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On the Ultramean Construction

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Abstract. We use the ultramean construction to prove linear compactness theorem. We also extend the Rudin-Keisler ordering to maximal probability charges and characterize it by embeddings of power ultrameans..

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1. Introduction

An important application of ultrafilters is the ultraproduct construction and Łos theorem which also leads to an algebraic proof of the compactness theorem. A basic fact used in this construction is that ultralimits of sequences of elements in compact sets exist. In particular, if $(x_i)_{i\in I}$, is a sequence of elements in the unit interval and $\mathcal{U}$ is an ultrafilter on I, then $\lim_{i,\mathcal{U}} x_i$ exists (i.e. the unique x such that for every open $x \in U$, $\{i : x_i \in U\} \in \mathcal{U}$). Moreover, ultralimits are preserved by continuous functions. It turns out that, in the interval case, ultralimits coincide with integration with respect to the corresponding $0-1$ valued measure. Integration preserves only linear maps. However, it makes sense for arbitrary finitely additive measures (see [4]). This fact was used in [2] to prove a linear variant of Łos theorem. In this paper, we use this theorem to

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prove a linear variant of compactness theorem for the fragment of continuous logic obtained by restricting to linear connectives.

Ultrafilters are also widely used in set theory. Rudin-Keisler ordering (see [8] and [7]) is a way of comparing complexity of ultrafilters. This relation can be characterized by means of embeddings between ultrapowers of first order models. So, higher in this ordering, bigger ultrapowers generates the ultrafilter. This ordering can be defined for maximal probability charges in a similar way. We show that the extended ordering is characterized by means of embeddings between power ultrameans in a similar way.

In the next section, we introduce linear formulas and state the ultramean theorem. We then use it to prove the linear compactness theorem and also axiomatizability theorem. In the third section we study Rudin-Keisler ordering on maximal probability charges.

2. Ultramean and compactness

We assume the reader is familiar with continuous logic (see [3]). In the standard presentation, the unit interval is used as value space. We first introduce the fragment of continuous logic obtained by restricting to linear connectives. For this purpose, we need to shift from the unit interval to the real line. So, we use $\mathbb{R}$ as value space and, addition $+$ and scalar multiplication by r, for each $r \in \mathbb{R}$, as connectives. We also use the quantifier symbol 'sup'. The quantifier 'inf' is an abbreviation for $-\sup -$. As usual, we have an infinite list $x, y, z, \ldots$ of individual variables. Compactness of the value space is a crucial requirement which is retained locally by imposing bounds on formulas.

Let $\mathcal{L}$ be a first order language consisting of constant symbols and function and relation symbols of various arities. We always assume it contains a distinguished binary relation symbol d for metric. Suppose to each function symbol F is assigned a Lipschitz constant $\lambda_F \geqslant 0$ and also to each relation symbol R is assigned a Lipschitz constant $\lambda_R \geqslant 0$ and a bound $\mathbf{b}_R \geqslant 0$ (we may put $\mathbf{b}_d = \lambda_d = 1$). With such a data $\mathcal{L}$ is called a *Lipschitz language*.

Let $\mathcal{L}$ be a Lipschitz language. Below, if (M, d) is a metric space, we put the metric on M^n defined by $d(\bar{a}, \bar{b}) = \sum_i d(a_i, b_i)$.

Definition 2.1. *An* $\mathcal{L}$-prestructure *is a pseudo-metric space* (M, d) *of diameter at most* $\mathbf{b}_d$ *equipped with:*

- for each $c \in \mathcal{L}$*, an element* $c^M \in M$

- for each n*-ary function symbol* F *a function* $F^M : M^n \to M$ *such that*

$$d(F^M(\bar{a}), F^M(\bar{b})) \leqslant \lambda_F d(\bar{a}, \bar{b}) \qquad \forall \bar{a}, \bar{b}$$

- for each n*-ary relation symbol* R *a function* $R^M : M^n \to [-\mathbf{b}_R, \mathbf{b}_R]$ *such that*

$$R^M(\bar{a}) - R^M(\bar{b}) \leqslant \lambda_R d(\bar{a}, \bar{b}) \qquad \forall \bar{a}, \bar{b}.$$

So, for example, R^M is λ_R-Lipschitz and bounded by $\mathbf{b}_R$. $\mathcal{L}$*-terms* and their Lipschitz constants are inductively defined as follows:

- constant symbols and variables are terms with Lipschitz constants respectively 0 and 1;

- if F is a n-ary function symbol and $\mathtt{t}_1, \ldots, \mathtt{t}_n$ are terms with Lipschitz constants respectively $\lambda_{\mathtt{t}_1}, \ldots, \lambda_{\mathtt{t}_n}$, then $\mathtt{t} = F(\mathtt{t}_1, \ldots, \mathtt{t}_n)$ is a term with Lipschitz constant $\lambda_{\mathtt{t}} = \lambda_F \cdot (\lambda_{\mathtt{t}_1} + \cdots + \lambda_{\mathtt{t}_n})$.

$\mathcal{L}$*-formulas* and their Lipschitz constants and bounds are inductively defined as follows:

- for each $r \in \mathbb{R}$, r is an atomic formula with Lipschitz constant 0 and bound $|r|$;

- if R is a n-ary relation symbol and $\mathtt{t}_1, \ldots, \mathtt{t}_n$ are terms then $R(\mathtt{t}_1, \ldots, \mathtt{t}_n)$ is an atomic formula with Lipschitz constant $\lambda_R \cdot (\lambda_{\mathtt{t}_1} + \ldots + \lambda_{\mathtt{t}_n})$ and bound $\mathbf{b}_R$;

- if ϕ, ψ are formulas and $r, s \in \mathbb{R}$ then $r\phi + s\psi$ is a formula with Lipschitz constant $|r|\lambda_\phi + |s|\lambda_\psi$ and bound $|r|\mathbf{b}_\phi + |s|\mathbf{b}_\psi$;

- $\sup_x \phi$ is a formula with Lipschitz constant λ_ϕ and bound $\mathbf{b}_\phi$.

The notion of free variable and the notation $\phi(x_1, \ldots, x_n)$ have their obvious meanings. A *sentence* is a formula without free variable. For every prestructure M, the value of a term $\mathtt{t}(\bar{x})$ at $\bar{a} \in M$, denoted by $\mathtt{t}^M(\bar{a})$, is defined in the usual way (by induction). For each formula $\phi(\bar{x})$, the interpretation of ϕ in M is defined by induction and is a real valued function denoted by $\phi^M(\bar{x})$. It is easily seen that with $|\bar{x}| = n$

Proposition 2.2. *$\phi^M(\bar{x})$ is λ_ϕ-Lipschitz function on M^n bounded by $\mathbf{b}_\phi$.*

Interesting prestructures are those which are complete metric spaces. They are called $\mathcal{L}$*-structures*. Every prestructure can be easily transformed to an $\mathcal{L}$-structure by first forming the quotient metric and then completing. By uniform continuity, interpretations of function and relation symbols induce well-defined function and relations on the resulting complete metric space. We state the main result and refer the reader to [3] for further details.

Proposition 2.3. *Let M be a prestructure in a language $\mathcal{L}$. Then there exists an $\mathcal{L}$-structure $\overline{M}$ and a map $\pi : M \to \overline{M}$ such that $\pi[M]$ is dense in $\overline{M}$ and for every formula $\phi(\bar{x})$*

$$\phi^M(a_1, \ldots, a_k) = \phi^{\overline{M}}(\pi a_1, \ldots, \pi a_k) \qquad \forall a_1, \ldots, a_k \in M.$$

From now on $M, N, \ldots$ denote structures in the language $\mathcal{L}$.

Definition 2.4. *M, N are* elementarily equivalent*, denoted by $M \equiv N$, if for every sentence σ, $\sigma^M = \sigma^N$. A function $f : M \to N$ is an* embedding *if for* *each quantifier-free formula* $\phi(\bar{x})$ *and* $\bar{a} \in M$*, owe have* $\phi^M(\bar{a}) = \phi^N(f(\bar{a}))$. *It is an* elementary embedding *if this condition holds for every formula. Substructure relation* $M \subseteq N$ *and elementary substructure relation* $M \preceq N$ *are defined in the obvious way.*

Let $\mathcal{L}$ be a (Lipschitz) language. Expressions of the form $\phi \leqslant \psi$ and $\phi = \psi$, where ϕ, ψ are formulas, are called a *conditions*. They are called closed if ϕ, ψ are sentences. A *theory* is a set of closed conditions. The notions $M \vDash (\phi \leqslant \psi)(\bar{a})$ and $M \vDash T$ are defined in the obvious way. A theory T is said to be *linearly closed* if for every finite number of conditions $\phi_1 \leqslant \psi_1$, ..., $\phi_k \leqslant \psi_k$ in T and real numbers $r_1, ..., r_k \geqslant 0$, the condition $\sum_i r_i\phi_i \leqslant \sum_i r_i\psi_i$ belongs to T. The set of all such combinations is called the *linear closure* of T. Note that T and its linear closure are equivalent, i.e. have the same models. If T consists of equalities, we may allow the above coefficients to be negative. The corresponding closure set is again equivalent to T. A theory T is *linearly satisfiable* if every member of its linear closure has a model.

We now define the ultramean construction (see [2] for further details) and use it to prove linear compactness theorem. Let I be a nonempty set. A probability charge on I is a finitely additive measure $\wp : \mathcal{B} \to [0, 1]$ where $\mathcal{B}$ is a Boolean algebra of subsets of I. It is called maximal if $\mathcal{B} = \mathcal{P}(I)$. Ultrafilters may be regarded as $0 - 1$ valued maximal charges. Let $\wp$ be a maximal probability charge on I. For each $i \in I$ let (M_i, d_i) be an $\mathcal{L}$-structure. For $a, b \in \prod_i M_i$ set

$$d(a, b) = \int d_i(a_i, b_i)\ d\wp$$

where the integration is according to the theory of D-integration with respect to finitely additive measures [4]. Then d is a pseudometric on $\prod_i M_i$ and $d(a, b) = 0$ an equivalence relation on it. The equivalence class of (a_i) is denoted by $[a_i]$ and the quotient set by $N = \prod_\wp M_i$. The metric induced on N is denoted again by d. We define an $\mathcal{L}$-structure on N as follows. For each relation symbol R (say unary for simplicity) set

$$R^N([a_i]) = \int R^{M_i}(a_i) d\wp.$$

Then R^N is a λ_R-Lipschitz function:

$$R^{M_i}(a_i) \leqslant R^{M_i}(b_i) + \lambda_R d(a_i, b_i)$$

$$\int R^{M_i}(a_i)\ di \leqslant \int R^{M_i}(b_i)\ di + \lambda_R \int d(a_i, b_i)\ di$$

$$R^N([a_i]) \leqslant R^N([b_i]) + \lambda_R d([a_i], [b_i]).$$

Also, for each function symbol F (say unary again) set

$$F^N([a_i]) = [F^{M_i}(a_i)].$$

One then verifies easily that for each term $t(\bar{x})$ and $a_i^1, ..., a_i^n \in M_i$ one has that

$$t^N([a_i^1], ..., [a_i^n]) = [t^{M_i}(a_i^1, ..., a_i^n)].$$

Proof of the following theorem can be found in [2].

Theorem 2.5. (Ultramean theorem) *For every linear formula* $\phi(x_1, ...x_n)$ *and* $[a_i^1], ..., [a_i^n]$

$$\phi^N([a_i^1], ..., [a_i^n]) = \int \phi^{M_i}(a_i^1, ..., a_i^n) d\wp.$$

If $M_i = M$ for all i, the ultramean is denoted by $M^\wp$ and is called the power ultramean.

Corollary 2.6. *The diagonal embedding* $\mathfrak{d} : M \to M^\wp$ *is elementary.*

To prove the linear compactness theorem, we need two known results from analysis. Let E be partially ordered vector space. A subspace G is called majorizing if for every $x \in E$ there is a $y \in G$ such that $x \leqslant y$.

Theorem 2.7. (Kantorovich, [1], Th. 1.32) *Let E be an ordered vector spaces and F a Dedekind complete Riesz space. Let G be a majorizing vector subspace of E and $\Lambda : G \to F$ a positive linear map. Then Λ has an extension to a positive linear map on E.*

Theorem 2.8. ([4], Th 4.7.4) *Let $\mathcal{F}$ be a field of subsets of a set I and Λ be a continuous positive linear functional on $C(I, \mathcal{F})$ (the set of $\mathcal{F}$-continuous functions on I). Then there exists a unique bounded positive charge μ on $\mathcal{F}$ such that for every $f \in C(I, \mathcal{F})$*

$$\Lambda(f) = \int f d\mu.$$

Further, $||\Lambda|| = \sup\{|\Lambda(f)| \ ; \ |f| \leq 1\} = \mu(I)$.

In case $\mathcal{F} = \mathcal{P}(I)$, $C(I, \mathcal{F})$ is the space of all bounded real functions on I and is a Banach lattice with the sup-norm.

Theorem 2.9. *Let T be a set of conditions of the form $\phi = r$. Assume every linear combination of conditions in T is satisfiable. Then T is satisfiable.*

Proof. Without loss of generality assume that T is linearly closed and contains the conditions $r = r$ for all $r \in \mathbb{R}$. Let $I = \{\phi : \ \exists r \ \phi = r \in T\}$ and for each $\phi \in I$ let ϕ^T be the unique r such that $\phi = r \in T$. For each $\phi \in I$ let M_ϕ be a model of $\phi = \phi^T$ and set

$$\bar{\phi} : I \to \mathbb{R}, \qquad \bar{\phi}(\psi) = \phi^{M_\psi}$$

Let $G = \{\bar{\phi} : \ \phi \in I\}$. Then G is a majorizing linear subspace of $C(I, P(I))$. Let

$$\Lambda_0(\bar{\phi}) = \phi^T.$$

Then Λ_0 is a positive linear functional on G with $\Lambda_0(1) = 1$ and hence by Theorem 2.7 has an extension to positive linear functional Λ on $C(I, \mathcal{P}(I))$ (which is necessarily continuous). By Theorem 2.8, there is a probability charge $\wp$ on $\mathcal{P}(I)$ such that for each $f \in C(I, \mathcal{P}(I))$

$$\Lambda(f) = \int f d\wp.$$

Therefore, if $\phi = r \in T$, we have that

$$r = \Lambda(\bar{\phi}) = \int \phi^{M_i} d\wp = \phi^{\prod_\wp M_i}$$

and hence $\prod_\wp M_i$ (as well as its completion) is a model of T. ☐

To obtain a more general form of linear compactness, we need first to complete T. Note that, by using a suitable ultraproduct over $\mathbb{N}$, one can show that if for each $\epsilon > 0$, the theory $T, \theta \geqslant -\epsilon$ is linearly satisfiable, then $T, \theta \geqslant 0$ is linearly satisfiable. A linearly satisfiable theory T is *complete* if for every sentence ϕ there is a (unique) real number r such that $\phi = r \in T$.

Lemma 2.10. *Every linearly satisfiable theory is contained in a complete one.*

Proof. We may assume T is linearly closed. We first show that for each θ, either $T, \theta \geqslant 0$ or $T, -\theta \geqslant 0$ is linearly satisfiable. Suppose not. Then, there are $r, r', s, s' \geqslant 0$, $\epsilon > 0$ and $\eta \geqslant 0$, $\eta' \geqslant 0$ in T such that every model satisfies $r\eta + s\theta \leqslant -\epsilon$ and $r'\eta' - s'\theta \leqslant -\epsilon$. Since s, s' and hence $r + r'$ can be not zero, this leads easily to a contradiction. Now, use Zorn's lemma to find a maximal linearly satisfiable $\bar{T} \supseteq T$. Given a sentence ϕ, let $\alpha = \sup\{r : r \leqslant \phi \in \bar{T}\}$ and $\beta = \inf\{s : \phi \leqslant s \in \bar{T}\}$. So, by maximality, $\alpha \leqslant \phi \leqslant \beta$ belongs to $\bar{T}$. Also, for each $\epsilon > 0$ we have that $(\alpha + \epsilon) \leqslant \phi \notin \bar{T}$. So, $\phi \leqslant \alpha + \epsilon \in \bar{T}$. Therefore, $\beta \leqslant \alpha + \epsilon$ and since ϵ is arbitrary, $\beta = \alpha$. ☐

Theorem 2.11. (Linear compactness) *Let T be a linearly satisfiable set of conditions of the form $\phi \leqslant \psi$. Then T is satisfiable.*

As a special case, if both $\sigma \leqslant 0$ and $\sigma \geqslant 0$ have model, then $\sigma = 0$ has a model. Note that finite satisfiability implies linear satisfiability so that linear compactness is a stronger result than ordinary compactness. One can also easily verify that if $T \vDash \phi \geqslant 0$ then for each $\epsilon > 0$ there exists a finite $\Delta \subseteq T$ such that $\Delta \vDash \phi \geqslant -\epsilon$. A class $\mathcal{K}$ of $\mathcal{L}$-structures is elementary if there exists a theory T such that $\mathcal{K} = Mod(T)$.

Theorem 2.12. (Axiomatizability) *A class $\mathcal{K}$ is elementary if and only if it is closed under ultramean and elementary equivalence.*

Proof. Let us prove the nontrivial direction. Assume $\mathcal{K}$ is closed under ultramean and elementary equivalence. Let $T = Th(\mathcal{K})$, i.e. the set of conditions holding in every $M \in \mathcal{K}$. We show that $\mathcal{K} = Mod(T)$. Clearly, $\mathcal{K} \vDash T$. Conversely assume M is an arbitrary model of T. Note that for each $\sigma \geqslant 0$ in $Th(M)$, there exists $N \in \mathcal{K}$ such that $N \vDash \sigma \geqslant 0$. Since, otherwise, $\mathcal{K} \vDash \sigma \leqslant -\epsilon$ for some $\epsilon > 0$ and hence $M \vDash \sigma \leqslant -\epsilon$ which is a contradiction. Now, every condition in $Th(M)$ is satisfiable in $\mathcal{K}$. Hence, by the proof of Theorem 2.11, M is elementarily equivalent to an ultramean of members of $\mathcal{K}$. So, by the assumptions, $M \in \mathcal{K}$. □

We end this section by reviewing the situation in many sorted case. Usually, structures in analysis are unbounded and must be considered as many sorted ones. For example, a Banach space X can be considered as a structure $\big((B_n|n \geqslant 1), 0, \{I_{mn}\}_{m<n}, \{\mathsf{r}\cdot\}_{r\in\mathbb{R}}, +, \|\ \|\big)$ where B_n is the ball of radius n in X, $I_{mn} : B_m \to B_n$ is the inclusion map, $+ : B_m^2 \to B_{2m}$ etc. In the many sorted situation, terms are defined in the natural way. If R is a relation symbol of sort $\langle s_1, ..., s_n\rangle$ and $t_1, ..., t_n$ are terms of sorts $s_1, ..., s_n$ respectively, then $R(t_1, ..., t_n)$ is a formula. We can then add formulas, multiply them by scalars or quantify over variables of any sort as usual.

Let $M_i = \{M_{s,i}\}_{s\in S}$, $i \in I$, be a family of many sorted structures in a language $\mathcal{L}$. Let $\wp$ be a maximal probability charge on I. For each $s \in S$, set $M_s = \prod_\wp M_{s,i}$ as a metric space. Then an $\mathcal{L}$-structure is defined on $M = \{M_s\}_{s\in S}$ in the natural way. For example, let R be a relation symbol of sort $(s_1, ..., s_k)$. Then, for $[a^k]$ of sort s_k, $k = 1, ..., n$,

$$R^M([a^1], ..., [a^n]) = \int R^{M_i}(a_i^1, ..., a_i^n)\ d\wp.$$

The reader can then state similar variants of the ultramean, diagonal embedding, compactness and axiomatizability theorems. The last one helps us to prove that the theory of Hilbert spaces is not expressible linearly. Let $I = \{0, 1\}$ with $\wp(0) = \wp(1) = \frac{1}{2}$. If $\mathbb{H}$ is a Hilbert space, then $\prod_\wp \mathbb{H}$ is a Banach space but its norm does not satisfy the parallelogram law. For example, if $\mathbb{H} = \mathbb{R}$ and $a = (1, 0)$, $b = (0, 1)$ then $2 = \|a + b\|^2 + \|a - b\|^2 \neq 2\|a\|^2 + 2\|b\|^2 = 1$.

3. Rudin-Keisler ordering for maximal charges

The Rudin-Keisler ordering is an important ordering defined on ultrafilters. Let $\mathcal{U}$ and $\mathcal{F}$ be ultrafilters on I and J respectively. Write $\mathcal{F} \leq_{RK} \mathcal{U}$ if there exists $f : I \to J$ such that for every $X \subseteq J$, $X \in \mathcal{F}$ if and only if $f^{-1}(X) \in \mathcal{U}$. Then $\leq_{RK}$ is a preordering on the class of all ultrafilters. Set $\mathcal{U} \equiv_{RK} \mathcal{F}$ if $\mathcal{F} \leq_{RK} \mathcal{U}$ and $\mathcal{U} \leq_{RK} \mathcal{F}$. This defines an equivalence relation on ultrafilters. Rudin-Keisler ordering has a close connection to elementary embeddings in first order logic. It can be shows that $\mathcal{F} \leq_{RK} \mathcal{U}$ if and only if for every first order language $\mathcal{L}$ and $\mathcal{L}$-structure M, $M^{\mathcal{F}}$ is elementarily embeddable in $M^{\mathcal{U}}$. Similarly, $\mathcal{U} \equiv_{RK} \mathcal{F}$ if and only if $M^{\mathcal{U}} \simeq M^{\mathcal{F}}$ for all M (see [7], [5]). We extend this ordering to the class of maximal probability charges and investigate linear variants of these theorems.

Let $(I, \mathcal{B}, \wp)$ be a charge space and $f : I \to J$ a function. Then $f(\wp)$ is the charge on J defined by $f(\wp)(X) = \wp(f^{-1}(X))$ for every $X \subseteq J$ for which $f^{-1}(X) \in \mathcal{B}$. To relate maximal probability charges to linear elementary embeddings, we need the change of variables formula for finitely additive measures.

Proposition 3.1. (Change of variables) *Let $(I, \wp)$, (J, ν) be charge spaces and $f : I \to J$ be a function such that $f(\wp) = \nu$. Then for each bounded D-integrable function $u : J \to \mathbb{R}$*

$$\int_J u \; d\nu = \int_I u \circ f \; d\wp.$$

Proof. The claim holds for simple functions and can be extended to other integrable functions as in the classical case (see [9] Prop 15.1). □

Definition 3.2. *Let $\wp$ and ν be maximal probability charges on I and J respectively. We write $\nu \leq \wp$ if there exists $f : I \to J$ such that $\nu = f(\wp)$. We write $\wp \equiv \nu$ if $\wp \leq \nu$ and $\nu \leq \wp$.*

It is clear that $\leq$ coincides with the Rudin-Keisler ordering on ultrafilters. Moreover, $\leq$ is a preordering and $\equiv$ is an equivalence relation on the class of maximal probability charges. If $\wp$ is an ultrafilter and $\nu \leq \wp$ then ν is an ultrafilter. So, ultrafilters form an initial segment in this partial ordering.

Let $\nu \leq \wp$ via the function $f : I \to J$ and M be an $\mathcal{L}$-structure. Define a map $f^* : M^\nu \to M^\wp$ as follows:

$$f^*([a]_\nu) = [a \circ f]_\wp \qquad \forall a \in \prod_J M.$$

Lemma 3.3. *f^* is an elementary embedding.*

Proof. Let $\phi(x_1, ..., x_n)$ be a formula and $a^1, ..., a^n \in \prod_J M$. Then by the ultramean theorem

$$\phi^{M^\nu}([a^1]_\nu, ..., [a^n]_\nu) = \int \phi^M(a^1(j), ..., a^n(j)) \; d\nu.$$

$$= \int \phi^M\big((a^1 \circ f)(i), ..., (a^n \circ f)(i)\big) \; d\wp$$

$$= \phi^{M^\wp}([a^1 \circ f]_\wp, ..., [a^n \circ f]_\wp)$$

$$= \phi^{M^\wp}(f^*([a^1]_\nu), ..., f^*([a^n]_\nu)).$$

□

Let J be a nonempty set and $\mathcal{L}$ be the first order language consisting of relation and function symbols for every relation (i.e. a subset on J^n) and operation on J. Then J is endowed with a first order $\mathcal{L}$-structure in the natural way called the *complete structure* on J. Let us denote the complete structure on J by M. Note that $\mathcal{L}$ is indeed a Lipschitz language. Moreover, every $a \in M$ is the interpretation of a constant symbol (i.e. a 0-ary function symbol). Therefore, every structure linearly elementary equivalent to M contains an elementary substructure isomorphic to M.

Lemma 3.4. *Let $\wp$ and ν be maximal probability charges on I and J respectively. Let M be the complete structure on J. Then for every elementary embedding*

$$\xi : M^\nu \to M^\wp$$

there exists a unique (up to $\wp$-null sets) $f : I \to J$ such that $\nu = f(\wp)$ and $\xi = f^*$.

Proof. Let id be the identity map on J. Then id determines an element $[\mathsf{id}]_\nu$ of M^ν and $\xi([\mathsf{id}]_\nu) \in M^\wp$. Let $f : I \to M$ be such that $[f]_\wp = \xi([\mathsf{id}]_\nu)$. We show that $\nu = f(\wp)$ and $\xi = f^*$.

Let $A \subseteq J$. Then we have

$$\chi_A^{M^\nu}([\mathsf{id}]_\nu) = \int_J \chi_A^M(j)\ d\nu = \nu(A).$$

On the other hand, since ξ is an elementary embedding, we have that

$$\chi_A^{M^\nu}([\mathsf{id}]_\nu) = \chi_A^{M^\wp}(\xi([\mathsf{id}]_\nu)) = \chi_A^{M^\wp}([f]_\wp)$$

$$= \int_I \chi_A^M(f(i))\ d\wp = \int_J \chi_A^M(j)\ d(f(\wp)) = f(\wp)(A).$$

This shows that $\nu = f(\wp)$. Now, we show that $\xi = f^*$. Let $[a]_\nu \in M^\nu$. Then, a is also a unary operation on J and

$$[a]_\nu = [a \circ \mathsf{id}]_\nu = [a^M(j)]_\nu = a^{M^\nu}([\mathsf{id}]_\nu).$$

Again, since ξ is elementary, we have that

$$\xi([a]_\nu) = \xi(a^{M^\nu}([\mathsf{id}]_\nu)) = a^{M^\wp}(\xi([\mathsf{id}]_\nu)) = a^{M^\wp}([f]_\wp)$$

$$= [a^M(f(i))]_\wp = [a \circ f]_\wp = f^*([a]_\nu).$$

Therefore, $\xi = f^*$.

For uniqueness, let $g : I \to J$ be another function such that $\xi = g^*$. Then

$$[f]_\wp = \xi([\mathsf{id}]_\nu) = g^*([\mathsf{id}]_\nu) = [g]_\wp.$$

This means that $f = g$ a.e., i.e. $\wp\{i : |f(i) - g(i)| > \epsilon\} = 0$ for every $\epsilon > 0$. $\square$

It is also clear that f^* is an isomorphism of structures if and only if f induces an isomorphism of the corresponding measure algebras on I and J (denoted by $\wp \simeq \nu$). A consequence of the previous lemmas is that

Theorem 3.5. *Let $\wp$ and ν be maximal probability charges on I and J respectively. Then $\nu \leq \wp$ if and only if M^ν is elementarily embedded in $M^\wp$ for every M.*

Let M be the complete structure on J. Assume $\xi : M^\nu \to M^\wp$ is an isomorphism and $\eta : M^\wp \to M^\nu$ is its inverse. Let $f : I \to J$ and $g : J \to I$ be such that $f^* = \xi$ and $g^* = \eta$. Then $(g \circ f)^* = f^* \circ g^* = (\mathsf{id}_I)^*$ and hence $g \circ f = \mathsf{id}_I$ a.e. Similarly, $f \circ g = \mathsf{id}_J$ a.e. We deduce that if $M^\wp \simeq M^\nu$ for all M, then $\wp \simeq \nu$ and hence $\wp \equiv \nu$. The converse this observation holds for ultrafilters. This is essentially because ultrafilters are rigid, i.e. have no nontrivial automorphism (see [5], [6]). This property does not hold for arbitrary maximal probability charges. Even more, it is quite possible that f is measure preserving without f^* being surjective (consider $x \mapsto 2x$ mod 1 on the unit interval, if we suppose all subsets are measurable). For this reason, a positive answer to the following question seems to be difficult.

Question Is it true that $\wp \equiv \nu$ implies $M^\wp \simeq M^\nu$ for all M?

On the other hand, the existence of ultrafilters (and hence maximal probabilities) depend on the axiom of choice. So, a possible counterexample to this question must be nonconstructive and hence difficult. Moreover, even in the language of pure metric spaces, there is no effective way of verifying two metric spaces are isometric.

Example 3.6. *Let $\wp$ be a maximal probability charge on a set X. For example, $\wp$ may be the Lebesgue measure on the unit interval if we put set theoretical assumptions guaranteeing all subsets of the reals are measurable. Then, $\mathbb{R} \preceq \mathbb{R}^\wp$ (here we assume $\mathbb{R}$ is formalized in the many sorted language of Banach lattices as in* [3]*). Since bounded functions are dense in $L^1(\wp)$, we conclude that $\mathbb{R} \preceq \mathbb{R}^\wp \preceq L^1(\wp)$. Moreover, by Theorem 3.5, if $\nu \leq \wp$ then $\mathbb{R}^\nu \preceq \mathbb{R}^\wp$ and hence $L^1(\nu) \preceq L^1(\wp)$.*

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