}$

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but $Range(\varphi) \setminus \mathcal{K} \neq \emptyset$.
(d) For each topological choice function f *for* Γ *, there is a finite subfamily* $\mathcal{F}$ *of* Γ *such that* $\overline{S} = S^{Lmc} = \bigcup_{\mathcal{L}\in\mathcal{F}}(\overline{f(\mathcal{L})})^\circ$.

Proof. We show first that (a) implies (b),(c) and (d), so we assume (a) holds. Observe that if $\widetilde{p}, \widetilde{q} \in S^{Lmc}$ and $\widetilde{p} \notin [\widetilde{q}]$, then for some $f \in wap(S)$, $\widehat{f}(\widetilde{p}) \neq \widehat{f}(\widetilde{q})$. Since $\widehat{f}$ is continuous, there is a neighborhood of $\widetilde{p}$ missing $[\widetilde{q}]$, that is, each $[\widetilde{q}]$ is closed in S^{Lmc}. Thus by Theorem 1.7 condition (d) holds.

To establish (b), let $\varphi \in \mathcal{W}(S)$ and let $\mathcal{L} \in \Gamma$ and assume $\mathcal{L} \bigcup Range(\varphi)$ has the finite intersection property. Pick $\widetilde{p} \in S^{Lmc}$ such that pure $z-$filter generated by $\mathcal{L} \bigcup Range(\varphi)$ contained in $\widetilde{p}$. Let $\mathcal{K} = \bigcap[\widetilde{p}]$. Suppose that $Range(\varphi) - \mathcal{K} \neq \emptyset$, pick $\widetilde{q} \in [\widetilde{p}]$ and $r \in Q^+$ such that $\widetilde{q} \notin \overline{\varphi(r)}$. Define f as in Lemma 2.4(c), then $\widehat{f}(\widetilde{p}) = 0$ while $\widehat{f}(\widetilde{q}) \geq r$, a contradiction.

To establish (c), let $\mathcal{L}$ and $\mathcal{K}$ be distinct members of Γ and pick $\widetilde{p}$ and $\widetilde{q}$ in S^{Lmc} such that $\mathcal{L} = \bigcap[\widetilde{p}]$ and $\mathcal{K} = \bigcap[\widetilde{q}]$. Pick $f \in wap(S)$ such that $\widehat{f}(\widetilde{p}) \neq \widehat{f}(\widetilde{q})$ and let $a = \widehat{f}(\widetilde{p})$. Define φ as in Lemma 2.4. Let $\epsilon =\mid \widehat{f}(\widetilde{p}) - \widehat{f}(\widetilde{q}) \mid$ and pick $\nu \in Q^+$ such that $\nu < \epsilon$. Then $\widetilde{q} \notin (\overline{\varphi(\nu)})^\circ$ so $\varphi(\nu) \notin \mathcal{K}$. Further, if $\widetilde{r} \in [\widetilde{p}]$, then $\widehat{f}(\widetilde{r}) = a$. Since $\widetilde{r} \in (\overline{\varphi(\nu)})^\circ$ for every $\nu \in \mathbb{Q}^+$ therefore $\varphi(\nu) \in \widetilde{r}$, [10, Lemma 2.8] , and so $Range(\varphi) \subseteq \widetilde{r}$. Therefore $Range(\varphi) \subseteq \mathcal{L}$.

Now we assume that (b),(c) and (d) hold and prove (a). Since (d) holds, by Theorem 1.7 each $\widetilde{p} \in S^{Lmc}$ contains some $\mathcal{L} \in \Gamma$. Therefore, it suffices to show that if $\widetilde{p} \in S^{Lmc}$ and $\mathcal{L} \in \Gamma$ with $\mathcal{L} \subseteq \widetilde{p}$, then $\mathcal{L} = \bigcap[\widetilde{p}]$. To this end , let $\widetilde{p} \in S^{Lmc}$ and let $\mathcal{L} \in \Gamma$ with $\mathcal{L} \subseteq \widetilde{p}$. Let $\widetilde{q} \in [\widetilde{p}]$ and suppose $\mathcal{L} \setminus \widetilde{q} \neq \emptyset$. By condition (d), pick $\mathcal{K} \in \Gamma$ such that $\mathcal{K} \subseteq q$. Pick φ as guaranteed by condition (c). Note that if we had $Range(\varphi) \subseteq \widetilde{q}$, we would have $\mathcal{K} \bigcup Range(\varphi)$ having the finite intersection property and thus we would have $Range(\varphi) \subseteq \mathcal{K}$, by condition (b) and this is a contradiction with (c). Thus we pick $r \in Q^+$ such that $\widetilde{q} \notin \overline{\varphi(r)}$. Define f as in Lemma 2.4. Then $\widehat{f}(\widetilde{q}) \geq r$ while $\widehat{f}(\widetilde{p}) = 0, a$ is a contradiction . Thus $\mathcal{L} \subseteq \bigcap[\widetilde{p}]$. To see that $\bigcap[\widetilde{p}] \subseteq \mathcal{L}$, let $\overline{\bigcap[\widetilde{p}]} \subseteq (\overline{A})^\circ$ and suppose that $\overline{\mathcal{L}} - (\overline{A})^\circ \neq \emptyset$. Pick $\widetilde{q} \in S^{Lmc}$ such that $\mathcal{L} \bigcup \{B \in Z(Lmc(S)) : \overline{S} - (\overline{A})^\circ \subseteq (\overline{B})^\circ\} \subseteq \widetilde{q}$ and pick $f \in wap(S)$ such that $\widehat{f}(\widetilde{p}) \neq \widehat{f}(\widetilde{q})$. Let $a = \widehat{f}(\widetilde{p})$ and define φ as in Lemma 2.4(c). Then $Range(\varphi) \subseteq \widetilde{p}$, by Lemma 2.4(a), so $\mathcal{L} \bigcup Range(\varphi)$ has the finite intersection property so by condition (b), $Range(\varphi) \subseteq \mathcal{L}$. But then $Range(\varphi) \subseteq \widetilde{q}$ so $\widehat{f}(\widetilde{p}) = \widehat{f}(\widetilde{q})$, is a contradiction.

Now assume that (a) holds. To show that Γ is a Hausdorff compact space and $e[S]$ is dense in Γ is suffices, by Theorem 1.10, to show that Γ is a quotient of S^{Lmc}, that is, conditions (a) and (b) of Theorem 1.8 hold. Since condition (a) of Theorem 1.8 and condition (d) of this Theorem are identical, it suffices to establish condition (b) of Theorem 1.8. To this end, let $\mathcal{L}$ and $\mathcal{K}$ be distinct members of Γ and pick $\widetilde{p}$ and $\widetilde{q}$ in S^{Lmc} such that $\mathcal{L} = \bigcap[\widetilde{p}]$ and $\mathcal{K} = \bigcap[\widetilde{q}]$. Pick $f \in wap(S)$ such that $\widehat{f}(\widetilde{p}) \neq \widehat{f}(\widetilde{q})$. Let $\epsilon =\mid \widehat{f}(\widetilde{p}) - \widehat{f}(\widetilde{q}) \mid$, let $A = \{s \in S :\mid f(s) - \widehat{f}(\widetilde{p}) \mid \leq \epsilon/3\}$ and $B = \{s \in S :\mid f(s) - \widehat{f}(\widetilde{q}) \mid \leq \epsilon/3\}$. Then

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$A, B \in Z(Lmc(S))$, $[\widetilde{p}] \subseteq (\overline{A})^\circ$ and $[\widetilde{q}] \subseteq (\overline{B})^\circ$ so $A \in \mathcal{L}^\circ$ and $B \in \mathcal{K}^\circ$. Let $\mathcal{C} \in \Gamma$ and pick $r \in S^{Lmc}$ such that $\mathcal{C} = \bigcap[r]$. If $\mid \widehat{f}(r) - \widehat{f}(\widetilde{p}) \mid \leq 2\epsilon/3$, then $\overline{\mathcal{C}} \subseteq \overline{S - B} \subseteq \overline{S} - (\overline{B})^\circ \subseteq (\overline{H})^\circ$ implies that $H \in \mathcal{C}$ for each $H \in Z(Lmc(S))$, and if $\mid \widehat{f}(r) - \widehat{f}(\widetilde{p}) \mid \geq \epsilon/3$ then $\overline{\mathcal{C}} \subseteq \overline{S - A} \subseteq \overline{S} - (\overline{A})^\circ \subseteq (\overline{T})^\circ$ implies $T \in \mathcal{C}$ for each $T \in Z(Lmc(S))$. □

We want to show that $\Gamma = \{\bigcap[\widetilde{p}] : \widetilde{p} \in S^{Lmc}\}$ is S^{wap}. In order to use Remark 1.9, we need that for each $\mathcal{L}$ and $\mathcal{K}$ in Γ, there exists $\mathcal{C} \in \Gamma$ with $\mathcal{C} \subseteq \mathcal{L} + \mathcal{K}$. For such $\mathcal{C}$, we have $\overline{\mathcal{L} + \mathcal{K}} \subseteq \overline{\mathcal{C}}$. By [10, Lemma 2.10], we have, if $\mathcal{L} = \bigcap[\widetilde{p}]$ and $\mathcal{K} = \bigcap[\widetilde{q}]$, then $\widetilde{p} * \widetilde{q} \in \overline{\mathcal{L} + \mathcal{K}}$. Since for distinct $\mathcal{C}$ and $\mathcal{D}$ in Γ, $\overline{\mathcal{C}} \bigcap \overline{\mathcal{D}} = \emptyset$, our only candidate for $\mathcal{C}$ in Γ with $\mathcal{C} \subseteq \mathcal{L} + \mathcal{K}$ is thus $\bigcap[\widetilde{p} * \widetilde{q}]$.

Lemma 3.8. *If for all $\widetilde{p}, \widetilde{q}$ and $\widetilde{r}$ in S^{Lmc}, $\widetilde{q} \equiv \widetilde{r}$ implies $(\widetilde{q} * \widetilde{p}) \equiv (\widetilde{r} * \widetilde{p})$, then for all $\widetilde{p}$ and $\widetilde{q}$ in S^{Lmc},*

$$\bigcap[\widetilde{q} * \widetilde{p}] \subseteq \bigcap[\widetilde{q}] + \bigcap[\widetilde{p}].$$

Proof. Let $\widetilde{p}$ and $\widetilde{q}$ be in S^{Lmc} and let $[\widetilde{q} * \widetilde{p}] \subseteq (\overline{A})^\circ$. We show that $A \in \bigcap[\widetilde{q}] + \bigcap[\widetilde{p}]$. To this end, we let $\widetilde{r} \in [\widetilde{q}]$ and show that for each $F \in Z(Lmc(S))$, $\Omega_{\bigcap[\widetilde{p}]}(A) \subseteq F$ implies $F \in \widetilde{r}$,(i.e. $A \in \cap[\widetilde{q}]$). Since $\widetilde{r} \in [\widetilde{q}]$ we have by assumption that $(\widetilde{r} * \widetilde{p}) \equiv (\widetilde{q} * \widetilde{p})$. Thus $[\widetilde{r} * \widetilde{p}] \subseteq (\overline{A})^\circ$. Pick, by Lemma 2.6, $f \in wap(S)$ and $\delta > 0$ such that $Range(f) \subseteq [0, 1]$, $\widehat{f}(\widetilde{r} * \widetilde{p}) = 0$ and $f^{-1}([0, \delta]) \subseteq A$. Let $B = f^{-1}([0, \delta/2])$. Then $B \in \widetilde{r} + \widetilde{p}$ and so for each $F \in Z(Lmc(S))$, $\Omega_{\widetilde{p}}(B) \subseteq F$ implies that $F \in \widetilde{r}$. For $s \in S$, define g_s by $g_s(t) = f(st) = f \circ \lambda_s(t)$. Then $g_s \in wap(S)$ and for $\gamma > 0$, $g_s^{-1}([0, \gamma]) = \lambda_s^{-1}(f^{-1}([0, \gamma])$. Now given $s \in \Omega_{\widetilde{p}}(B)$, we have $g_s^{-1}([0, \delta/2]) \in \widetilde{p}$ so $\widehat{g}_s(\widetilde{p}) \leq \delta/2$. Thus

$$\{t \in S :\mid g_s(t) - \widehat{g}_s(\widetilde{p}) \mid \leq \delta/2\} \subseteq g_s^{-1}([0, \delta/2])$$

so, by Lemma 2.6, $g_s^{-1}([0, \delta/2]) \in \bigcap[\widetilde{p}]$. Since $g_s^{-1}([0, \delta/2]) \subseteq (f \circ \lambda_s)^{-1}([0, \delta])$ we have

$$\Omega_{\widetilde{p}}(B) \subseteq \{s \in S : (f \circ \lambda_s)^{-1}([0, \delta]) \in \bigcap[\widetilde{p}]\}.$$

Since $f^{-1}([0, \delta]) \subseteq A$,

$$\Omega_{\widetilde{p}}(B) \subseteq \{s \in S : \lambda_s^{-1}(A) \in \bigcap[\widetilde{p}]\}.$$

So for each $H \in Z(Lmc(S))$,

$$\Omega_{\cap[\widetilde{p}]}(A) = \{s \in S : \lambda_s^{-1}(A) \in \bigcap[\widetilde{p}]\} \subseteq H$$

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implies $H \in \widetilde{r}$, and this conclude $H \in \bigcap[\widetilde{r}]$, hence $A \in \bigcap[\widetilde{r}] + \bigcap[\widetilde{p}]$. Now we can conclude

$$\begin{aligned}\overline{\bigcap[\widetilde{r}] + \bigcap[\widetilde{p}]} &= \bigcap_{B \in \bigcap[\widetilde{r}]+\bigcap[\widetilde{p}]} (\overline{B}) \\ &\subseteq \bigcap_{[\widetilde{q}*\widetilde{p}] \subseteq (\overline{B})^\circ} \overline{B} \\ &= [\widetilde{q} * \widetilde{p}]\end{aligned}$$

and this implies that

$$\bigcap[\widetilde{q} * \widetilde{p}] \subseteq \bigcap[\widetilde{q}] + \bigcap[\widetilde{p}].$$

□

Lemma 3.9. *Let $\widetilde{p}, \widetilde{q} \in S^{Lmc}$ and let $A \in Z(Lmc(S))$. If $[\widetilde{q} * \widetilde{p}] \subseteq (\overline{A})^\circ$, then for each $B \in Z(Lmc(S))$, $\{t \in S : [\widetilde{q}] \subseteq \overline{r_t^{-1}(A)}\} \subseteq B$ implies $B \in \widetilde{p}$.*

Proof. Let $[\widetilde{q} * \widetilde{p}] \subseteq (\overline{A})^\circ$, and pick by Lemma 2.6, $f \in wap(S)$ and $\delta > 0$ such that $Range(f) \subseteq [0, 1]$, $\widehat{f}(\widetilde{q} * \widetilde{p}) = 0$ and $f^{-1}([0, \delta]) \subseteq A$. Let $B = f^{-1}([0, \delta/3])$. Then $B \in \bigcap[\widetilde{q} * \widetilde{p}] \subseteq \widetilde{q} * \widetilde{p}$ so, in particular, for each $T \in Z(Lmc(S))$, $\Omega_{\widetilde{p}}(B) \subseteq T$ implies $T \in \widetilde{q}$.

Given $t \in S$, define $g_t \in wap(S)$ by $g_t(s) = f \circ r_t(s) = f(st)$. Then for $\gamma > 0$, $g_t^{-1}([0, \gamma]) = r_t^{-1}(f^{-1}([0, \gamma]))$. Note that if $g_t^{-1}([0, \delta/2]) \in \widetilde{q}$, then $\widehat{g}_t(\widetilde{q}) \leq \delta/2$ so that

$$\{s \in S :| \ g_t(s) - \widehat{g}_t(\widetilde{q}) \ | < \delta/2\} \subseteq g_t^{-1}([0, \delta])$$

and hence by Lemma 2.6, $g_t^{-1}([0, \delta]) \in \bigcap[\widetilde{q}]$. Thus it suffices to show, with $D = f^{-1}([0, \delta/2])$, that for each $H \in Z(Lmc(S))$,

$$\{t \in S : [\widetilde{q}] \subseteq \overline{r_t^{-1}(D)}\} \subseteq H$$

implies $H \in \widetilde{p}$.(Because $\{t \in S : [\widetilde{q}] \subseteq \overline{r_t^{-1}(D)}\} \subseteq \{t \in S : \widetilde{q} \in \overline{r_t^{-1}(D)}\}$.) Suppose instead that there exists $H \in Z(Lmc(S))$ such that $\{t \in S : \widetilde{q} \in \overline{r_t^{-1}(D)}\} \subseteq H$ and $H \notin \widetilde{p}$. Hence there exists a $z-$ultrafilter $p \subseteq Z(Lmc(S))$ such that $H \notin p$ and by [10, Lemma 2.5(10)], for each $F \in Z(Lmc(S))$, $H^c \subseteq F$ implies that $F \in p$. So for each $F \in Z(Lmc(S))$, $H^c \subseteq \{t \in S : \widetilde{q} \notin \overline{r_t^{-1}(D)}\} \subseteq F$ implies that $F \in p$.

Let $t_1 \in E = \{t \in S : \widetilde{q} \notin \overline{r_t^{-1}(D)}\}$. Inductively pick

$$s_n \in r_{\widetilde{p}}^{-1}((\overline{B})^\circ) \cap \bigcap_{k=1}^{n} (S - r_{t_k}^{-1}(D)),$$

(if $r_{\widetilde{p}}^{-1}((\overline{B})^\circ) \cap \bigcap_{k=1}^{n}(S - r_{t_k}^{-1}(D)) = \emptyset$ then

$$\varepsilon^{-1}(r_{\widetilde{p}}^{-1}((\overline{B})^\circ)) \subseteq \bigcup_{k=1}^{n} r_{t_k}^{-1}(D)$$

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and so $\widetilde{q} \in \overline{\bigcup_{k=1}^{n} r_{t_k}^{-1}(D)} = \bigcup_{k=1}^{n} \overline{r_{t_k}^{-1}(D)}$ is a contradiction.) and

$$t_{n+1} \quad \in \quad E \cap \bigcap_{k=1}^{n} (\overline{\lambda_{s_k}^{-1}(B)})^{\circ}.$$

($E \cap \bigcap_{k=1}^{n} (\overline{\lambda_{s_k}^{-1}(B)})^{\circ} \neq \emptyset$, because $\widetilde{p}$ is a closure point of E and interior point of $\bigcap_{k=1}^{n} (\overline{\lambda_{s_k}^{-1}(B)})^{\circ}$).

Then, if $k \leq n$ we have $s_n t_k \notin D$ so that $f(s_n t_k) \geq \delta/2$. Also, if $n < k$ then $s_n t_k \in B$ so that $f(s_n t_k) \leq \delta/3$. Thus, thinning the sequence $< t_n >_{n=1}^{\infty}$ and $< s_n >_{n=1}^{\infty}$ so that all limits exist, we have $lim_{k\to\infty} lim_{n\to\infty} f(s_n t_k) \geq \delta/2$ while $lim_{n\to\infty} lim_{k\to\infty} f(s_n t_k) \leq \delta/3$ so that $f \notin wap(S)$. □

Theorem 3.10. *For all $\widetilde{p}$ and $\widetilde{q}$ in S^{Lmc}*

$$\bigcap [\widetilde{q} * \widetilde{p}] \subseteq \bigcap [\widetilde{q}] + \bigcap [\widetilde{p}].$$

Proof. Let $\widetilde{p}, \widetilde{q}$ and $\widetilde{r}$ be in S^{Lmc} and assume $\widetilde{q} \equiv \widetilde{r}$. By Lemma 2.8 it suffices to show that $(\widetilde{q} * \widetilde{p}) \equiv (\widetilde{r} * \widetilde{p})$. Suppose instead we have $f \in wap(S)$ such that $\widehat{f}(\widetilde{q} * \widetilde{p}) \neq \widehat{f}(\widetilde{r} * \widetilde{p})$ and let $\epsilon = \mid \widehat{f}(\widetilde{q} * \widetilde{p}) - \widehat{f}(\widetilde{r} * \widetilde{p}) \mid$. Let

$$A = \{s \in S : \mid f(s) - \widehat{f}(\widetilde{q} * \widetilde{p}) \mid \leq \epsilon/3\}$$

and let

$$B = \{s \in S : \mid f(s) - \widehat{f}(\widetilde{r} * \widetilde{p}) \mid \leq \epsilon/3\}.$$

By Lemma 2.6, $[\widetilde{q} * \widetilde{p}] \subseteq (\overline{A})^{\circ}$ and $[\widetilde{r} * \widetilde{p}] \subseteq (\overline{B})^{\circ}$, in particular $B \in \widetilde{r} * \widetilde{p}$. By Lemma 2.9, for each $T \in Z(Lmc(S))$,

$$\{t \in S : [\widetilde{q}] \subseteq \overline{r_t^{-1}(A)}\} \subseteq T$$

implies that $T \in \widetilde{p}$. Since $[\widetilde{q}] = [\widetilde{r}]$, we have for each $T \in Z(Lmc(S))$,

$$\{t \in S : \widetilde{r} \in \overline{r_t^{-1}(A)}\} \subseteq T$$

implies that $T \in \widetilde{p}$. Let $C = \varepsilon^{-1}(r_{\widetilde{p}}^{-1}((\overline{B})^{\circ})) \subseteq \Omega_{\widetilde{p}}(B)$ (see [10, Lemma 2.7],) and let $D = \{t \in S : \widetilde{r} \in \overline{r_t^{-1}(A)}\}$. Pick $t_1 \in D$. Inductively let $s_n \in C \cap (\bigcap_{k=1}^{n} r_{t_k}^{-1}(A))$ and let $t_{n+1} \in D \cap (\bigcap_{k=1}^{n} (\overline{\lambda_{s_k}^{-1}(B)})^{\circ})$. Thinning the sequences $< t_n >_{n=1}^{\infty}$ and $< s_n >_{n=1}^{\infty}$ we get

$$\mid lim_{n\to\infty} lim_{k\to\infty} f(s_k t_n) - \widehat{f}(\widetilde{q} * \widetilde{p}) \mid \leq \epsilon/3$$

and

$$\mid lim_{k\to\infty} lim_{n\to\infty} f(s_k t_n) - \widehat{f}(\widetilde{r} * \widetilde{p}) \mid \leq \epsilon/3,$$

a contradiction. □

Theorem 3.11. *Let $\Gamma = \{\bigcap [\widetilde{p}] : \widetilde{p} \in S^{Lmc}\}$. Then Γ is a semitopological semigroup.*

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Proof. Let $\widetilde{p} \in S^{Lmc}$ and let $\mathcal{L} = \bigcap[\widetilde{p}]$. We need only show $\lambda_{\mathcal{L}}$ is continuous. Do not suppose and pick $\widetilde{q} \in S^{Lmc}$ such that $\lambda_{\mathcal{L}}$ is not continuous at $\bigcap[\widetilde{q}]$, and pick $A \in \bigcap[\widetilde{p} * \widetilde{q}]$ such that $[\widetilde{p} * \widetilde{q}] \subseteq (\overline{A})^{\circ}$ and $\lambda_{\mathcal{L}}^{-1}(\gamma((\overline{A})^{\circ}))$ contains no neighborhood of $\bigcap[\widetilde{q}]$. Pick $\delta > 0$ and $f \in wap(S)$ such that $Range(f) \subseteq [0,1]$, $\widehat{f}(\widetilde{p} * \widetilde{q}) = 0$ and $f^{-1}([0,\delta]) \subseteq A$.

We show first that for all $D \in (\bigcap[\widetilde{q}])^{\circ}$, there is some $t \in D$ such that

$$\widetilde{p} \in \overline{\{s \in S : f(st) \geq \delta/2\}}.$$

Suppose instead we have $D \in (\bigcap[\widetilde{q}])^{\circ}$ such that for all $t \in D$,

$$\widetilde{p} \notin \overline{\{s \in S : f(st) \geq \delta/2\}}$$

and so by [10, Lemma 2.5(12)], for all $t \in D$ and for every $F \in Z(Lmc(S))$,

$$\{s \in S : f(st) < \delta/2\} \subseteq F$$

implies that $F \in \widetilde{p}$. Since $\gamma((\overline{D})^{\circ})$ is not contained in $\lambda_{\mathcal{L}}^{-1}(\gamma((\overline{A})^{\circ}))$, pick $\widetilde{r} \in S^{Lmc}$ such that $D \in (\bigcap[\widetilde{r}])^{\circ}$ and $A \notin \bigcap[\widetilde{p} * \widetilde{r}]$. Now if $\widehat{f}(\widetilde{p} * \widetilde{r}) < \delta$, we would have, with $\mu = \delta - \widehat{f}(\widetilde{p} * \widetilde{r})$,

$$\{s \in S :\mid f(s) - \widehat{f}(\widetilde{p} * \widetilde{r}) \mid < \mu\} \subseteq A$$

and hence $A \in \bigcap[\widetilde{p} * \widetilde{r}]$ by Lemma 2.6. Thus $\widehat{f}(\widetilde{p} * \widetilde{r}) \geq \delta$. Thus

$$\{s \in S : f(st) \geq 2\delta/3\} \in \widetilde{p} * \widetilde{r}.$$

Let $B = \{s \in S : f(st) \geq 2\delta/3\}$ and let

$$C = \varepsilon^{-1}(r_{\widetilde{r}}^{-1}((\overline{B})^{\circ})) \subseteq \Omega_{\widetilde{r}}(B) = \{s \in S : \lambda_s^{-1}(B) \in \widetilde{r}\}.$$

Then for each $F \in Z(Lmc(S))$, $C \subseteq F$ implies that $F \in \widetilde{p}$. Let $s_1 \in C$, and, inductively, let

$$t_n \in D \cap (\bigcap_{k=1}^{n} (\lambda_{s_k}^{-1}(B))),$$

and let

$$s_{n+1} \in C \cap (\bigcap_{k=1}^{n} \{s \in S : f(st_k) < \delta/2\}).$$

Then, after thinning we have

$$lim_{n\to\infty} lim_{k\to\infty} f(s_k t_n) \leq \delta/2$$

while

$$lim_{k\to\infty} lim_{n\to\infty} f(s_k t_n) \geq 2\delta/3.$$

This contradiction establishes the claim.

Let $E = f^{-1}([0,\delta/3])$ and let $F = \varepsilon^{-1}(r_{\cap[\widetilde{q}]}^{-1}((\overline{E})^{\circ}))$. Then $E \in \cap[\widetilde{p} * \widetilde{q}]$ so, by Theorem 2.10, for each $T \in Z(Lmc(S))$, $F \subseteq T$ implies that $T \in \cap[\widetilde{p}]$

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and hence $T \in \widetilde{p}$. Pick $s_1 \in F$. Inductively, $\cap_{k=1}^{n}\lambda_{s_k}^{-1}(E) \in (\cap[\widetilde{q}])^{\circ}$ so pick $t_n \in \cap_{k=1}^{n}\lambda_{s_k}^{-1}(E)$ such that $\widetilde{p} \in \overline{\{s \in S : f(st_n) \geq \delta/2\}}$. Pick

$$s_{n+1} \in F \cap \bigcap_{k=1}^{n}\{s \in S : f(st_k) \geq \delta/2\}.$$

Again, after thinning we obtain

$$lim_{n\to\infty}lim_{k\to\infty}f(s_kt_n) \geq \delta/2$$

while $lim_{k\to\infty}lim_{n\to\infty}f(s_kt_n) \leq \delta/3$. □

The next theorem says that $\{\cap[\widetilde{p}] : \widetilde{p} \in S^{Lmc}\}$ is S^{wap}.

Theorem 3.12. *Let* $\Gamma = \{\cap[\widetilde{p}] : \widetilde{p} \in S^{Lmc}\}$. *Then*
(1) e *is a continuous homomorphism from* S *to* Γ,
(2) Γ *is a compact Hausdorff semitopological semigroup,*
(3) $e(S)$ *is dense in* Γ, *and*
(4) if (T, φ) *satisfies (1) and (2), with* T *replacing* Γ *and* φ *replacing* e, *there is a continuous homomorphism* $\eta : \Gamma \to T$ *such that* $\eta \circ e = \varphi$.

Proof. Statements (1), (2) and (3) follow from Remark 1.9, Theorem 2.7, 2.10 and 2.11. (Theorem 2.10 need to show that the hypotheses of Remark 1.9 are satisfied.)

Let (T, φ) satisfy (1) and (2). We first show there is a continuous $\eta : \Gamma \to T$ such that $\eta \circ e = \varphi$. By Lemma 2.1 it suffices to show for this that for each $\widetilde{p} \in S^{Lmc}$, $\widetilde{\varphi}$ is constant on $[\widetilde{p}]$, where $\widetilde{\varphi} : S^{Lmc} \to T$ is the continuous extension of $\varphi : S \to T$. Suppose instead that we have $\widetilde{p} \equiv \widetilde{q}$ with $\widetilde{\varphi}(\widetilde{p}) \neq \widetilde{\varphi}(\widetilde{q})$. Since T is completely regular, pick $f \in \mathcal{C}(T)$ such that $f(\widetilde{\varphi}(\widetilde{p})) \neq f(\widetilde{\varphi}(\widetilde{p}))$. By Lemma 1.3(ii), $f \in wap(T)$. Consequently $f \circ \varphi \in wap(S)$.(Given $< t_n >_{n=1}^{\infty}$ and $< s_n >_{n=1}^{\infty}$ in S such that $lim_{n\to\infty}lim_{k\to\infty}f \circ \varphi(t_ns_k)$ and $lim_{k\to\infty}lim_{n\to\infty}f \circ \varphi(t_ns_k)$ exist, we have

$$\begin{aligned} lim_{n\to\infty}lim_{k\to\infty}f \circ \varphi(t_ns_k) &= lim_{n\to\infty}lim_{k\to\infty}f(\varphi(t_n)\varphi(s_k)) \\ &= lim_{k\to\infty}lim_{n\to\infty}f(\varphi(t_n)\varphi(s_k)) \\ &= lim_{k\to\infty}lim_{n\to\infty}f \circ \varphi(t_ns_k)). \end{aligned}$$

$f \circ \widetilde{\varphi}$ is a continuous extension of $f \circ \varphi$ to S^{Lmc} so $f \circ \widetilde{\varphi} = \widehat{f \circ \varphi}$. But then $\widehat{f \circ \varphi}(\widetilde{p}) \neq \widehat{f \circ \varphi}(\widetilde{q})$ so that $\widetilde{p}$ is not equivalent with $\widetilde{q}$, a contradiction.

To complete the proof, we show that η is a homomorphism. Since $\widetilde{\varphi}$ is a homomorphism also since $\eta \circ \gamma$ is a continuous extension of φ, we have $\eta \circ \gamma = \widetilde{\varphi}$.

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Thus, given $\widetilde{p}$ and $\widetilde{q}$ in S^{Lmc}, we have

$$\begin{aligned}\eta(\cap[\widetilde{p}]\dot{+}\cap[\widetilde{q}]) &= \eta(\cap[\widetilde{p}*\widetilde{q}])\\ &= \eta\circ\gamma(\widetilde{p}*\widetilde{q})\\ &= \widetilde{\varphi}(\widetilde{p}*\widetilde{q})\\ &= \widetilde{\varphi}(\widetilde{p})\widehat{\varphi}(\widetilde{q})\\ &= \eta\circ\gamma(\widetilde{p})\eta\circ\gamma(\widetilde{q})\\ &= \eta(\cap[\widetilde{p}])\eta(\cap[\widetilde{q}]).\end{aligned}$$

□

Theorem 3.13. *Let Γ be a set of pure $z-$filter. There exist an operator $*$ on Γ and a function $h : S \to \Gamma$ satisfying conditions (a) of Theorem 1.10 such that $(\Gamma, *, h)$ is S^{wap} if and only if Γ satisfies conditions (a), (c) and (d) of Theorem 2.7.*

Proof. (Sufficiency) By Theorem 2.7, $\Gamma = \{\cap[\widetilde{p}] : \widetilde{p} \in S^{Lmc}\}$ so Theorem 2.12 applies.

(Necessity) By Theorem 1.10, Γ is a quotient of S^{Lmc}. Thus by Remark 1.9, the function $\gamma : S^{Lmc} \to \Gamma$ defined by $\gamma(\widetilde{p}) \subseteq \widetilde{p}$ is (well defined and) a quotient map. By condition $(a)(i)$, for each $s \in S$, $s \in \cap h(s)$. That is, $h(s) \subseteq \widehat{s}$. Then $h(s) = \gamma(\widehat{s}) = e(s)$ so $h = e$.

Define an equivalence relation R on S by pRq if and only if $\gamma(\widetilde{p}) = \gamma(\widetilde{p})$. It suffices to show that $\widetilde{p}R\widetilde{q}$ if and only if $\widetilde{p} \equiv \widetilde{q}$. For then we get $\Gamma = \{\cap[\widetilde{p}] : \widetilde{p} \in S^{Lmc}\}$ and Theorem 2.7 applies. To this end, let $\widetilde{p}, \widetilde{q} \in S^{Lmc}$ and assume $\widetilde{p}R\widetilde{q}$. Suppose $\widetilde{p} \equiv q$ is not true and pick $f \in wap(S)$ such that $\widehat{f}(\widetilde{p}) \neq \widehat{f}(\widetilde{q})$. By Theorem 1.2, there exists $g \in \mathcal{C}(\Gamma)$ such that $g \circ e = f$. Then $g \circ \gamma$ is a continuous extension of f to S^{Lmc}. (For $s \in S$, $(g \circ \gamma)(\widehat{s}) = g(e(s)) = f(s)$.) Thus $g\circ\gamma = \widehat{f}$. Since $\widetilde{p}R\widetilde{q}$, then $\widehat{f}(\widetilde{p}) = g(\gamma(\widetilde{p})) = g(\gamma(\widetilde{q})) = \widehat{f}(\widetilde{q})$, contradiction. Now assume that $\widetilde{p} \equiv \widetilde{q}$ and suppose that $\gamma(\widetilde{p}) \neq \gamma(\widetilde{q})$. Pick $g \in \mathcal{C}(\Gamma)$ such that $g(\gamma(\widetilde{p})) \neq g(\gamma(\widetilde{q}))$. Define $f \in C(S)$ by $f(s) = g(e(s))$. Then, since f extends continuously to Γ, $f \in wap(S)$ by Theorem 1.2. But, as above $g \circ \gamma = f$ so $\widehat{f}(\widetilde{p}) \neq \widehat{f}(\widetilde{q})$, a contradiction. □

We now proceed to apply our results to a determination of size of S^{wap}.

Definition 3.14. Let $A \subseteq S$. We say A is an unbounded subset of S if

$$\overline{A} \cap S^* \neq \emptyset,$$

where $S^* = S^{Lmc} - S$. Also a sequence in S is unbounded if range of sequence is unbounded.

Lemma 3.15. *Let $A \in Z(Lmc(S))$. Then*

a) Let there exist $\widetilde{p}$ and $\widetilde{q}$ in S^* such that $A \in \widetilde{p} + \widetilde{q}$. Then there exist one-to-one sequences $\{t_n\}$ and $\{s_n\}$ in S such that $\{s_k t_n : k \leq n\} \subseteq A$.

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b) Let there exist one-to-one unbounded sequences $\{t_n\}$ and $\{s_n\}$ in S such that $\{s_k t_n : k \leq n\} \subseteq A$. Then there exist $\widetilde{p}$ and $\widetilde{q}$ in S^* such that $\widetilde{p} * \widetilde{q} \in \overline{A}$.

Proof. (*a*) Pick $\widetilde{p}$ and $\widetilde{q}$. Let $B = \Omega_{\widetilde{p}}(A) = \{s \in S : \lambda_s^{-1}(A) \in \widetilde{p}\}$. Then $\widetilde{q} \in \overline{B}$ and since $\widetilde{q} \notin S$, B is unbounded. Let $\{s_n\}$ be an one-to-one sequence in B. For each n, we have $\bigcap_{k=1}^{n} \lambda_{s_k}^{-1}(A) \in \widetilde{p}$ and is hence unbounded. Pick $t_n \in \bigcap_{k=1}^{n} \lambda_{s_k}^{-1}(A)$ such that $t_n \notin \{t_k : k < n\}$.

(*b*) It is obvious. □

Lemma 3.16. *Let S be a commutative semitopological semigroup and let $\widetilde{p} \in S^* - cl_{S^{Lmc}}(S^* * S^*)$. Then for each $\widetilde{q} \in S^{Lmc} - \{\widetilde{p}\}$ there exists $f \in wap(S)$ such that Range$(f) = [0,1]$, $\widehat{f}(\widetilde{p}) = 1$ and $\widehat{f}(\widetilde{q}) = 0$. Consequently $[\widetilde{p}] = \{\widetilde{p}\}$.*

Proof. Pick $A \in (\widetilde{p})^\circ$ such that $\overline{A} \cap cl_{S^{Lmc}}(S^* * S^*) = \emptyset$. Pick $B \in (\widetilde{p})^\circ$ such that $\widetilde{q} \notin \overline{B}$. Then there exists $f \in Lmc(S)$ such that $\widehat{f}|_{A \cap B} = 1$, $\widehat{f}((\overline{A})^c) \subseteq [0,1)$, $\widehat{f} = 0$ on $cl_{S^{Lmc}}(S^* * S^*) \cup \{\widetilde{q}\}$ and $f(S) = [0,1]$. It suffices to show that $f \in wap(S)$. Suppose, instead, we have sequences $\{t_n\}$ and $\{s_n\}$ such that $lim_{k\to\infty} lim_{n\to\infty} f(s_k t_n) = 1$ and $lim_{n\to\infty} lim_{k\to\infty} f(s_k t_n) = 0$. We may assume by thinning that $\{t_n\}$ and $\{s_n\}$ are one-to-one and then

$$\{s_k t_n : k \leq n \text{ for all sufficiently large } n, k \in \mathbb{N}\} \subseteq \overline{A}$$

so if $\{s_n\}$ and $\{t_n\}$ are unbounded sequences, by Lemma 2.15(b), $\overline{A} \cap (S^* * S^*) \neq \emptyset$ and this is a contradiction.
If $\{s_n\}$ or $\{t_n\}$ are bounded in S and limits exist then limits are equal. □

Theorem 3.17. *Let S be a Hausdorff non-compact commutative semitopological semigroup and assume $A \in Z(Lmc(S))$ be an unbounded set such that*
(i) card(S^{Lmc}) =card$(\overline{A})$ and
(ii) $\overline{A} \cap cl_{S^{Lmc}}(S^* * S^*) = \emptyset$.
Then card(S^{wap}) =card(S^{Lmc}).

Proof. It is obvious that card(S^{wap}) ≤card(S^{Lmc}). By Lemma 2.16, for each $\widetilde{p} \in \overline{A}$, $[\widetilde{p}] = \{\widetilde{p}\}$. Therefore $card(S^{wap}) \geq card(\overline{A}) = card(S^{Lmc})$. □

Example 3.18. Let S be a dense subsemigroup of $((1, +\infty), +)$. Then card(S^{wap}) = card(S^{Lmc}). Since $cl_{S^{Lmc}}((1,2] \cap S)$ is infinite. Let $\widetilde{p}, \widetilde{q} \in cl_{S^{Lmc}}((1,2] \cap S) - S$ then there exist two nets $\{x_\alpha\}$ and $\{y_\beta\}$ in $(1,2]$ such that $x_\alpha \to \widetilde{p}$ and $y_\beta \to \widetilde{q}$. This implies that $\widetilde{p} * \widetilde{q} \notin cl_{S^{Lmc}}((1, \frac{3}{2}] \cap S)$. Therefore

$$S^* - cl_{S^{Lmc}}(S^* * S^*) \neq \emptyset.$$

It is obvious $card(cl_{S^{Lmc}}((1,2] \cap S)) = card(cl_{S^{Lmc}}[2, +\infty) \cap S)$ and so

$$card(cl_{S^{Lmc}}((1,2] \cap S)) = card(S^{Lmc}).$$

Now let $A = (1,2] \cap S$, then A satisfies in the assumptions of Theorem 2.17. Thus by Theorem 2.17, card(S^{wap}) =card(S^{Lmc}).

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Lemma 3.19. *Let S be a semitopological semigroup and let*

$$\widetilde{p} \in S^* - cl_{S^{Lmc}}(S^* * S^{Lmc}).$$

Then for each $\widetilde{q} \in S^{Lmc} - \{\widetilde{p}\}$, there exists $f \in wap(S)$ such that Range$(f) = [0,1]$, $\widehat{f}(\widetilde{p}) = 1$ and $\widehat{f}(\widetilde{q}) = 0$. Consequently $[\widetilde{p}] = \{\widetilde{p}\}$.

Proof. Pick $A \in (\widetilde{p})^\circ$ such that $\overline{A} \cap cl_{S^{Lmc}}(S^* * S^{Lmc}) = \emptyset$. Pick $B \in (\widetilde{p})^\circ$ such that $\widetilde{q} \notin \overline{B}$. Then there exists $f \in Lmc(S)$ such that $\widehat{f}|_{A \cap B} = 1$, $\widehat{f}((\overline{A})^c) \subseteq [0,1)$, $\widehat{f} = 0$ on $cl_{S^{Lmc}}(S^* * S^{Lmc}) \cup \{\widetilde{q}\}$ and $f(S) = [0,1]$. It suffices to show that $f \in wap(S)$. Suppose, instead, we have sequences $\{t_n\}$ and $\{s_n\}$ such that $lim_{k\to\infty} lim_{n\to\infty} f(s_k t_n) = 1$ and $lim_{n\to\infty} lim_{k\to\infty} f(s_k t_n) = 0$. We may assume by thinning that $\{t_n\}$ and $\{s_n\}$ are one-to-one and that,

$$\{s_k t_n : k \leq n \text{ for all sufficiently large } n, k \in \mathbb{N}\} \subseteq \overline{A},$$

so if $\{s_n\}$ and $\{t_n\}$ are unbounded sequences, by Lemma 2.15(b), $\overline{A} \cap (S^* * S^*) \neq \emptyset$ and this is a contradiction.
Let $\{s_n\}$ and $\{t_n\}$ are bounded sequences in S or $\{s_n\}$ be a bounded sequence and $\{t_n\}$ be an unbounded sequence in S, then

$$lim_{k\to\infty} lim_{n\to\infty} f(s_k t_n) = lim_{n\to\infty} lim_{k\to\infty} f(s_k t_n).$$

Now let $\{s_n\}$ be an unbounded sequence and $\{t_n\}$ be a bounded sequences in S, so there exist two nets $\{s_\alpha\} \subseteq \{s_n : n \in \mathbb{N}\}$ and $\{t_\beta\} \subseteq \{t_n : n \in \mathbb{N}\}$ such that $s_\alpha \to u \in S^*$ and $t_\beta \to t \in S$. Therefore

$$\begin{aligned} lim_{k\to\infty} lim_{n\to\infty} f(s_k t_n) &= lim_\alpha lim_\beta f(s_\alpha t_\beta) \\ &= lim_\alpha f(s_\alpha t) \\ &= \widehat{f}(ut) \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} lim_{n\to\infty} lim_{k\to\infty} f(s_k t_n) &= lim_\beta lim_\alpha f(s_\alpha t_\beta) \\ &= lim_\beta \widehat{f}(ut_\beta) \\ &= 0. \end{aligned}$$

Thus $f \in wap(S)$ and this complete the proof. □

Theorem 3.20. *Let S be a Hausdorff non-compact semitopological semigroup and assume $A \in Z(Lmc(S))$ be an unbounded set such that*
(i) card(S^{Lmc}) =card$(\overline{A})$ and
(ii) $\overline{A} \cap cl_{S^{Lmc}}(S^* * S^{Lmc}) = \emptyset$.
Then card(S^{wap}) =card(S^{Lmc}).

Proof. It is obvious that card$(S^{wap}) \leq$card(S^{Lmc}). By Lemma 2.19, for each $\widetilde{p} \in \overline{A}$, $[\widetilde{p}] = \{\widetilde{p}\}$. Therefore $card(S^{wap}) \geq card(\overline{A}) = card(S^{Lmc})$. □

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Theorem 3.21. *Let S be a semitopological semigroup. Let S^{wap} be the one point compactification and $card(S^*) > 1$. Then $S^* * S^{Lmc}$ is dense in S^*.*

Proof. Let S^{wap} be the one point compactification so $[\widetilde{p}] = S^*$ for each $\widetilde{p} \in S^*$. Let $\widetilde{p} \in S^* - cl_{S^{Lmc}}(S^* * S^{Lmc})$ then for each $\widetilde{q} \in S^{Lmc} - \{\widetilde{p}\}$ there exists $f \in wap(S)$ such that Range$(f) = [0,1]$, $\widehat{f}(\widetilde{p}) = 1$ and $\widehat{f}(\widetilde{q}) = 0$, by Lemma 2.19. Consequently $[\widetilde{p}] = \{\widetilde{p}\}$. This is a contradiction. So $S^* - cl_{S^{Lmc}}(S^* * S^{Lmc}) = \emptyset$. □

Example 3.22. Let G be the linear group $Sl(2,\mathbb{R})$. For this group G^{wap} = the one-point compactification,(see [6]), and so $card(G^{wap}) = card(G)$ while $card(G^{Lmc}) = card(G^{\mathcal{LUC}}) = 2^{2^{\kappa(G)}}$, where $G^{\mathcal{LUC}}$ is the maximal ideal space of the C^*-algebra of bounded left norm continuous function on G and $\kappa(G) = \omega$ (in this case) is the compact covering number of G, i.e., the minimal number of compact sets needed to cover G. Thus $[\widetilde{p}] = G^*$ for each $\widetilde{p} \in G^* = G^{\mathcal{LUC}} - G = G^{Lmc} - G$. Hence $G^*G^{\mathcal{LUC}} = G^*G^{Lmc}$ is dense in G^*, by Theorem 2.21.

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References

1. M. Amini, A. R. Medghalchi, Spectrum of the Fourier-Stieltjes algebra of a semigroup, *Iranian Journal of Mathematical Sciences and Informatics*, **1** (2), (2006), 1-8.
2. J. F. Berglund, H. D. Junghenn, P. Milnes, *Analysis on Semigroups: Function Spaces, Compactifications, Representations*, Wiley, New York, 1989.
3. J. F. Berglund, N. Hindmani, Filters and the weak almost periodic compactification of discrete semigroup, *Trans. Amer. Math. Soc.*, **284**, (1984), 1-38.
4. W. Comfort, S. Negrepontis, *The Theory of Ultrafilters,* Springer-Verlag, Berlin, 1974.
5. L. Gilman, M. Jerison, *Ring of Continuous Functions,* Van Nostrand, Princeton, N. J., 1960.
6. M. Filali, *On the action of a locally compact group on same of its semigroup compactifications*, 2007, Math. Proc. Camb. Phil. Soc., **143**, 25-39.
7. N. Hindman, D. Strauss, *Algebra in the Stone-Čech Compactification, Theory and Application,* Springer Series in Computational Mathematics, Walter de Gruyter, Berlin, 1998.
8. R. Safari, S. M. Bagheri, On the Ultramean Construction, *Iranian Journal of Mathematical Sciences and Informatics*, **9** (2), (2014), 109-119.
9. M. A. Tootkaboni, *Comfort order on topological space,* Ultramath Con. 2008.
10. M. A. Tootkaboni, Quotient space of $\mathcal{LMC}$-compactification as a space of $z-$filters, *Acta Math Scientia.*, **32** (5), (2012), 2010-2020.
11. M. A. Tootkaboni, Z. Eslami, Density in Locally Compact Topological Groups, *Topology and its Applications*, **165**, (2014), 2638
12. M. A. Tootkaboni, A. Riazi, Ultrafilters on Semitopological Semigroup, *Semigroup Forum*, **70** (3), (2005), 317-328.
13. M. A. Tootkaboni, H. R. E. Vishki, Filters and semigroup compactification properties, *Semigroup Forum*, **78** (2), (2009), 349-359.

[Downloaded from ijmsi.ir on 2026-11-04]

[DOI: 10.7508/ijmsi.2015.01.005]

14. Y. Zelenyuk, *Ultrafilters and Topologies on Groups,* Walter de Gruyter, Berlin, 2011.

[Downloaded from ijmsi.ir on 2026-11-04]

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