OD-characterization of Almost Simple Groups Related to $D_4(4)$

G. R. Rezaeezadeh$^a$*, M. R. Darafsheh$^b$, M. Bibak$^a$, M. Sajjadi$^a$

$^a$Faculty of Mathematical Sciences, Shahrekord University, P.O.Box:115, Shahrekord, Iran.
$^b$School of Pure Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran.

E-mail: rezaeezadeh@sci.sku.ac.ir
E-mail: darafsheh@ut.ac.ir
E-mail: m.bibak62@gmail.com
E-mail: sajadi_mas@yahoo.com

ABSTRACT. Let $G$ be a finite group and $\pi_e(G)$ be the set of orders of all elements in $G$. The set $\pi_e(G)$ determines the prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ whose vertex set is $\pi(G)$. The set of primes dividing the order of $G$, and two vertices $p$ and $q$ are adjacent if and only if $pq \in \pi_e(G)$. The degree $\text{deg}(p)$ of a vertex $p \in \pi(G)$, is the number of edges incident on $p$. Let $\pi(G) = \{p_1, p_2, ..., p_k\}$ with $p_1 < p_2 < ... < p_k$. We define $D(G) := (\text{deg}(p_1), \text{deg}(p_2), ..., \text{deg}(p_k))$, which is called the degree pattern of $G$. The group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $M$ satisfying conditions $|G| = |M|$ and $D(G) = D(M)$. Usually a 1-fold OD-characterizable group is simply called OD-characterizable. In this paper, we classify all finite groups with the same order and degree pattern as an almost simple groups related to $D_4(4)$.

Keywords: Degree pattern, $k$-fold OD-characterizable, Almost simple group.

*Corresponding Author

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Let $G$ be a finite group, $\pi(G)$ the set of all prime divisors of $|G|$ and $\pi_e(G)$ be the set of orders of elements in $G$. The prime graph (or Grunberg-Kegel graph) $\Gamma(G)$ of $G$ is a simple graph with vertex set $\pi(G)$ in which two vertices $p$ and $q$ are joined by an edge (and we write $p \sim q$) if and only if $G$ contains an element of order $pq$ (i.e. $pq \in \pi_e(G)$).

The degree $\deg(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident on $p$. If $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ with $p_1 < p_2 < \ldots < p_k$, then we define $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, which is called the degree pattern of $G$, and leads a following definition.

**Definition 1.1.** The finite group $G$ is called $k$-fold OD-characterizable if there exist exactly $k$ non-isomorphic groups $H$ satisfying conditions $|G| = |H|$ and $D(G) = D(H)$. In particular, a 1-fold OD-characterizable group is simply called OD-characterizable.

The interest in characterizing finite groups by their degree patterns started in [7] by M. R. Darafsheh and et. all, in which the authors proved that the following simple groups are uniquely determined by their order and degree patterns: All sporadic simple groups, the alternating groups $A_p$ with $p$ and $p - 2$ primes and some simple groups of Lie type. Also in a series of articles (see [4, 6, 8, 9, 14, 17]), it was shown that many finite simple groups are OD-characterizable.

Let $A$ and $B$ be two groups then a split extension is denoted by $A : B$. If $L$ is a finite simple group and $\text{Aut}(L) \cong L : A$, then if $B$ is a cyclic subgroup of $A$ of order $n$ we will write $L : n$ for the split extension $L : B$. Moreover if there are more than one subgroup of orders $n$ in $A$, then we will denote them by $L : n_1$, $L : n_2$, etc.

**Definition 1.2.** A group $G$ is said to be an almost simple group related to $S := L : D_4(4)$ if and only if $S \leq \text{Aut}(S)$, for some non-abelian simple group $S$.

In many papers (see [2, 3, 10, 13, 15, 16]), it has been proved, up to now, that many finite almost simple groups are OD-characterizable or $k$-fold OD-characterizable for certain $k \geq 2$.

We denote the socle of $G$ by $	ext{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of $G$. For $p \in \pi(G)$, we denote by $G_p$ and $\text{Syl}_p(G)$ a Sylow $p$-subgroup of $G$ and the set of all Sylow $p$-subgroups of $G$ respectively, all further unexplained notation are standard and can be found in [11].

In this article our main aim is to show the recognizability of the almost simple groups related to $L := D_4(4)$ by degree pattern in the prime graph and
order of the group. In fact, we will prove the following Theorem.

**Main Theorem** Let $M$ be an almost simple group related to $L := D_4(4)$. If $G$ is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, then the following assertions hold:

(a) If $M = L$, then $G \cong L$.
(b) If $M = L : 2_1$, then $G \cong L : 2$ or $L : 2_3$.
(c) If $M = L : 2_2$, then $G \cong L : 2_2$ or $Z_2 \times L$.
(d) If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.
(e) If $M = L : 2_4$, then $G \cong L : 3$ or $Z_3 \times L$.
(f) If $M = L : 2^2$, then $G \cong L : 2^2$ or $L : 2_1$.

2. **Preliminary Results**

It is well-known that $\text{Aut}(D_4(4)) \cong D_4(4) : D_{12}$ where $D_{12}$ denotes the dihedral group of order 12. We remark that $D_{12}$ has the following non-trivial proper subgroups up to conjugacy: three subgroups of order 2, one cyclic subgroup each of order 3 and 6, two subgroups isomorphic to $Z_6 \cong S_3$ and one subgroup of order 4 isomorphic to the Klein’s four group denoted by $2^2$. The field and the duality automorphisms of $D_4(4)$ are denoted by $2_1$ and $2_2$ respectively, and we set $2_3 = 2_1.2_2$ (field+duality which is called the diagonal automorphism). Therefore up to conjugacy we have the following almost simple groups related to $D_4(4)$.

**Lemma 2.1.** If $G$ is an almost simple group related to $L := D_4(4)$, then $G$ is isomorphic to one of the following groups: $L, L : 2_1, L : 2_2, L : 2_3, L : 3, L : 2^2, L : (D_6)_1, L : (D_6)_2, L : 6, L : D_{12}$.

**Lemma 2.2** ([5]). Let $G$ be a Frobenius group with kernel $K$ and complement $H$. Then:

(a) $K$ is a nilpotent group.
(b) $|K| \equiv 1(\text{mod}|H|)$. 

Let $p \geq 5$ be a prime. We denote by $\mathcal{S}_p$, the set of all simple groups with prime divisors at most $p$. Clearly, if $q \leq p$, then $\mathcal{S}_q \subseteq \mathcal{S}_p$. We list all the simple groups in class $\mathcal{S}_{17}$ with their order and the order of their outer automorphisms in TABLE 1, taken from [12].

| $S$ | $|S|$ | $|\text{Out}(S)|$ | $S$ | $|S|$ | $|\text{Out}(S)|$
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<tbody>
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<td>$A_5$</td>
<td>$2^2 \cdot 3 \cdot 5$</td>
<td>2</td>
<td>$G_2(3)$</td>
<td>$2^6 \cdot 3^6 \cdot 7 \cdot 13$</td>
<td>2</td>
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<td>4</td>
<td>$3D_4(2)$</td>
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<td>$L_2(64)$</td>
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<td>$U_3(5)$</td>
<td>$2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 13$</td>
<td>4</td>
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<tr>
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<td>$L_3(9)$</td>
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<td>$S_6(3)$</td>
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<td>6</td>
<td>$S_4(8)$</td>
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<td>12</td>
<td>$O_2^+(4)$</td>
<td>$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13$</td>
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<td>2</td>
<td>$L_3(3)$</td>
<td>$2^9 \cdot 3^{10} \cdot 5 \cdot 11 \cdot 13$</td>
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<td>$A_{13}$</td>
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<td>8</td>
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<td>$SU_2$</td>
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<td>$A_{16}$</td>
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<td>6</td>
<td>$F_{412}$</td>
<td>$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$</td>
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<td>2</td>
<td>$L_2(17)$</td>
<td>$2^4 \cdot 3^2 \cdot 17$</td>
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<td>$M_{11}$</td>
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<td>1</td>
<td>$L_2(16)$</td>
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<td>$M_{12}$</td>
<td>$2^6 \cdot 3^3 \cdot 5 \cdot 11$</td>
<td>2</td>
<td>$S_4(4)$</td>
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<td>4</td>
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<td>$U_3(2)$</td>
<td>$2^{10} \cdot 3^5 \cdot 5 \cdot 11$</td>
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<td>$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 13$</td>
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<td>$M_{22}$</td>
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<td>2</td>
<td>$O_{16}^*(2)$</td>
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<td>$M^{23}$</td>
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<td>2</td>
<td>$S_8(2)$</td>
<td>$2^{16} \cdot 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 17$</td>
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<td>$H_8$</td>
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<td>$U_4(4)$</td>
<td>$2^{12} \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$</td>
<td>4</td>
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<tr>
<td>$A_{12}$</td>
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<td>$2^9 \cdot 3^5 \cdot 5 \cdot 7 \cdot 13 \cdot 17^5$</td>
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<td>$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$</td>
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<td>$L_2(25)$</td>
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<td>$S_4(13)$</td>
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<td>$U_3(4)$</td>
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<td>$L_3(16)$</td>
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<td>$S_4(5)$</td>
<td>$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$</td>
<td>2</td>
<td>$S_6(4)$</td>
<td>$2^{18} \cdot 3^4 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 17$</td>
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<tr>
<td>$L_4(3)$</td>
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<td>2</td>
<td>$O_{27}^*(4)$</td>
<td>$2^{24} \cdot 3^7 \cdot 5 \cdot 7 \cdot 13 \cdot 17^2$</td>
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<td>$F_4(2)$</td>
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<td>$F_4(2)$</td>
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<td>$L_2(13)$</td>
<td>$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$</td>
<td>2</td>
<td>$A_{17}$</td>
<td>$2^{14} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 17$</td>
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<tr>
<td>$L_2(27)$</td>
<td>$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$</td>
<td>6</td>
<td>$A_{18}$</td>
<td>$2^{15} \cdot 3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 17$</td>
<td>2</td>
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</tbody>
</table>
Definition 2.3. A completely reducible group will be called a CR-group. The center of a CR-group is a direct product of the abelian factor in the decomposition. Hence, a CR-group is centerless, that is, has trivial center, if and only if it is a direct product of non-abelian simple groups. The following Lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.3 ([11]). Let \( R \) be a finite centerless CR-group and write \( R = R_1 \times R_2 \times \ldots \times R_k \), where \( R_i \) is a direct product of \( n_i \) isomorphic copies of a simple group \( H_i \), and \( H_i \) and \( H_j \) are not isomorphic if \( i \neq j \). Then \( \text{Aut}(R) = \text{Aut}(R_1) \times \text{Aut}(R_2) \times \ldots \times \text{Aut}(R_k) \cong \text{Aut}(H_i) \wr \mathbb{S}_{n_i} \), where in this wreath product \( \text{Aut}(H_i) \) appears in its right regular representation and the symmetric group \( \mathbb{S}_{n_i} \) in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms \( \text{Out}(R) \cong \text{Out}(R_1) \times \text{Out}(R_2) \times \ldots \times \text{Out}(R_k) \) and \( \text{Out}(R_i) \cong \text{Out}(H_i) \wr \mathbb{S}_{n_i} \).

3. OD-Characterization of Almost Simple Groups Related to \( D_4(4) \)

In this section, we study the problem of characterizing almost simple groups by order and degree pattern. Especially we will focus our attention on almost simple groups related to \( L = D_4(4) \), namely, we will prove the Main Theorem of Sec. 1. We break the proof into a number of separate propositions.

By assumption, we depict all possibilities for the prime graph associated with \( G \) by use of the variables for some vertices in each proposition. Also, we need to know the structure of \( \Gamma(M) \) to determine the possibilities for \( G \) in some proposition, therefore we depict the prime graph of all extension of \( L \) in pages 18 to 20. Note that the set of order elements in each of the following propositions is calculated using Magma.

Proposition 3.1. If \( M = L \), then \( G \cong L \).

Proof. By TABLE 1 \(|L| = 2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2 \cdot \pi_e(L) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 21, 30, 34, 51, 63, 65, 85, 255\} \), so \( D(L) = (3, 4, 4, 1, 1, 3) \). Since \(|G| = |L| \) and \( D(G) = D(L) \), we conclude that the prime graph of \( G \) has following form:

![Figure 3.1](image)

where \( \{a, b\} = \{7, 13\} \).
We will show that $G$ is isomorphic to $L = D_4(4)$. We break up the proof into a several steps.

**Step 1.** Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

First we show that $K$ is a $17^i$-group. Assume the contrary and let $17 \in \pi(K)$. Then $13$ does not divide the order of $K$. Otherwise, we may suppose that $T$ is a Hall $\{13, 17\}$-subgroup of $K$. It is seen that $T$ is a nilpotent subgroup of order $13.17^i$ for $i = 1$ or $2$. Thus, $13.17 \in \pi_c(K) \subseteq \pi_c(G)$, a contradiction. Thus $\{17\} \subseteq \pi(K) \subseteq \pi(G) - \{13\}$. Let $K_{17} \in \text{Syl}_{17}(K)$. By Frattini argument, $G = K\text{N}_G(K_{17})$. Therefore, $\text{N}_G(K_{17})$ contains an element $x$ of order $13$. Since $G$ has no element of order $13.17$, $(x)$ should act fixed point freely on $K_{17}$, that is implying $(x)K_{17}$ is a Frobenius group. By Lemma 2.2(b), $|\langle x \rangle|(|K_{17}| - 1)$. It follows that $13|17^i - 1$ for $i = 1$ or $2$, which is a contradiction.

Next, we show that $K$ is a $p^i$-group for $p \in \{a, b\}$. Let $p||K|$ and $K_p \in \text{Syl}_p(K)$. Now by Frattini argument, $G = K\text{N}_G(K_p)$, so $17$ must divide the order of $\text{N}_G(K_p)$. Therefore, the normalizer $\text{N}_G(K_p)$ contains an element of order $17$, say $x$. So $(x)K_p$ is a cyclic subgroup of $G$ of order $17.p$, and so $p \sim 17$ in $\Gamma(G)$, which is a contradiction. Therefore $K$ is a $\{2, 3, 5\}$-group. In addition, since $K$ is a proper subgroup of $G$, it follows that $G$ is non-solvable.

**Step 2.** The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group isomorphic to $L := D_4(4)$.

Let $\overline{G} = G/K$. Then $S := \text{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where $P_i$s are finite non-abelian simple groups and $S \leq \frac{G}{K} \leq \text{Aut}(S)$. If we show that $m = 1$, the proof of Step 2 will be completed.

Suppose that $m \geq 2$. In this case, we claim that $13$ does not divide $|S|$. Assume the contrary and let $13 \mid |S|$, on the other hand, $\{2, 3\} \subset \pi(P_i)$ for every $i$ (by TABLE 1), hence $2 \sim 13$ and $3 \sim 13$, which is a contradiction.

Now, by step 1, we observe that $13 \in \pi(\overline{G}) \subseteq \pi(\text{Aut}(S))$. But $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times ... \times \text{Aut}(S_t)$, where the groups $S_j$ are direct products of isomorphic $P_i$'s such that $S = S_1 \times S_2 \times ... \times S_t$. Therefore, for some $j$, $13$ divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in \mathcal{S}_{17}$, it follows that $|\text{Out}(P_i)|$ is not divisible by $13$ (see TABLE 1). Now, by Lemma 2.3, we obtain $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq 13$ and so $2^{26}$ must divide the order of $G$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^\gamma.7.13.17^2$, where $2 \leq \alpha \leq 24$, $1 \leq \beta \leq 5$ and $0 \leq \gamma \leq 4$. Now, using collected results contained in TABLE 1, we deduce that $S \cong D_4(4)$ and by Step 2, $L \leq G/K \leq \text{Aut}(L)$ is completed. As $|G| = |L|$, we deduce $K = 1$, so $G \cong L$ and the proof is completed.

□
Proposition 3.2. If \( M = L : 2_1 \), then \( G \cong L : 2_1 \) or \( L : 2_3 \).

Proof. As \( |L : 2_1| = 2^{25}.3^5.5^4.7.13.17^2 \) and \( \pi_c(L : 2_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 40, 42, 51, 60, 63, 65, 85, 102, 126, 130, 170, 255\} \), then \( D(L : 2_1) = (4, 4, 2, 1, 3) \). Since \( |G| = |L : 2_1| \) and \( D(G) = D(L : 2_1) \), we conclude that there exist several possibilities for \( \Gamma(G) \):

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & 3 & 17 & a \\
7 & a & 17 & 13 & \\
\end{array}
\]

Figure 3.2

where \( \{a, b, c\} = \{2, 3, 5\} \).

Step 1. Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\}\)-group. In particular, \( G \) is non-solvable.

By a similar argument to that in Proposition 3.1, we can obtain this assertion.

Step 2. The quotient \( G/K \) is an almost simple group. In fact, \( S \leq G/K \leq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

The proof is similar to Step 2 of Proposition 3.1.

By TABLE 1 and Step 1, it is evident that \( |S| = 2^\alpha.3^\beta.5.7.13.17^2 \), where \( 2 \leq \alpha \leq 25, 1 \leq \beta \leq 5 \) and \( 0 \leq \gamma \leq 4 \). Now, using collected results contained in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \leq G/K \leq \text{Aut}(L) \). As \( |G| = |L : 2_1| = 2|L| \), we deduce \( |K| = 1 \) or \( 2 \).

If \( |K| = 1 \), then \( G \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \). Obviously, \( G \cong L : 2_1 \) or \( L : 2_3 \) because \( \text{deg}(2) = 5 \) in \( \Gamma(L : 2_2) \) (see page 16).

If \( |K| = 2 \), then \( K \leq Z(G) \) and so \( \text{deg}(2) = 5 \), which is a contradiction. \( \square \)

Proposition 3.3. If \( M = L : 2_2 \), then \( G \cong L : 2_2 \) or \( \mathbb{Z}_2 \times L \).

Proof. As \( |L : 2_2| = 2^{25}.3^5.5^4.7.13.17^2 \) and \( \pi_c(L : 2_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 40, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\} \), then \( D(L : 2_2) = (5, 4, 4, 2, 2, 3) \). By assumption \( |G| = |L : 2_2| \) and \( D(G) = D(L : 2_2) \), so the prime graph of \( G \) has following form:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & 5 \\
3 & 5 & 2 \\
\end{array}
\]

Figure 3.3

where \( \{a, b\} = \{7, 13\} \).
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable. By similar arguments as in the proof of Step 1 in Proposition 3.1, we conclude that $K$ is a $\{2, 3, 5\}$-group and $G$ is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \subseteq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Let $G/K = \frac{G}{K}$. Then $S := \text{Soc}(G/K)$, $S = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$'s are finite non-abelian simple groups and $S \leq \frac{G}{K} \subseteq \text{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. We claim $a$ does not divide $|S|$. Assume the contrary and let $a \mid |S|$, we conclude that a just divide the order of one of the simple groups $P_i$'s. Without loss of generality, we assume that $a \mid |P_1|$. Then the rest of the $P_i$'s must be $\{2, 3\}$-group (because only 2 and 3 are adjacent to a in $\Gamma(G)$), this is a contradiction because $P_i$'s are finite non-abelian simple groups. Now, by Step 1, we observe that $a \in \pi(G/K) \subseteq \pi(\text{Aut}(S))$. But $\text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \ldots \times \text{Aut}(S_r)$, where the groups $S_j$ are direct products of isomorphic $P_i$'s such that $S = S_1 \times S_2 \times \ldots \times S_r$. Therefore, for some $j$, $a$ divides the order of an automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. Since $P_i \in S_2$, it follows that $|\text{Out}(P_i)|$ is not divisible by a (see TABLE 1), so $a$ does not divide the order of $\text{Out}(P_i)$. Now, by Lemma 2.3, we obtain $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^t$. Therefore, $t \geq a$ and so $3^a$ must divide the order of $G$, which is a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a.3^b.5^c.7.13.19^2$, where $2 \leq a \leq 25$, $1 \leq b \leq 5$ and $0 \leq c \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \cong \frac{G}{K} \cong \text{Aut}(L)$. As $|G| = |L : 2_2| = 2|L|$, we deduce $|K| = 1$ or 2.

If $|K| = 1$, then $G \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ because $|G| = 2|L|$. It is obvious that $G \cong L : 2_2$, because $\text{deg}(13) = 1$ in $\Gamma(L : 2_1)$ and $\Gamma(L : 2_3)$ (see page 17).

If $|K| = 2$, then $G/K \cong L$ and $K \leq Z(G)$. It follows that $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1. But this is a contradiction, so we obtain that $G$ split over $|K|$. Hence $G \cong Z_2 \times L$.

$\square$

**Proposition 3.4.** If $M = L : 2_3$, then $G \cong L : 2_3$ or $L : 2_1$.

**Proof.** As $|L : 2_3| = 2^{25}.3^5.5^4.7.13.17^2$ and $\pi_r(L : 2_3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 51, 63, 65, 85, 255\}$, then $D(L : 2_3) = (4, 4, 4, 2, 1, 3)$. Since $|G| = |L : 2_3|$ and $D(G) = D(L : 2_3)$, we conclude that $\Gamma(G)$ has the following form similarly to Proposition 3.2:
where \( \{a, b, c\} = \{2, 3, 5\} \).

**Step 1.** Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

We can prove this by the similar way to that in Proposition 3.2.

**Step 2.** The quotient \( \overline{G} \) is an almost simple group. In fact, \( S \leq \overline{G} \lesssim \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

By using a similar argument, as in the proof of Proposition 3.2, we can verify that \( \overline{G} \) is an almost simple group.

By TABLE 1 and Step 1, it is evident that \( |S| = 2^\alpha 3^\beta 5^\gamma 7^\beta 13^\alpha \), where \( 2 \leq \alpha \leq 25, 1 \leq \beta \leq 5, 0 \leq \gamma \leq 4 \). Now, using collected results contained in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \triangleleft \overline{G} \lesssim \text{Aut}(L) \). As \( |G| = |L : 2_3| = 2|L| \), we deduce \( |K| = 1 \) or 2.

If \( |K| = 1 \), then \( G \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \) because \( |G| = 2|L| \). Obviously, \( G \cong L : 2_3 \) or \( L : 2_1 \), because \( \text{deg}(2) = 5 \) in \( \Gamma(L : 2_2) \) (see page 16).

If \( |K| = 2 \), then \( K \leq Z(G) \) and so \( \text{deg}(2) = 5 \), which is a contradiction. \( \square \)

**Proposition 3.5.** If \( M = L : 3 \), then \( G \cong L : 3 \) or \( \mathbb{Z}_3 \times L \).

**Proof.** As \( |L : 3| = 2^{24} 3^6 5^4 7 13 17^2 \) and \( \pi_e(L : 3) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 18, 20, 21, 24, 30, 34, 39, 45, 51, 63, 65, 85, 255\} \), then \( D(L : 3) = (3, 5, 4, 1, 2, 3) \). Since \( |G| = |L : 3| \) and \( D(G) = D(L : 3) \), we conclude that \( \Gamma(G) \) has the following form (like \( \Gamma(L : 3) \)):

\[
\begin{array}{c}
3 \\
\end{array}
\begin{array}{c}
5 \\
\end{array}
\begin{array}{c}
2 \\
\end{array}
\begin{array}{c}
7 \\
\end{array}
\begin{array}{c}
13 \\
\end{array}
\begin{array}{c}
17 \\
\end{array}
\end{array}
\]

**Figure 3.5**

**Step 1.** Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3\} \)-group. In particular, \( G \) is non-solvable.

First, we show that \( K \) is a \( p \)-group for \( p = 7, 13 \) and 17. Since the proof is quite similar to the proof of Step 1 in Proposition 3.1, so we avoid here full explanation of all details.
Next we consider \( K \) is a 5-prime group. Assume the contrary, \( 5 \in \pi_e(K) \). Let \( K_5 \in \text{Syl}_5(K) \). By Frattini argument, \( G = Kn_G(K_5) \). Therefore, \( N_G(K_5) \) has an element \( x \) of order 7. Since \( G \) has no element of order 5, \( \langle x \rangle \) should act fixed point freely on \( K_5 \), implying \( \langle x \rangle K_5 \) is a Frobenius group. By Lemma 2.2(b), \( |\langle x \rangle||(|K_5| - 1) \), which is impossible. Therefore \( K \) is a \( \{2, 3\} \)-group. In addition since \( K \) is a proper subgroup of \( G \), then \( G \) is non-solvable and the proof of this step is completed.

**Step 2.** The quotient \( G/K \) is an almost simple group. In fact, \( S \leq G/K \cong \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group.

In a similar way as in the proof of Step 2 in Proposition 3.1, we conclude that \( G/K \) is an almost simple group.

By Table 1 and Step 1, it is evident that \( |S| = 2^{\alpha}3^25^47.13.17^2 \), where \( 2 \leq \alpha \leq 24 \) and \( 1 \leq \beta \leq 6 \). Now, using collected results contained in Table 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \leq G/K \cong \text{Aut}(L) \). As \( |G| = |L : 3| = 3|L| \), we deduce \( |K| = 1 \) or \( 3 \).

If \( |K| = 1 \), then \( G \cong L : 3 \).

If \( |K| = 3 \), then \( G/K \cong L \). In this case we have \( G/C_G(K) \cong \text{Aut}(K) \cong \mathbb{Z}_2 \). Thus \( |G/C_G(K)| = 1 \) or \( 2 \). If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \), that is, \( G \) is a central extension of \( K \) by \( L \). If \( G \) is a non-split extension of \( K \) by \( L \), then \( |K| \) must divide the Schur multiplier of \( L \), which is 1. But this is a contradiction, so we obtain that \( G \) split over \( K \). Hence \( G \cong \mathbb{Z}_4 \times L \). If \( |G/C_G(K)| = 2 \), then \( K \cong C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L \), which is a contradiction since \( L \) is simple.

**Proposition 3.6.** If \( M = L : 2^2 \), then \( G \cong L : 2^2 \), \( \mathbb{Z}_2 \times (L : 2_1) \), \( \mathbb{Z}_2 \times (L : 2_2) \), \( \mathbb{Z}_2 \times (L : 2_3) \), \( \mathbb{Z}_2 \times L \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times L \).

**Proof.** As \( |L : 2^2| = 2^{10}3^55^47.13.17^2 \) and \( \pi_e (L : 2^2) = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 42, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255 \} \), then \( D(L : 2^2) = \{ 5, 4, 4, 2, 2, 3 \} \). Since \( |G| = |L : 2^2| \) and \( D(G) = D(L : 2^2) \), so the prime graph of \( G \) has following form similarly to Proposition 3.3:

![Figure 3.6](image)

where \( \{ a, b \} = \{ 7, 13 \} \).
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3, 5\}$-group. In particular, $G$ is non-solvable.

According to Step 1 in Proposition 3.3, we have $K$ is a $\{2, 3, 5\}$-group and $G$ is non-solvable.

Step 2. The quotient $G/K$ is an almost simple group. In fact, $S \leq G/K \lesssim \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

We can prove this by the similar argument in Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a 3^b 5^c 7^d 13^{e} 17^f$, where $2 \leq a \leq 6$, $1 \leq b \leq 5$ and $0 \leq c \leq 4$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_{4}(4)$ and by Step 2, $L \leq G/K \lesssim \text{Aut}(L)$. As $|G| = |L : 2^2| = 4|L|$, we deduce $|K| = 1, 2$ or 4.

If $|K| = 1$, then $G \cong L : 2^2$.

If $|K| = 2$, then $K \leq Z(G)$. In this case $G$ is a central extension of $\mathbb{Z}_2$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If $G$ splits over $K$ then $G \cong \mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$ or $\mathbb{Z}_2 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2_1$, $L : 2_2$ and $L : 2_3$, which is impossible.

If $|K| = 4$, then $G/K \cong L$. In this case we have $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$ or $S_3$. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L$. If $G$ is a non-split extension of $K$ by $L$, then $|K|$ must divide the Schur multiplier of $L$, which is 1, but this is a contradiction. Therefore $G$ splits over $K$. Hence $G \cong K \times L$. So we have $G \cong \mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ because $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $|G/C_G(K)| = 2$, 3 or 6, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L$. Which is a contradiction, since $L$ is simple.

Proposition 3.7. If $M = L : (D_{6})_1$, then $G \cong L : (D_{6})_1$, $L : 6$, $\mathbb{Z}_3 \times (L : 2_1)$, $\mathbb{Z}_3 \times (L : 2_2)$ or $(\mathbb{Z}_3 \times L).\mathbb{Z}_2$.

Proof. As $|L : (D_{6})_1| = 2^{25} 3^6 5^4 7^3 13^2 17^2$ and $\pi_e(L : (D_{6})_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 21, 24, 30, 34, 39, 42, 45, 51, 60, 63, 65, 85, 255\}$, then $D(L : (D_{6})_1) = (4, 5, 4, 2, 2, 3)$. Since $|G| = |L : (D_{6})_1|$ and $D(G) = D(L : (D_{6})_1)$, we conclude that there exist several possibilities for $\Gamma(G)$:

![Figure 3.7](image-url)
where \( \{a, b\} = \{7, 13\} \).

**Step 1.** Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \( \{2, 3, 5\} \)-group. In particular, \( G \) is non-solvable.

By the similar argument to that in Step 1 in Proposition 3.1, we can obtain this assertion.

**Step 2.** The quotient \( \frac{G}{K} \) is an almost simple group. In fact, \( S \leq \frac{G}{K} \), where \( S \) is a finite non-abelian simple group.

The proof is similar to Step 2 in Proposition 3.3.

By TABLE 1 and Step 1, it is evident that \( |S| = 2^a \cdot 3^b \cdot 5^c \cdot 7 \cdot 11 \cdot 17^2 \), where \( 2 \leq a \leq 25, 1 \leq b \leq 6 \) and \( 0 \leq c \leq 4 \). Now, using collected results contained in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \leq \frac{G}{K} \cong \text{Aut}(L) \).

As \( |G| = |L : D_6|_1 = 6|L| \), we deduce \( |K| = 1, 2, 3 \) or 6.

If \( |K| = 1 \), then \( G \cong L : (D_6)_1, L : (D_6)_2 \) or \( L : 6 \) because \( |G| = 6|L| \).

Obviously, \( G \cong L : (D_6)_1 \) or \( L : 6 \) because \( \text{deg}(2) = 5 \) in \( \Gamma(L : (D_6)_2) \).

If \( |K| = 2 \), then \( K \leq Z(G) \) and so \( \text{deg}(2) = 5 \), which is a contradiction (see page 18).

If \( |K| = 3 \), then \( G/K \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \). But \( G/C_G(K) \not\cong \text{Aut}(K) \cong Z_2 \). Thus \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \), that is, \( G \) is a central extension of \( K \) by \( L : 2_1, L : 2_2 \) or \( L : 2_3 \). If \( G \) splits over \( K \), then \( G \cong Z_3 \times (L : 2_1) \) or \( Z_3 \times (L : 2_3) \) because in \( \Gamma(Z_3 \times (L : 2_2)) \) the degree of 2 is 5. Otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L : 2_1, L : 2_2 \) and \( L : 2_3 \), which is impossible. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \), we obtain \( C_G(K)/K \cong L \). Since \( K \leq Z(C_G(K)) \), \( C_G(K) \) is a central extension of \( K \) by \( L \). If \( C_G(K) \) splits over \( K \), then \( C_G(K) \cong Z_3 \times L \), otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L \), which is impossible. Therefore, \( G \cong (Z_3 \times L).Z_2 \).

If \( |K| = 6 \), then \( G/K \cong L \) and \( K \cong Z_6 \) or \( D_6 \).

If \( K \cong Z_6 \), then \( G/C_G(K) \cong Z_2 \) and so \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \). It follows that \( \text{deg}(2) = 5 \), a contradiction. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L \), which is a contradiction because \( L \) is simple.

If \( K \cong D_6 \), then \( K \cap C_G(K) = 1 \) and \( G/C_G(K) \not\cong D_6 \). Thus \( C_G(K) \neq 1 \). Hence, \( 1 \neq C_G(K) \cong C_G(K)/K \leq G/K \cong L \). It follows that \( L \cong G/K \cong C_G(K) \) because \( L \) is simple. Therefore, \( G \cong D_6 \times L \), which implies that \( \text{deg}(2) = 5 \), a contradiction.

**Proposition 3.8.** If \( M = L : (D_6)_2 \), then \( G \cong L : (D_6)_2, Z_2 \times (L : 3), Z_3 \times (L : 2_2), (Z_3 \times L).Z_2, Z_6 \times L \) or \( S_3 \times L \).
Proof. As $|L : (D_6)_2| = 2^{25}.3^6.5^4.7.13.17^2$ and $\pi_e(L : (D_6)_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255\}$, then $D(L : (D_6)_2) = (5, 5, 4, 2, 3, 3)$. Since $|G| = |L : (D_6)_2|$ and $D(G) = D(L : (D_6)_2)$, we conclude that $\Gamma(G)$ has the following form (like $\Gamma(L : (D_6)_2)$):

![Figure 3.8](image)

Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

Let $G = \frac{G}{K}$. Then $S := \text{Soc}(G)$, $S = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$’s are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \text{Aut}(S)$. We are going to prove that $m = 1$ and $S = P_1$. Suppose that $m \geq 2$. By the same argument in Step 2 of Proposition 3.3 and considering 7 instead of a, we get a contradiction. Therefore $m = 1$ and $S = P_1$.

By TABLE 1 and Step 1, it is evident that $|S| = 2^\alpha.3^\beta.5^4.7.13.17^2$, where $2 \leq \alpha \leq 25$ and $1 \leq \beta \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \lesssim \text{Aut}(L)$. As $|G| = |L : (D_6)_2| = 6|L|$, we deduce $|K| = 1, 2, 3$ or 6.

If $|K| = 1$, then $G \cong L : (D_6)_1, L : (D_6)_2$ or $L : 6$ because $|G| = 6|L|$. Obviously $G \cong L : (D_6)_2$ because in $\Gamma(L : (D_6)_1)$ and $\Gamma(L : 6)$, we have $deg(13) = 2$(see page 17).

If $|K| = 2$, then $K \leq Z(G)$ and $G/K \cong L : 3$. Hence $G$ is a central extension of $K$ by $L : 3$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_2 \times (L : 3)$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 3$, which is impossible.

If $|K| = 3$, then $G/K \cong L : 2_1, L : 2_2$ or $L : 2_3$. But $G/C_G(K) \lesssim \text{Aut}(K) \cong \mathbb{Z}_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2_1, L : 2_2$ or $L : 2_3$. If $G$ splits over $K$, then only $G \cong \mathbb{Z}_3 \times (L : 2)$ because $2 \approx 13$ in $\Gamma(\mathbb{Z}_3 \times (L : 2))$ and $\Gamma(\mathbb{Z}_3 \times (L : 2))$. Otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L : 2_1, L : 2_2$ and $L : 2_3$, which is impossible. If
If \(|G/C_G(K)| = 2\), then \(K < C_G(K)\) and \(1 \neq C_G(K)/K \leq G/K \cong L: 2_1, L: 2_2\) or \(L: 2_3\); we obtain \(C_G(K)/K \cong L: 21, L: 2\). Since \(K \leq Z(C_G(K))\), \(C_G(K)\) is a central extension of \(K\) by \(L\). If \(C_G(K)\) splits over \(K\), then \(C_G(K) \cong \mathbb{Z}_3 \times L\), otherwise we get a contradiction because \(|K|\) must divide the Schure multiplier of \(L\), which is impossible. Therefore, \(G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2\).

If \(|K| = 6\), then \(G/K \cong L\) and \(K \cong \mathbb{Z}_6\) or \(D_6\). If \(K \cong \mathbb{Z}_6\), then \(G/C_G(K) \cong \mathbb{Z}_2\) and so \(|G/C_G(K)| = 1\) or \(2\). If \(|G/C_G(K)| = 1\), then \(K \leq Z(G)\) and \(G/K \cong L\). Therefore \(G\) is a central extension of \(K\) by \(L\). If \(G\) is a non-split extension of \(K\) by \(L\), then \(|K|\) must divide the Schure multiplier of \(L\), which is 1. But this is a contradiction. So we obtain that \(G\) splits over \(K\). Hence \(G \cong \mathbb{Z}_6 \times L\). If \(|G/C_G(K)| = 2\), then \(K < C_G(K)\) and \(1 \neq C_G(K)/K \leq G/K \cong L\), which is a contradiction because \(L\) is simple. If \(K \cong D_6\), then \(K \cap C_G(K) = 1\) and \(G/C_G(K) \cong D_6\). Thus \(C_G(K) \neq 1\). Hence, \(1 \neq C_G(K) \cong C_G(K)/K \leq G/K \cong L\). It follows that \(L \cong G/K \cong C_G(K)\) because \(L\) is simple. Therefore \(G \cong D_6 \times L\).

**Proposition 3.9.** If \(M = L: 6\), then \(G \cong L: 6, L: (D_6)_1, \mathbb{Z}_3 \times (L: 2_1), \mathbb{Z}_3 \times (L: 2_3)\) or \((\mathbb{Z}_3 \times L).\mathbb{Z}_2\).

**Proof.** As \(|L: 6| = 2^{25}.3^6.5^4.7.13.17^2\) and \(\pi_e(L: 6) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 30, 34, 36, 39, 42, 45, 48, 51, 63, 65, 85, 255\}\), then \(D(L: 6) = (4, 5, 4, 2, 2, 3)\). Since \(|G| = |L: 6|\) and \(D(G) = D(L: 6)\), there exist several possibilities for \(\Gamma(G)\) similarly to Proposition 3.7.

![Figure 3.9](image_url)

where \(\{a, b\} = \{7, 13\}\).

**Step 1.** Let \(K\) be the maximal normal solvable subgroup of \(G\). Then \(K\) is a \(\{2, 3, 5\}\)-group. In particular, \(G\) is non-solvable.

The proof is similar to that in Proposition 3.3.

**Step 2.** The quotient \(\frac{G}{K}\) is an almost simple group. In fact, \(S \leq \frac{G}{K} \cong \text{Aut}(S)\), where \(S\) is a finite non-abelian simple group.

Again we refer to Step 2 of proposition 3.3 to get the proof.

By TABLE 1 and Step 1, it is evident that \(|S| = 2^a.3^b.5^c.7.13.17^2\), where \(2 \leq a \leq 25, 1 \leq b \leq 6\) and \(0 \leq c \leq 4\). Now, using collected results contained
in TABLE 1, we conclude that \( S \cong D_4(4) \) and by Step 2, \( L \leq \frac{G}{K} \leq \text{Aut}(L) \). As \( |G| = |L : 6| = 6|L| \), we deduce \( |K| = 1, 2, 3 \) or 6.

If \( |K| = 1 \), then \( G \cong L : 6, L : (D_6)_1 \) or \( L : (D_6)_2 \) because \( |G| = 6|L| \). Obviously, \( G \cong L : 6 \) or \( L : (D_6)_1 \) because \( \text{deg}(2) = 5 \) in \( \Gamma(L : (D_6)_2) \) (see page 18).

If \( |K| = 2 \), then \( K \leq Z(G) \) and so \( \text{deg}(2) = 5 \), which is a contradiction.

If \( |K| = 3 \), then \( G/K \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \). But \( G/C_G(K) \leq \text{Aut}(K) \cong \mathbb{Z}_2 \). Thus \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \), that is, \( G \) is a central extension of \( K \) by \( L : 2_1, L : 2_2 \) or \( L : 2_3 \). If \( G \) splits over \( K \), then \( G \cong \mathbb{Z}_3 \times (L : 2_1) \) or \( \mathbb{Z}_3 \times (L : 2_3) \) because in \( \Gamma(\mathbb{Z}_3 \times (L : 2_2)) \) the degree of 2 is 5. Otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L : 2_1, L : 2_2 \) and \( L : 2_3 \), which is impossible. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L : 2_1, L : 2_2 \) or \( L : 2_3 \), we obtain \( C_G(K)/K \cong L \). Since \( K \leq Z(C_G(K)) \), \( C_G(K) \) is a central extension of \( K \) by \( L \). If \( C_G(K) \) splits over \( K \), then \( C_G(K) \cong \mathbb{Z}_3 \times L \), otherwise we get a contradiction because \( |K| \) must divide the Schur multiplier of \( L \), which is impossible. Therefore, \( G \cong (\mathbb{Z}_3 \times L).\mathbb{Z}_2 \).

If \( |K| = 6 \), then \( G/K \cong L \) and \( K \cong \mathbb{Z}_6 \) or \( D_6 \). If \( K \cong \mathbb{Z}_6 \), then \( G/C_G(K) \cong \mathbb{Z}_2 \) and so \( |G/C_G(K)| = 1 \) or 2. If \( |G/C_G(K)| = 1 \), then \( K \leq Z(G) \). It follows that \( \text{deg}(2) = 5 \), a contradiction. If \( |G/C_G(K)| = 2 \), then \( K < C_G(K) \) and \( 1 \neq C_G(K)/K \leq G/K \cong L \), which is a contradiction because \( L \) is simple. If \( K \cong D_6 \), then \( K \cap C_G(K) = 1 \) and \( G/C_G(K) \leq D_6 \). Thus \( C_G(K) \neq 1 \). Hence, \( 1 \neq C_G(K) \cong (C_G(K)/K/K \leq G/K \cong L \). It follows that \( L \cong G/K \cong C_G(K) \) because \( L \) is simple. Therefore, \( G \cong D_6 \times L \), which implies that \( \text{deg}(2) = 5 \), a contradiction.

\( \square \)

**Proposition 3.10.** If \( M = L : D_{12} \), then \( G \cong L : D_{12}, Z_2 \times (L : (D_6)_1), Z_2 \times (L : (D_6)_2), Z_2 \times (L : 6), Z_3 \times (L : 2^2), (Z_3 \times (L : 2_1)).Z_2, (Z_3 \times (L : 2_3)).Z_2, Z_4 \times (L : 3), (Z_2 \times Z_2) \times (L : 3), (Z_4 \times L).Z_3, ((Z_3 \times Z_2) \times L).Z_3, Z_6 \times (L : 2_1), Z_6 \times (L : 2_2), Z_6 \times (L : 2_3), (Z_6 \times Z_6).Z_2, S_3 \times (L : 2_1), S_3 \times (L : 2_2), S_3 \times (L : 2_3), Z_{12} \times L, (Z_2 \times Z_6) \times L, D_{12} \times L, (Z_2 \times L).D_6, A_4 \times L, L.A_4 \) or \( T \times L \).

**Proof.** As \( |L : D_{12}| = 2^{26} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2 \) and \( \pi_e(L : (D_{12})) = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 20, 21, 24, 26, 30, 34, 39, 40, 42, 45, 51, 60, 63, 65, 68, 85, 102, 126, 130, 170, 255 \} \), then \( D(L : D_{12}) = (5, 5, 4, 2, 3, 3) \). Since \( |G| = |L : D_{12}| \) and \( D(G) = D(L : D_{12}) \), we conclude that \( \text{Aut}(G) \) has the following form (like \( \Gamma(L : D_{12}) \)):...
Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{2, 3\}$-group. In particular, $G$ is non-solvable.

The proof is similar to Step 1 in Proposition 3.5.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \precsim \text{Aut}(S)$, where $S$ is a finite non-abelian simple group.

To get the proof, follow the way in the proof of Step 2 in Proposition 3.5.

By TABLE 1 and Step 1, it is evident that $|S| = 2^a \cdot 3^b \cdot 5^c \cdot 7 \cdot 13^d \cdot 17^e$, where $2 \leq a \leq 26$ and $1 \leq b \leq 6$. Now, using collected results contained in TABLE 1, we conclude that $S \cong D_4(4)$ and by Step 2, $L \leq \frac{G}{K} \precsim \text{Aut}(L)$. As $|G| = |L : D_{12}| = 12|L|$, we deduce $|K| = 1, 2, 3, 4, 6$ or 12.

If $|K| = 1$, then $G \cong L : D_{12}$.

If $|K| = 2$, then $G/K \cong L : (D_6)_1 \times (D_6)_2$ or $L : 6$ and $K \leq Z(G)$. It follows that $G$ is a central extension of $K$ by $L : (D_6)_1 \times (D_6)_2$ or $L : 6$. If $G$ splits over $K$, then $G \cong L \times (L : (D_6)_1 \times (D_6)_2)$ or $L \times (L : 6)$.

Otherwise $G \cong L \times (L : (D_6)_1 \times (D_6)_2)$.

If $|K| = 3$, then $G/K \cong L : 2^3$. But $G/C_G(K) \precsim \text{Aut}(K) \cong Z_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 2^2$. If $G$ splits over $K$, then $G \cong L \times (L : 2^2)$. Otherwise we get a contradiction because $[K]$ must divide the Schure multiplier of $L : 2^2$, which is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \precsim G/K \cong L : 2^2$, and we obtain $C_G(K)/K \cong L : 2_1 \times 2_2$ or $L : 2_3$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L : 2_1 \times 2_2$ or $L : 2_3$. Thus $C_G(K) \cong Z_3 \times (L : 2_1)$, $Z_3 \times (L : 2_2)$ or $Z_3 \times (L : 2_3)$, otherwise we get a contradiction because 3 must divide the Schure multiplier of $L : 2_1$, $L : 2_2$ or $L : 2_3$, which is impossible. Therefore, $G \cong (Z_3 \times (L : 2_1))Z_2$, $(Z_3 \times (L : 2_2))Z_2$ or $(Z_3 \times (L : 2_3))Z_2$.

If $|K| = 4$, then $G/K \cong L : 3$ and $K \cong Z_4$ or $Z_2 \times Z_2$. In this case we have $G/C_G(K) \precsim \text{Aut}(K) \cong Z_2$ or $S_3$. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is, $G$ is a central extension of $K$ by $L : 3$. If $G$ splits over $K$ by $L : 3$, then $G \cong Z_4 \times (L : 3)$ or $(Z_2 \times Z_2) \times (L : 3)$. Otherwise we get a contradiction because $[K]$ must divide the Schure multiplier of $L : 3$, which is impossible. If $|G/C_G(K)| \neq 1$, since $|G/C_G(K)| = 2, 3$ or 6, it follows that $K < C_G(K)$. As $L$ is simple, we conclude that $1 \neq C_G(K)/K$ must...
be an extension of $L$. Hence $|G/C_G(K)| = 3$ and therefore $C_G(K)/K \cong L$. Now, since $K \leq Z(C_G(K))$, we conclude that $C_G(K)$ is a central extension of $K$ by $L$. Thus $C_G(K) \cong \mathbb{Z}_4 \times L$, or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$, otherwise $|K|$ must divide the Schur multiplier of $L$, which is 1 and it is impossible. Therefore, $G \cong (\mathbb{Z}_4 \times L).\mathbb{Z}_3$ or $((\mathbb{Z}_2 \times \mathbb{Z}_2) \times L).\mathbb{Z}_3$.

If $|K| = 6$, then $G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ and $K \cong \mathbb{Z}_6$ or $D_6$. If $K \cong \mathbb{Z}_6$, then $G/C_G(K) \cong \mathbb{Z}_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is $G$ is a central extension of $\mathbb{Z}_6$ by $L : 2_1$, $L : 2_2$ or $L : 2_3$. If $G$ splits over $K$, we obtain $G \cong \mathbb{Z}_6 \times (L : 2_1)$, $\mathbb{Z}_6 \times (L : 2_2)$ or $\mathbb{Z}_6 \times (L : 2_3)$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is 1 and it is impossible. If $|G/C_G(K)| = 2$, then $K < C_G(K)$ and $1 \neq C_G(K)/K \leq G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$, and we obtain $C_G(K)/K \cong L$. Since $K \leq Z(C_G(K))$, $C_G(K)$ is a central extension of $K$ by $L$. Thus $C_G(K) \cong \mathbb{Z}_6 \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$. Therefore $G \cong (\mathbb{Z}_6 \times L).\mathbb{Z}_2$. If $K \cong D_6$, then $G/C_G(K) \cong D_6$ and so $|G/C_G(K)| = 1, 2, 3$ or 6. If $|G/C_G(K)| = 1$, then $K \leq Z(G)$, that is a contradiction. If $|G/C_G(K)| = 2$, then we have $|KCC_G(K)| = 6.\frac{|G|}{2} = 3|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. If $|G/C_G(K)| = 3$, then we have $|KCC_G(K)| = 6.\frac{|G|}{3} = 2|G|$ because $K \cap C_G(K) = 1$, which is a contradiction. Hence, $1 \neq C_G(K) \not\cong G/K \leq G/K \cong L : 2_1$, $L : 2_2$ or $L : 2_3$. It follows that $C_G(K) \cong L : 2_1$, $L : 2_2$ or $L : 2_3$ because $L$ is simple. Therefore, $G \cong D_6 \times (L : 2_1)$, $D_6 \times (L : 2_2)$ or $D_6 \times (L : 2_3)$.

Before processing the last case, we recall the following facts.

There exist five non-isomorphic groups of order 12. Two of them are abelian and three are non-abelian. The non-abelian groups are: alternating group $A_4$, dihedral group $D_{12}$ and the dicyclic group $T$ with generators $a$ and $b$, subject to the relations $a^6 = 1$, $a^3 = b^2$ and $b^{-1}ab = a^{-1}$.

If $|K| = 12$, then $G/K \cong L$ and $K \cong \mathbb{Z}_{12}$, $\mathbb{Z}_2 \times \mathbb{Z}_6$, $D_{12}$, $A_4$ or $T$. But $C_G(K)K/K \not\leq G/K \cong L$. If $C_G(K)K/K = 1$, then $C_G(K) \leq K$ and hence $|L| = |G/K|||G/C_G(K)|||\text{Aut}(K)||$. Thus $|L|||\text{Aut}(K)||$, a contradiction. Therefore, $C_G(K)/K \not\cong 1$ and since $L$ is simple group, we conclude that $G = C_G(K)K$ and hence, $G/C_G(K) \cong K/Z(K)$. Now, we should consider the following cases:

If $K \cong \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$, then $G/C_G(K) = 1$. Therefore $K \leq Z(G)$, that is $G$ is a central extension of $\mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$ by $L$. If $G$ splits over $K$, we obtain $G \cong \mathbb{Z}_{12} \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_6) \times L$, otherwise we get a contradiction because $|K|$ must divide the Schur multiplier of $L$, which is 1 and it is impossible.
If $K \cong D_{12}$, then $G = K.L$ and $G/C_{G}(K) \cong D_{6}$. Since $C_{G}(K)/Z(K) \cong G/K \cong L$ and $Z(K) \leq Z(C_{G}(K))$, we conclude that $C_{G}(K)$ is a central extension of $Z(K) \cong \mathbb{Z}_{2}$ by $L$. If $C_{G}(K)$ is a non-split extension, then 2 must divide the Schure multiplier of $L$, which is 1 and it is impossible. Thus $C_{G}(K) \cong \mathbb{Z}_{2} \times L$ and hence, $G$ is a split extension of $K$ by $L$. Now, since $\text{Hom}(L, \text{Aut}(D_{12}))$ is trivial, we have $G \cong D_{12} \times L$.

If $K \cong A_{4}$, then $G/C_{G}(K) \cong A_{4}$. As $G = C_{G}(K)K$, it follows that $C_{G}(K) \cong L$. Therefore $G \cong L \times A_{4}$ or $L.A_{4}$.

If $K \cong T$, then by the similar way in case $K \cong D_{12}$, we can conclude that $G$ is a split extension of $K$ by $L$. Also, since $\text{Hom}(L, \text{Aut}(T))$ is trivial, we have $G \cong T \times L$. \hfill \qed

According to what we said before the proof, here we depict $\Gamma(M)$ by $|M|$ and $\pi_{e}(M)$, where $M$ is an almost simple group related to $L = D_{4}(4)$.
OD-characterization of Almost Simple Groups Related to $D_4(4)$

\[\Gamma(L : 2_3)\]

\[\Gamma(L : 3)\]

\[\Gamma(L : 2^2)\]

\[\Gamma(L : (D_6)_1)\]
4. ACKNOWLEDGMENTS

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OD-characterization of Almost Simple Groups Related to $D_4(4)$

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