

## ***p*-Analog of the Semigroup Fourier-Stieltjes Algebras**

Marzieh Shams Yousefi

Department of Pure Mathematics, Faculty of Mathematical Science,  
University of Guilan, Rasht, Iran.

E-mail: m.shams@guilan.ac.ir

**ABSTRACT.** In this paper we define the  $p$ -analog of the restricted representations and the  $p$ -analog of the Fourier–Stieltjes algebras on inverse semigroups. Also we improve some results about Herz algebras on Clifford semigroups and we give a necessary and sufficient condition for amenability of these algebras on Clifford semigroups.

**Keywords:** Restricted fourier–Stieltjes algebras, Restricted inverse semigroup, Restricted representations,  $QSL_p$ -spaces,  $p$ -Analog of the Fourier–Stieltjes algebras.

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### 1. INTRODUCTION AND PRELIMINARIES

An inverse semigroup  $S$  is a discrete semigroup such that for each  $s \in S$  there exists a unique element  $s^* \in S$  such that  $ss^*s = s$ ,  $s^*ss^* = s^*$ . The set  $E(S)$  of idempotents of  $S$  consists of elements of the form  $ss^*$ ,  $s \in S$ . Actually for each abstract inverse semigroup  $S$  there is a  $*$ -semigroup homomorphism from  $S$  into the inverse semigroup of partial isometries on some Hilbert space[18].

Dunkl and Ramirez in [8] and T. M. Lau in [15] attempted to define a suitable substitution for Fourier and Fourier–Stieltjes algebras on semigroups. Each definition has its own difficulties. Amini and Medghalchi introduced and

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\*Corresponding Author

extensively studied the theory of restricted semigroups and restricted representations and restricted Fourier and Fourier–Stieltjes algebras,  $A_{r,e}(S), B_{r,e}(S)$  in [2] and [3]. Also they studied the spectrum of the Fourier Stieltjes algebra for a unital foundation topological  $*$ -semigroup in [4]. In this section we mention some of their results.

Throughout this paper  $S$  is an inverse semigroup. Given  $x, y \in S$ , the restricted product of  $x, y$  is  $xy$  if  $x^*x = yy^*$ , and undefined, otherwise. The set  $S$  with its restricted product forms a groupoid [16, 3.1.4] which is called the associated groupoid of  $S$ . If we adjoin a zero element 0 to this groupoid, and put  $0^* = 0$ , we will have an inverse semigroup  $S_r$  with the multiplication rule

$$x \bullet y = \begin{cases} xy & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for  $x, y \in S \cup \{0\}$ , which is called the restricted semigroup of  $S$ . A restricted representation  $\{\pi, \mathcal{H}_\pi\}$  of  $S$  is a map  $\pi : S \rightarrow \mathcal{B}(\mathcal{H}_\pi)$  such that  $\pi(x^*) = \pi(x)^*$  ( $x \in S$ ) and

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for  $x, y \in S$ . Let  $\Sigma_r = \Sigma_r(S)$  be the family of all restricted representations  $\pi$  of  $S$  with  $\|\pi\| \leq 1$ . Now it is clear that, via a canonical identification,  $\Sigma_r(S) = \Sigma_0(S_r)$ , consist of all  $\pi \in \Sigma(S_r)$  with  $\pi(0) = 0$ , where the notation  $\Sigma$  has been used for all  $*$ -homomorphism from  $S$  into  $\mathcal{B}(\mathcal{H})$  [2]. One of the central concepts in the analytic theory of inverse semigroups is the left regular representation  $\lambda : S \rightarrow \mathcal{B}(\ell^2(S))$  defined by

$$\lambda(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* \geq yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for  $\xi \in \ell^2(S), x, y \in S$ . The restricted left regular representation  $\lambda_r : S \rightarrow \mathcal{B}(\ell^2(S))$  is defined in [2] by

$$\lambda_r(x)\xi(y) = \begin{cases} \xi(x^*y) & \text{if } xx^* = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for  $\xi \in \ell^2(S), x, y \in S$ . The main objective of [2] is to change the convolution product on the semigroup algebra to restore the relation with the left regular representation.

For each  $f, g \in \ell^1(S)$ , define

$$(f \bullet g)(x) = \sum_{x^*x = yy^*} f(xy)g(y^*) \quad (x \in S),$$

and for all  $x \in S$ ,  $\tilde{f}(x) = \overline{f(x^*)}$ .  $\ell_r^1(S) := (\ell^1(S), \bullet, \sim)$  is a Banach  $*$ -algebra with an approximate identity. The left regular representation  $\lambda_r$  lifts to a faithful representation  $\tilde{\lambda}$  of  $\ell_r^1(S)$ . We call the completion  $C_{\lambda_r}^*(S)$  of  $\ell_r^1(S)$  with the

norm  $\|\cdot\|_{\lambda_r} := \|\tilde{\lambda}_r(\cdot)\|$  which is a  $C^*$ -norm on  $\ell_r^1(S)$ , the restricted reduced  $C^*$ -algebra and its completion with the norm  $\|\cdot\|_{\Sigma_r} := \sup\{\|\tilde{\pi}(\cdot)\|, \pi \in \Sigma(S_r)\}$  the restricted full  $C^*$ -algebra and show it by  $C_r^*(S)$ . The dual space of  $C^*$ -algebra  $C_r^*(S)$  is a unital Banach algebra which is called the restricted Fourier–Stieltjes algebra and is denoted by  $B_{r,e}(S)$ . The closure of the set of finitely support functions in  $B_{r,e}(S)$  is called the restricted Fourier algebra and is denoted by  $A_{r,e}(S)$  [2].

In [10], Figà-Talamanca introduced a natural generalization of the Fourier algebra, for a compact abelian group  $G$ , by replacing  $L_2(G)$  by  $L_p(G)$ . In [11], Herz extended the notion to an arbitrary group, to get the commutative Banach algebra  $A_p(G)$ , called the Figà–Talamanca–Herz algebra. Figà–Talamanca–Herz algebra and Eymard’s Fourier algebra have very similar behavior. For example, Leptin’s theorem is valid:  $G$  is amenable if and only if  $A_p(G)$  has a bounded approximate identity [12]. The  $p$ -analog,  $B_p(G)$  of the Fourier–Stieltjes algebra is defined as the multiplier algebra of  $A_p(G)$ , by some authors, as mentioned in [5] and [19]. Runde in [20] defined and studied  $B_p(G)$ , the  $p$ -analog of the Fourier–Stieltjes algebra on the locally compact group  $G$ . He developed the theory of representations and defined the suitable coefficient functions on them.

For  $p \in (1, \infty)$ , Medghalchi and Pourmahmood Aghababa developed the theory of restricted representations on  $\ell_p(S)$  and defined the Banach algebra of  $p$ -pseudomeasures  $PM_p(S)$  and the Figà–Talamanca–Herz algebras  $A_p(S)$ . They showed that  $A_q(S)^* = PM_p(S)$  for dual pairs  $p, q$ . They characterized  $PM_p(S)$  and  $A_p(S)$  for Clifford semigroups, in the sense of  $p$ -pseudomeasures and Figà–Talamanca–Herz algebras of maximal semigroups of  $S$ , respectively [17].

Amini also worked on quantum version of Fourier transforms in [1].

In this paper we will combine what Medghalchi–Pourmahmood Aghababa and Runde have done. We will define the restricted representations on  $QSL_p$ -spaces and the  $p$ -analog of the Fourier–Stieltjes algebra on the restricted inverse semigroup.

Section 2 is a review of the theory of  $QSL_p$ -spaces. In Section 3 we define the restricted representations on  $QSL_p$ -spaces and study their tensor product. In Sections 4 and 5 we construct the  $p$ -analog of the restricted Fourier–Stieltjes algebra and study its order structure. The last section will be about Clifford semigroups and the  $p$ -analog of their restricted Fourier–Stieltjes algebra. Some new results which improves the results of [17] and [22] will be given in Section 6.

## 2. REVIEW OF THE THEORY OF $QSL_p$ -SPACES

This section is a review of the paper of Runde [20].

**Definition 2.1.** A Banach space  $\mathcal{E}$  is called

- (i) an  $L_p$ -space if it is of the form  $L_p(X)$ , for some measure space  $X$ .
- (ii) a  $QSL_p$ -space if it is isometrically isomorphic to a quotient of a subspace of an  $L_p$ -space (or equivalently, a subspace of a quotient of an  $L_p$ -space [20, Section 1, Remark 1]).

If  $E$  is a  $QSL_p$ -space and if  $p' \in (1, \infty)$  is such that  $\frac{1}{p} + \frac{1}{p'} = 1$ , the dual space  $E^*$  is an  $QSL_{p'}$ -spaces. In particular, every  $QSL_p$ -space is reflexive.

By [14, Theorem 2], the  $QSL_p$ -spaces are precisely the  $p$ -spaces in the sense of [11], i.e. those Banach spaces  $E$  such that for any two measure spaces  $X$  and  $Y$  the amplification map

$$B(L_p(X), L_p(Y)) \rightarrow B(L_p(X, E), L_p(Y, E)), T \rightarrow T \otimes id_E$$

is an isometry. In particular, an  $L_q$ -space is a  $QSL_p$ -space if and only if  $2 \leq q \leq p$  or  $p \leq q \leq 2$ . Consequently, if  $2 \leq q \leq p$  or  $p \leq q \leq 2$ , then every  $QSL_q$ -space is a  $QSL_p$ -space.

Runde equipped the algebraic tensor product of two  $QSL_p$ -spaces with a suitable norm, which comes in the following.

**Theorem 2.2.** [20, Theorem 3.1] *Let  $E$  and  $F$  be  $QSL_p$ -spaces. Then there exists a norm  $\|\cdot\|_p$  on the algebraic tensor product  $E \otimes F$  such that:*

- (i)  $\|\cdot\|_p$  dominates the injective norm;
- (ii)  $\|\cdot\|_p$  is a cross norm;
- (iii) the completion  $E \tilde{\otimes}_p F$  of  $E \otimes F$  with respect to  $\|\cdot\|_p$  is a  $QSL_p$ -space. The Banach space  $E \tilde{\otimes}_p F$  will be called  $p$ -projective tensor product of  $E$  and  $F$ .

### 3. RESTRICTED REPRESENTATION ON A BANACH SPACE

In this section we give an analog of the theory of group representations on a Hilbert space for the restricted representations for an inverse semigroup on a  $QSL_p$ -space.

**Definition 3.1.** A representation of a discrete inverse semigroup  $S$  on a Banach space  $E$  is a pair  $(\pi, E)$  consisting of a map  $\pi : S \rightarrow B(E)$  satisfying  $\pi(x)\pi(y) = \pi(xy)$ , for  $x, y \in S$  and  $\|\pi(x)\| \leq 1$ , for all  $x \in S$ .

**Definition 3.2.** A restricted representation of a discrete inverse semigroup  $S$  on a Banach space  $E$  is a pair  $(\pi, E)$  consisting of a map  $\pi : S \rightarrow B(E)$  satisfying

$$\pi(x)\pi(y) = \begin{cases} \pi(xy) & \text{if } x^*x = yy^*, \\ 0 & \text{otherwise} \end{cases}$$

for  $x, y \in S$ , and  $\|\pi(x)\| \leq 1$ , for all  $x \in S$ .

**Definition 3.3.** Let  $S$  be an inverse semigroup, and let  $(\pi, E)$  and  $(\rho, F)$  be restricted representations of  $S$ , then these restricted representations are said to be equivalent if there exists a surjective isometry  $T : E \rightarrow F$  such that

$$T\pi(x)T^{-1} = \rho(x), \quad (x \in S).$$

For any inverse semigroup  $S$  and  $p \in (1, \infty)$ , we denote by  $\Sigma_{p,r}(S)$  the collection of all (equivalence classes) of restricted representations of  $S$  on a  $QSL_p$ -space.

*Remark 3.4.* By [17] for  $p \in (1, \infty)$  the restricted left regular representation  $\lambda_p : S \longrightarrow B(\ell^p(S))$

$$\lambda_p(s)(\delta_t) = \begin{cases} \delta_{st} & \text{if } s^*s = tt^*, \\ 0 & \text{otherwise} \end{cases}$$

for  $s, t \in S$  is a restricted representation so it belongs to  $\Sigma_{p,r}(S)$ .

The following propositions are easy to check, similar to [2].

**Proposition 3.5.** *For an inverse semigroup  $S$  and its related restricted semigroup  $S_r$ , each restricted representation of  $S$  on a Banach space is a representation on  $S_r$  which is zero on  $0 \in S_r$ , i.e. it is multiplicative with respect to the restricted multiplication.*

**Proposition 3.6.** *For an inverse semigroup  $S$ , each restricted representation  $\pi$  of  $S$  on a Banach space lifts to a representation of  $\ell_r^1(S)$ , via*

$$\tilde{\pi}(f) = \sum_{x \in S} f(x)\pi(x).$$

#### 4. BANACH ALGEBRA $B_{p,r}(S)$

In this section we define the  $p$ -analog of the Fourier–Stieltjes algebra on a inverse semigroup. We show that for  $p = 2$  we get the known algebra  $B_{r,e}(S)$ , defined in [2].

**Theorem 4.1.** *Let  $(\pi, E), (\rho, F) \in \Sigma_{p,r}(S)$  then  $(\pi \otimes \rho, E \tilde{\otimes}_p F) \in \Sigma_{p,r}(S)$ .*

*Proof.* By the definition of  $\pi \otimes \rho$  we have  $\pi \otimes \rho(x)(\xi \otimes \eta) = \pi(x)\xi \otimes \rho(x)\eta$ . For  $x, y \in S$ ,  $x^*x = yy^*$ ,

$$\begin{aligned} \pi \otimes \rho(xy)(\xi \otimes \eta) &= \pi(xy)\xi \otimes \rho(xy)\eta \\ &= \pi(x)\pi(y)\xi \otimes \rho(x)\rho(y)\eta \\ &= \pi(x)(\pi(y)\xi) \otimes \rho(x)(\rho(y)\eta) \\ &= \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta) \\ &= \pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta) \end{aligned}$$

when  $x^*x \neq yy^*$

$$\begin{aligned} \pi \otimes \rho(x)\pi \otimes \rho(y)(\xi \otimes \eta) &= \pi \otimes \rho(x)(\pi(y)\xi \otimes \rho(y)\eta) \\ &= \pi(x)(\pi(y)\xi) \otimes \rho(x)(\rho(y)\eta) \end{aligned}$$

which is equal to zero. Now it is enough to show that  $\pi(x) \in B(E)$  and  $\rho(y) \in B(F)$ ,  $\pi(x) \otimes \rho(y)$  could be extend to  $E \tilde{\otimes}_p F$ . This is shown as in the group case [20, Thorem 3.1].  $\square$

**Definition 4.2.** Let  $S$  be an inverse semigroup, and let  $(\pi, E) \in \Sigma_{p,r}(S)$ . A coefficient function of  $(\pi, E)$  is a function  $f : S \rightarrow \mathbb{C}$  of the form

$$f(x) = \langle \pi(x)\xi, \phi \rangle \quad (x \in S),$$

where  $\xi \in E$  and  $\phi \in E^*$ .

**Definition 4.3.** Let  $S$  be an inverse semigroup, let  $p \in (1, \infty)$ , and let  $q \in (1, \infty)$  be the dual scalar to  $p$ , i.e.  $\frac{1}{p} + \frac{1}{q} = 1$ . We define

$$B_{p,r}(S) := \{f : S \rightarrow \mathbb{C} : f \text{ is a coefficient function of some } (\pi, E) \in \Sigma_{q,r}(S)\}.$$

**Proposition 4.4.** Let  $S$  be an inverse semigroup, let  $p \in (1, \infty)$ , and let  $q \in (1, \infty)$  be the dual scalar to  $p$ , i.e.  $\frac{1}{p} + \frac{1}{q} = 1$ , and let  $f : S \rightarrow \mathbb{C}$  defined by

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \phi_n \rangle, \quad (x \in S),$$

where  $((\pi_n, E_n))_{n=1}^{\infty}$ ,  $(\xi_n)_{n=1}^{\infty}$ , and  $(\phi_n)_{n=1}^{\infty}$  are sequences with  $(\pi_n, E_n) \in \Sigma_{q,r}(S)$ ,  $\xi_n \in E_n$ , and  $\phi_n \in E_n^*$ , for  $n \in \mathbb{N}$  such that

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty.$$

Then  $f$  lies in  $B_{p,r}(S)$ .

*Proof.* The proof is similar to [20]. Without loss of generality, we may suppose that

$$\sum_{i=1}^{\infty} \|\xi_n\|_q < \infty, \quad \text{and} \quad \sum_{i=1}^{\infty} \|\phi_n\|_p < \infty.$$

Then  $E := \ell_q \oplus \bigoplus_{n=1}^{\infty} E_n$  is a  $QSL_q$ -space and for  $\xi := (\xi_1, \xi_2, \dots)$  and  $\phi := (\phi_1, \phi_2, \dots)$ , we have  $\xi \in E$  and  $\phi \in E^*$ . Now the map  $\pi : S \rightarrow B(E)$  with  $\pi(x)\eta = (\pi_1(x)\eta, \pi_2(x)\eta, \dots)$  is a restricted representation of  $S$  on  $E$ , and  $f$  is the coefficient function of  $\pi$ .  $\square$

**Definition 4.5.** [17, Definition 3.1]. Let  $S$  be an inverse semigroup and let  $p, q \in (1, \infty)$  be dual pairs. The space  $A_q(S)$  consists of those  $u \in c_0(S)$  such that there exist sequences  $(f_n)_{n=1}^{\infty} \subseteq \ell_q(S)$  and  $(g_n)_{n=1}^{\infty} \subseteq \ell_p(S)$  with  $\sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_p \leq \infty$  and  $u = \sum_{n=1}^{\infty} f_n \bullet \check{g}_n$ . For  $u \in A_q(S)$ , let

$$\|u\| = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_q \|g_n\|_p : u = \sum_{n=1}^{\infty} f_n \bullet \check{g}_n \right\}$$

**Proposition 4.6.** [17, Proposition 3.2]. Let  $S$  be an inverse semigroup and let  $p \in (1, \infty)$ , then  $A_p(S)$  is a Banach space and is the closure of finite support functions on  $S$ .

**Proposition 4.7.** *Let  $S$  be an inverse semigroup, let  $p \in (1, \infty)$ . Then  $B_{p,r}(S)$  is a linear subspace of  $c_b(S)$  containing  $A_p(S)$ . Moreover, if  $2 \leq q \leq p$  or  $p \leq q \leq 2$ , we have  $B_{q,r}(S) \subseteq B_{p,r}(S)$ .*

*Proof.* Every thing is easy to check, and is similar to [20].  $\square$

**Definition 4.8.** Let  $S$  be an inverse semigroup, and let  $(\pi, E)$  be a restricted representation of  $S$  on the Banach space  $E$ . Then  $(\pi, E)$  is called cyclic if there exists  $x \in E$  such that  $\pi(\ell_r^1(S))x$  is dense in  $E$ . For  $p \in (1, \infty)$ , we set  $Cyc_{p,r}(S) := \{(\pi, E) : (\pi, E) \text{ is a cyclic restricted representation on a } QSL_p\text{-space } E\}$ .

**Definition 4.9.** Let  $S$  be an inverse semigroup, let  $p, q \in (1, \infty)$  be the dual scalars, and let  $f \in B_{p,r}(S)$ . We define  $\|f\|_{B_{p,r}(S)}$  as the infimum over all expressions  $\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\|$ , where, for each  $n \in \mathbb{N}$ , there is  $(\pi_n, E_n) \in Cyc_{q,r}(S)$  with  $\xi_n \in E_n$  and  $\phi_n \in E_n^*$  such that  $\sum_{n=1}^{\infty} \|\xi_n\| \|\phi_n\| < \infty$  and

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x) \xi_n, \phi_n \rangle, \quad (x \in S).$$

The proof of the following theorem is similar to the group case.

**Theorem 4.10.** *Let  $S$  be an inverse semigroup, let  $p \in (1, \infty)$ , and let  $f, g : S \rightarrow \mathbb{C}$  be coefficient functions of  $(\pi, E)$  and  $(\rho, F)$  in  $\Sigma_{p,r}(S)$ , respectively. Then the pointwise product of  $f$  and  $g$  is a coefficient function of  $(\pi \otimes \rho, E \otimes_p F)$ .*

In the next theorem we give some result about our new constructed space and also the relation between semigroup restricted Herz algebra and our new space.

**Theorem 4.11.** *Let  $S$  be an inverse semigroup, let  $p \in (1, \infty)$ . Then:*

- (i)  $B_{p,r}(S)$  is a commutative Banach algebra.
- (ii) the inclusion  $A_p(S) \subseteq B_{p,r}(S)$  is a contraction.
- (iii) for  $2 \leq p' \leq p$  or  $p \leq p' \leq 2$ , the inclusion  $B_{p',r}(S) \subseteq B_{p,r}(S)$  is a contraction.
- (iv) for  $p = 2$ ,  $B_{r,e}(S)$  is isometrically isomorphic to  $B_{p,r}(S)$  as Banach algebras.

*Proof.* (i) Let  $\frac{1}{p} + \frac{1}{q} = 1$ . The space  $B_{p,r}(S)$  is the quotient space of complete  $q$ -projective tensor product of  $E \tilde{\otimes}_q E^*$ , for the universal restricted representation  $(\pi, E)$ , on  $QSL_q$ -space  $E$ . Also Theorem 4.10 shows it is an algebra. The submultiplicative property for norm of  $B_{p,r}(S)$  is similar to the group case in [20] and it is only based on characteristic property of infimum.

(ii) By the definition of semigroup Herz algebra in [17] for conjugate numbers  $p, q$ , each  $f \in A_p(S)$  is a coefficient function of the restricted left regular representation on the  $\ell_q$ -space,  $\ell_q(S)$ . So  $A_p(S) \subseteq B_{p,r}(S)$ . By the definition of the norm of  $f \in B_{p,r}(S)$ , the infimum is taken on all expressions of  $f$  as the coefficient function of some restricted representation on a  $QSL_q$ -space, and

the norm on the  $A_p(S)$  is the infimum only on expressions of  $f$  as the coefficient function of restricted left regular representation, so the inclusion map is a contraction.

(iii) For  $2 \leq p' \leq p$  or  $p \leq p' \leq 2$  and  $q, q'$  conjugate scalars to  $p$  and  $p'$  respectively. Then each restricted representation on a  $QSL_{q'}$ -space is a restricted representation on a  $QSL_q$ -space.

(iv) By the definition, each element of  $B_{r,e}(S)$  is a coefficient function of a 2-restricted representation [3]. □

*Remark 4.12.* A very natural question is that when  $A_p(S)$  is an ideal in  $B_{p,r}(S)$ . Even in  $p = 2$  this question is not studied. If we want to go along the proof of the group case, a difficulty to prove this is that in general for  $p \in (1, \infty)$ , and  $(\pi, E) \in \Sigma_{p,r}(S)$ , the representations  $(\lambda_p \otimes \pi, \ell_p(S, E))$  and  $(\lambda_p \otimes id_E, \ell_p(S, E))$  are not equivalent. In fact we can not find a suitable substitution for representation  $id : S \rightarrow B(E)$ ,  $id(s) = id_E$  in the class of restricted representations. But in a special case, such as Clifford semigroups, we can give a better result.

## 5. ORDER STRUCTURE OF THE $p$ -ANALOG OF THE SEMIGROUP FOURIER–STIELTJES ALGEBRAS $B_{p,r}(S)$

Studying the ordered spaces and order structures has a long history. The natural order structure of the Fourier-Stieltjes algebras was favorite in 80s. In [21] the authors studied the order structure of Figà-Talamanca–Herz algebra and generalized results on Fourier algebras. In this section, we consider the  $p$ -analog of the restricted Fourier–Stieltjes algebra,  $B_{p,r}(S)$ , introduced in Section 4, and study its order structure given by the  $p$ -analog of positive definite continuous functions.

A *compatible couple* of Banach spaces in the sense of interpolation theory (see [3]) is a pair  $(\mathcal{E}_0, \mathcal{E}_1)$  of Banach spaces such that both  $\mathcal{E}_0$  and  $\mathcal{E}_1$  are embedded continuously in some (Hausdorff) topological vector space. In this case, the intersection  $\mathcal{E}_0 \cap \mathcal{E}_1$  is again a Banach space under the norm  $\|\cdot\|_{(\mathcal{E}_0, \mathcal{E}_1)} = \max\{\|\cdot\|_{\mathcal{E}_0}, \|\cdot\|_{\mathcal{E}_1}\}$ . For example, for a locally compact group  $G$ , the pairs  $(A_p(G), A_q(G))$  and  $(L_p(G), L_q(G))$  are compatible couples.

**Definition 5.1.** Let  $(\pi, E)$  be a restricted representation of  $S$  on a Banach space  $E$ , such that  $(\mathcal{E}, \mathcal{E}^*)$  is a compatible couple. We mean by a  $\pi_r$ -positive definite function on  $S$ , a function which has a representation as  $f(x) = \langle \pi(x)\xi, \xi \rangle$ ,  $(x \in S)$ , where  $\xi \in \mathcal{E} \cap \mathcal{E}^*$ . For dual scalars  $p, q \in (1, \infty)$ , we call each element in the closure of the set of all  $\pi_r$ -positive definite functions on  $S$  in  $B_{p,r}(S)$ , where  $\pi$  is a restricted representation of  $S$  on an  $L_q$ -space, a restricted  $p$ -positive definite function on  $S$  and the set of all restricted  $p$ -positive definite functions on  $S$ , will be denoted by  $P_{p,r}(S)$ .



It follows from [21] and the definition of  $P_{p,r}(S)$ , that for each  $f \in P_{p,r}(S)$ , associated to a representation  $(\pi, E)$ , for a  $QSL_q$ -space  $E$ , there exists a sequence  $(\pi_n, \mathcal{E}_n)_{n=1}^\infty$  of cyclic restricted representations of  $S$  on closed subspaces  $E_n$  of  $E \cap E^*$ , and  $\{\xi_n\}$  in  $\mathcal{E}_n$ , such that

$$f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x) \xi_n, \xi_n \rangle \quad (x \in S).$$

**Proposition 5.2.** *The linear span of all finite support elements in  $P_{p,r}(S)$  is dense in  $A_p(S)$ , and  $A_p(S)$  is an ordered space.*

*Proof.* From [17, Proposition 3.2]  $A_p(S)$  is a norm closure of the set of elements of the form  $\sum_{i=1}^n f_i \bullet \tilde{g}_i$  where  $f_i, g_i$  are finite support functions on  $S$ ,  $i = 1, \dots, n$ . Also  $f_i \bullet \tilde{g}_i(x) = \langle \lambda_r(x^*) f_i, g_i \rangle$ . Now by Polarization identity, we have the statement.  $\square$

Since  $A_p(S)$  is the set of coefficient functions of the restricted left regular representation of  $S$  on  $\ell_p(S)$ , we define the positive cone of  $A_p(S)$  as the closure in  $A_p(S)$ , of the set of all function of the form  $f = \sum_{i=1}^n \xi_i \bullet \tilde{\xi}_i$ , for a sequence  $(\xi_i)$  in  $\ell_p(S) \cap \ell_q(S)$ , and denote it by  $A_p(S)_+$ .

This order structure, in the case where  $p = 2$ , is the same as the order structure of  $A_{r,e}(S)$ , induced by the set  $P_{r,e}(S) \cap A_{r,e}(S)$ , as a positive cone. Because in the case  $p = 2$ , the extensible restricted positive definitive functions are exactly the closed linear span of  $h \bullet \tilde{h}$ , for  $h \in \ell^2(S)$ .

## 6. $p$ -ANALOG OF THE FOURIER–STILETJES ALGEBRAS ON CLIFFORD SEMIGROUPS

Let  $S$  be a semigroup. Then, by [13, Chapter 2], there is an equivalence relation  $\mathcal{D}$  on  $S$  by  $s\mathcal{D}t$  if and only if there exists  $x \in S$  such that

$$Ss \cup \{s\} = Sx \cup \{x\} \text{ and } tS \cup \{t\} = xS \cup \{x\}.$$

If  $S$  is an inverse semigroup, then by [13, Proposition 5.1.2(4)],  $s\mathcal{D}t$  if and only if there exists  $x \in S$  such that  $s^*s = xx^*$  and  $t^*t = x^*x$ .

**Proposition 6.1.** [17, Proposition 4.1]. *Let  $S$  be an inverse semigroup,*

(i) *and let  $D$  be a  $\mathcal{D}$ -class of  $S$ . Then  $\ell_p(D)$  is a closed  $\ell_r^1(S)$ -submodule of  $\ell_p(S)$ .*

(ii) *and let  $\{D_\lambda; \lambda \in \Lambda\}$  be the family of  $\mathcal{D}$ -classes of  $S$  indexed by some set  $\Lambda$ . Then there is an isometric isomorphism of Banach  $\ell_r^1(S)$ -bimodules*

$$\ell^p(S) \cong \ell^p - \bigoplus_{\lambda \in \Lambda} \ell_p(D_\lambda). \quad (6.1)$$

**Corollary 6.2.** *Let  $S$  be an inverse semigroup, and let  $\{D_\lambda; \lambda \in \Lambda\}$  be the family of  $\mathcal{D}$ -classes of  $S$  indexed by some set  $\Lambda$ . Then for a  $QSL_p$ -space  $E$  of functions on  $S$ , there is a family of  $QSL_p$ -spaces  $\{E_\lambda\}_{\lambda \in \Lambda}$ , where for each  $\lambda \in \Lambda$ ,  $E_\lambda$  consists of functions on  $D_\lambda$ , and  $E \cong \ell^p - \bigoplus_{\lambda \in \Lambda} E_\lambda$ .*

*Proof.* This is clear by the definition of a  $QSL_p$ -space, and the fact that the isomorphism 6.1 is compatible with taking quotients and subspaces of  $\ell_p(D_\lambda)$ s.  $\square$

An inverse semigroup  $S$  is called a Clifford semigroup if  $s^*s = ss^*$  for all  $s \in S$ . For  $e \in E(S)$  define  $G_e := \{s \in S | s^*s = ss^* = e\}$ . Then  $G_e$  is a group with identity  $e$ . Here each  $\mathcal{D}$ -class  $D$  contains a single idempotent (say  $e$ ) and we have  $D = G_e$ .

We modified the isometrical isomorphism derived in [17, Section 5.3] in the following theorem.

**Theorem 6.3.** *Let  $S$  be a Clifford semigroup with the family of  $\mathcal{D}$ -classes  $\{G_e\}_{e \in E(S)}$ , and let  $p \in (1, \infty)$ . Then*

$$B_{p,r}(S) \cong \ell^1 - \bigoplus_{e \in E(S)} B_p(G_e)$$

*Proof.* Let  $p, q$  are conjugate scalars. Fix  $e \in E(S)$ , assume that  $G_e = \{x \in S; x^*x = e\}$ . Define  $\pi : S \rightarrow B(\ell_q(G_e))$

$$\pi(s)(\delta_t) = \begin{cases} \delta_t & \text{if } s^*s = e, \\ 0 & \text{otherwise} \end{cases}$$

for  $s \in S$ . Then  $\pi$  is a restricted representation and  $\chi_{G_e}(s) = \langle \pi(s)\delta_t, \delta_t \rangle$ . Hence  $\chi_{G_e}$  is in  $B_{p,r}(S)$ , and indeed  $\chi_{G_e}$  is a restricted positive definite function. Now for each  $u \in B_{p,r}(S)$ ,  $u \cdot \chi_{G_e}$  is in  $B_{p,r}(S)$ . In fact the set  $\{u \in B_{p,r}(S); u(s) = 0 \text{ for all } s \in S \setminus G_e\}$  is a closed subspace of  $B_{p,r}(S)$  and it is isometrically isomorphic to  $B_p(G_e)$ . This follows from the fact that, each coefficient function of a restricted representation of  $S$  on a  $QSL_q$ -space that is zero on  $G_e^c$ , is a coefficient function of a representation on a  $QSL_q$ -space of  $G_e$ , using Corollary 6.2.

Let  $u \in B_{p,r}(S)$ , then we could decompose  $u$  to  $(u_e)_{e \in E(S)}$ , for some  $u_e \in B_p(G_e)$ , by the above paragraph. Now for all  $e \in E(S)$  and all explanations of  $u_e$  as  $u_e = \langle \pi_e(\cdot)\xi_e, \eta_e \rangle$ , where  $\pi_e \in \Sigma_q(G_e)$ ,  $\xi_e$  in some  $QSL_q$  and  $\eta_e$  in some  $QSL_p$ -space for dual scalars  $p, q$  we have  $\|u_e\| \leq \|\xi_e\| \|\eta_e\|$  and also  $u = (u_e)_{e \in E(S)} = \langle \oplus \pi_e(\cdot) \oplus \xi_e, \oplus \eta_e \rangle$  and then  $\oplus \pi_e$  is a restricted representation of  $S$  on a  $QSL_q$ -space and  $\sum_{e \in E(S)} \|u_e\| \leq \sum_{e \in E(S)} \|\xi_e\| \|\eta_e\| \leq (\sum_{e \in E(S)} \|\xi_e\|^q)^{\frac{1}{q}} (\sum_{e \in E(S)} \|\eta_e\|^p)^{\frac{1}{p}} = \|(\xi_e)\| \|(\eta_e)\|$ , the last equality comes from Proposition 6.1. Now we have:

$$\sum_{e \in E(S)} \|u_e\| \leq \| (u_e)_{e \in E(S)} \| = \| \sum_{e \in E(S)} u_e \|$$

by the definition of the norm in the Fourier–Stieltjes algebras. So we have an isometric isomorphism of Banach algebras.  $\square$

The following corollary improves [22, Proposition 2.6 ].

**Corollary 6.4.** *Let  $S$  be a Clifford semigroup with the family of  $\mathcal{D}$ -classes  $\{G_e\}_{e \in E(S)}$ , and let  $p \in (1, \infty)$ . Then  $A_p(S)$  is an ideal of  $B_{p,r}(S)$ , and  $B_{p,r}(S)$  is amenable if and only if  $B_p(G_e)$  is amenable for all  $e \in E(S)$ .*

**Theorem 6.5.** *Let  $S$  be a Clifford semigroup with the family of amenable  $\mathcal{D}$ -classes  $\{G_e\}_{e \in E(S)}$  and let  $p \in (1, \infty)$ . Then  $A_p(S)$  is equal to the closure of  $B_{p,r}(S) \cap F(S)$  in the norm of  $A_p(S)$ .*

*Proof.* Since for each  $e \in E(S)$  the group  $G_e$  is amenable, the natural embedding  $i_e : A_p(G_e) \rightarrow B_p(G_e)$  is an isometry by [20, Corollary 5.3]. Now let  $f \in B_{p,r}(S) \cap F(S)$ . Then by Theorem 6.3,  $f = \sum_{e \in E(S)} f_e$ , where  $f_e \in B_p(G_e) \cap F(G_e)$ , so  $f_e$  belongs to  $A_p(G_e)$  by [20]. Now since

$$A_p(S) \cong \ell_1 - \bigoplus_{e \in E(S)} A_p(G_e),$$

[17, Equation 5.1], we conclude that  $f \in A_p(S)$ . On the other hand let  $f \in A_p(S) \cap F(S)$ . Then for each  $e \in E(S)$  the function  $f_e$ , which has been defined in Theorem 6.3, belongs to  $A_p(G_e) \cap F(G_e)$  and so to  $B_p(G_e) \cap F(G_e)$  with the same norm because of amenability of  $G_e$ . Now, by Theorem 6.3 we have  $f \in B_{p,r}(S) \cap F(S)$ . Since  $F(S)$  is dense in  $A_p(S)$  with norm of  $A_p(S)$ , [17, Proposition 3.2 vi], result follows.  $\square$

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