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Integrating Goal Programming, Taylor Series, Kuhn-Tucker Conditions, and Penalty Function Approaches to Solve Linear Fractional Bi-level Programming Problems

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Abstract. In this paper, we integrate goal programming (GP), Taylor Series, Kuhn-Tucker conditions and Penalty Function approaches to solve linear fractional bi-level programming (LFBLP)problems. As we know, the Taylor Series is having the property of transforming fractional functions to a polynomial. In the present article by Taylor Series we obtain polynomial objective functions which are equivalent to fractional objective functions. Then on using the Kuhn-Tucker optimality condition of the lower level problem, we transform the linear bilevel programming problem into a corresponding single level programming. The complementary and slackness condition of the lower level problem is appended to the upper level objective with a penalty, that can be reduce to a single objective function. In the other words, suitable transformations can be applied to formulate FBLP problems. Finally a numerical example is given to illustrate the complexity of the procedure to the solution.

Keywords: Bi-level programming, Fractional programming, Taylor Series, Kuhn-Tucker conditions, Goal programming, Penalty function.

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1. Introduction

Bilevel programming problems provide a framework to deal with decision processes involving two decision makers with a hierarchical structure. Both of the leader at the upper level of the hierarchy and the follower at the lower level seek to optimize their individual objective functions and control their own set of decision variables. The hierarchical process means that the leader sets the value of his variables first and then the follower reacts, bearing in mind the selection of the leader. The goal of the leader is to optimize his own objective function but incorporating within the optimization scheme of the reaction of the follower to his course of action. The leader can influence, but can not control, the decisions of the follower. In formal terms, bilevel programming problems are mathematical programs in which a subset of the variables is required to be an optimal solution of another mathematical program. There are many approaches in the literatures towards BLP problems such as [1, 7, 9, 11, 12] There are may algorithms, such as, the Kth best approach [1], Kuhn-Tucker approach [4], complementarity pivot approach [2], penalty function approach [13], which have been proposed for solving linear BLP problems. Fractional programming has received remarkable attention in the literature[10]. Calvetea and Gal [3] considered the linear fractional bilevel programming (LFBP) problem in which both objective functions are linear fractional. A problem of fuzzy production inventory model with resalable returns by using fuzzy trapezoidal number as a parameter is investigated by Nagoorgania and Palaniammalb [8]. By using a generalized parametric vector companion form, the problem of eigenvalue assignment with minimum sensitivity is re-formulated as an unconstrained minimization problem is recently considered by Karbassi and Soltanian in [5].

In this paper, we integrate goal programming (GP), Taylor Series (TS), Kuhn-Tucker conditions (KKT) and Penalty Function (PF) approaches to solve linear fractional Bi-Level Programming problems. The paper is organized as follows; in next section we present the formulation of FBLP; in Section 3 a solution method for solving new problem are described; in Section 4 we present a numerical example in order to show implementation of the method; finnally, conclusion remarks are presented in Section 5.

2. Problem Formulation

In a FBLP problem, each decision-maker tries to optimize its own objective function(s) without considering the objective(s) of the other party, but the decision of each party affects the objective value(s) of the other party as well as the decision space. The general formulation of a fractional bi-level programming problem (FBLPP) is as follows:

$$\min_{x} \quad F(x,y) = \frac{c_{11}x + c_{12}y + \alpha_1}{d_{11}x + d_{12}y + \beta_1}$$

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$$\begin{aligned} &\text{s.t} \\ &\qquad G(x,y) \leqslant 0 \\ &\underset{y}{\text{Min}} \quad f(x,y) = \frac{c_{21}x + c_{22}y + \alpha_2}{d_{21}x + d_{22}y + \beta_2} \\ &\text{s.t} \\ &\qquad g(x,y) \leqslant 0 \end{aligned} \tag{2.1}$$

where $x \in R^{n1}$ and $y \in R^{n2}$. The variables of problem (1) are divided into two classes, namely the upper-level variables x and the lower-level variables y. Similarly, $c_{i1}, d_{i1} \in R^{n1}, c_{i2}, d_{i2} \in R^{n2}$; α_i and β_i, i = 1, 2 are scalars and it is further assumed that the denominators are positive, i.e., $d_{i1}x + d_{i2}y + \beta_i > 0$, i = 1,2 , respectively, while the vector-valued functions $G : R^{n1} \times R^{n2} \longrightarrow R^{m1}$ and $g : R^{n1} \times R^{n2} \longrightarrow R^{m2}$ are called the upper-level and lower-level constraints, respectively. All of the constraints and objective functions may be linear, quadratic, non-linear, fractional, etc. In this paper, we restrict our attention to linear fractional objective functions and linear constraints. The relaxed problem associated with (1) can be stated as:

$$\begin{aligned} &\underset{x,y}{\text{Min}} \quad F(x,y) = \frac{c_{11}x + c_{12}y + \alpha_1}{d_{11}x + d_{12}y + \beta_1} \\ &\text{s.t} \\ &\qquad G(x,y) \leqslant 0 \\ &\qquad g(x,y) \leqslant 0 \end{aligned} \tag{2.2}$$

and its optimal value is a lower bound for the optimal value of F(x, y) in (1). Similarly optimal value of:

$$\begin{aligned} &\underset{x,y}{\text{Min}} \quad f(x,y) = \frac{c_{21}x + c_{22}y + \alpha_2}{d_{21}x + d_{22}y + \beta_2} \\ &\qquad s.t \\ &\qquad g(x,y) \leqslant 0 \end{aligned} \tag{2.3}$$

is also a lower bound for f(x, y) in (1) [12].

3. Solution Method Representation

In this section, Linearization of the objective functions by using a Taylor series approach, formulation of BLP on using a GP approach, KKT conditions and PF approaches to solve FBLP problems will be explained in detail.

3.1. **Linearization of the objective functions on using a Taylor series approach**

In the linear fractional bi-level programming problem (1), the linear fractional objective functions from each levels is converted to a linear polynomial on using Taylor series. The proposed approach can be explained in two steps.

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Step 1. In this step we first maximize the upper level F(x, y) subject to the whole constraints of the upper and lower level to get the optimal solution as (x_1^*, y_1^*) and then we maximize the lower level f(x, y) subject to its own constraints in lower level to get the optimal solution as (x_2^*, y_2^*).

Step 2. Transform objective functions by using first-order Taylor polynomial series.

$$F(x,y) \cong \widehat{F}(x,y) = F(x_1^*, y_1^*) + \Big((x - x_1^*)\frac{\partial F(x_1^*, y_1^*)}{\partial x} + (y - y_1^*)\frac{\partial F(x_1^*, y_1^*)}{\partial y}\Big) \quad (3.1.1)$$

From this method f(x, y) can be easily obtained.

$$f(x,y) \cong \widehat{f}(x,y) = f(x_2^*, y_2^*) + \Big((x - x_2^*)\frac{\partial f(x_2^*, y_2^*)}{\partial x} + (y - y_2^*)\frac{\partial f(x_2^*, y_2^*)}{\partial y}\Big) \quad (3.1.2)$$

In the next stage, the linear fractional objective bi-level problem is converted to a linear objective bi-level problem.

3.2. **Formulation of BLP using a GP approach**

Li in [6] proposed a solution method for solving a goal programming (GP) problem which is described in the following theorem.

Theorem 3.1. *A GP problem minimize $Z = |f(X) - g|$ subject to: $X \in F$ (F is a feasible set) can be liberalized using the following form:*

$$\begin{aligned} & Min \quad f(X) - g + 2\delta \\ & s.t \\ & \quad g - f(X) - \delta \leqslant 0 \\ & \quad \delta \geqslant 0, X \in F \end{aligned} \quad (3.2.1)$$

Proof. (See Ref.[6]) □

Let F^*, f^* be the goal values for F(x, y), f(x, y) respectively, therefore BLP is transformed to the following form:

$$\begin{aligned} & \underset{x}{\text{Min}} \quad |F(x,y) - F^*| \\ & \text{s.t} \\ & \qquad G(x,y) \leqslant 0 \\ & \underset{y}{\text{Min}} \quad |f(x,y) - f^*| \\ & \text{s.t} \\ & \qquad g(x,y) \leqslant 0 \end{aligned} \quad (3.2.2)$$

By using Theorem 1 we have,

$$\begin{aligned} & \underset{x}{\text{Min}} \quad F(x,y) - F^* + 2\delta_F \\ & \text{s.t} \\ & \qquad G(x,y) \leqslant 0 \end{aligned} \quad (3.2.3)$$

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$$
\begin{aligned}
& F^* - F(x,y) - \delta_F \leqslant 0 \\
& \delta_F \geqslant 0 \\
\underset{y}{\text{Min}} \quad & f(x,y) - f^* + 2\delta_f \\
\text{s.t} \quad & \\
& g(x,y) \leqslant 0 \\
& f^* - f(x,y) - \delta_f \leqslant 0 \\
& \delta_f \geqslant 0
\end{aligned}
$$

3.3. **KKT conditions and PF approach to solve BLP problems**

Shi et al. [11] proposed an extended the Kuhn-Tucker approach to deal with linear bi-level problems. In their approach a linear BLP is considered as the following form:

$$\underset{x \in X}{\text{Min}} \quad F(x,y) = c_1 x + d_1 y \tag{3.3.1}$$

s.t

$$A_1 x + B_1 y \leqslant b_1 \tag{3.3.2}$$

$$\underset{y \in Y}{\text{Min}} \quad f(x,y) = c_2 x + d_2 y \tag{3.3.3}$$

s.t

$$A_2 x + B_2 y \leqslant b_2 \tag{3.3.4}$$

where $c_1, c_2 \in R^n, d_1, d_2 \in R^m, b_1 \in R^p, b_2 \in R^q, A_1 \in R^{p\times n}, B_1 \in R^{p\times m}, A_2 \in R^{q\times n}, B_2 \in R^{q\times m}$.

Let $u \in R^p, \nu \in R^q$ and $w \in R^m$ be the dual variables associated with constraints (3.3.2) and (3.3.4) with $y \geqslant 0$, respectively. We now have the following theorem.

Theorem 3.2. *A necessary and sufficient condition that (x^*, y^*) solves the linear BLP problem (3.3.1)-(3.3.4) is that there exist (row) vectors u^*, ν^* and w^* such that $(x^*, y^*, u^*, \nu^*, w^*)$ solves:*

$$\text{Min} \quad F(x,y) = c_1 x + d_1 y \tag{3.3.5}$$

s.t

$$A_1 x + B_1 y \leqslant b_1 \tag{3.3.6}$$

$$A_2 x + B_2 y \leqslant b_2 \tag{3.3.7}$$

$$u B_1 + \nu B_2 - w = -d_2 \tag{3.3.8}$$

$$u(b_1 - A_1 x - B_1 y) + \nu(b_2 - A_2 x - B_2 y) + wy = 0 \tag{3.3.9}$$

$$x \geqslant 0, y \geqslant 0, u \geqslant 0, \nu \geqslant 0, w \geqslant 0 \tag{3.3.10}$$

Proof. (See Ref.[11]) □

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All of the constraints except (3.3.9) are linear.We use following penalty function to transfer (3.3.9) to objective function and convert (3.3.5)-(3.3.10) to a Quadratic programming [7]:

$$\begin{aligned} \text{Min} \quad & F(x,y) = c_1x + d_1y + M(us + \nu r + wy) \\ & \text{s.t} \\ & A_1x + B_1y + s = b_1 \\ & A_2x + B_2y + r = b_2 \\ & uB_1 + \nu B_2 - w = -d_2 \\ & x \geqslant 0, y \geqslant 0, u \geqslant 0, \nu \geqslant 0, w \geqslant 0, s \geqslant 0, r \geqslant 0, with M \geqslant 0 \end{aligned} \tag{3.3.11}$$

(which can be consider as a penalty coefficient)

4. Numerical Example

To demonstrate the proposed procedure for solving FBLP problem, consider the following example:

$$\begin{aligned} \underset{x}{\text{Min}} \quad & F(x,y) = \frac{2x+y}{x+3y} \\ & \text{s.t} \\ & x + 2y \geqslant 3 \\ & 2x - y \leqslant 5 \end{aligned}$$

$$\begin{aligned} \underset{y}{\text{Min}} \quad & f(x,y) = \frac{x+2y}{3x+y+1} \\ & \text{s.t} \\ & -x + 3y \leqslant 4 \\ & 3x + 2y \leqslant 12 \\ & x \geqslant 0, y \geqslant 0 \end{aligned}$$

As we stated in step 1, we are now to maximize the first level with the constraints of both of the upper and lower level simultaneously and then maximize the lower level with the constraints appeared in the lower level to get the optimal solution for both of the levels as (x_1^*, y_1^*) and (x_2^*, y_2^*) respectively as below.

$$\begin{aligned} \text{Max} \quad & F(x,y) = \frac{2x+y}{x+3y} \\ & \text{s.t} \\ & x + 2y \geqslant 3 \\ & 2x - y \leqslant 5 \\ & -x + 3y \leqslant 4 \\ & 3x + 2y \leqslant 12 \\ & x \geqslant 0, y \geqslant 0 \end{aligned} \qquad \begin{aligned} \text{Max} \quad & f(x,y) = \frac{x+2y}{3x+y+1} \\ & \text{s.t} \\ & -x + 3y \leqslant 4 \\ & 3x + 2y \leqslant 12 \\ & x \geqslant 0, y \geqslant 0 \end{aligned}$$

If we solve the above problem by charnes and cooper method then F(2.6, 0.2) and f(0, 1.33) are obtained as an optimal solution. Then objective functions

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are transformed by using first-order Taylor polynomial series to the following form.

$$F(x,y) \cong \widehat{F}(x,y) = F(2.6,0.2) + \Big((x-2.6)\frac{\partial F(2.6,0.2)}{\partial x} + (y-0.2)\frac{\partial F(2.6,0.2)}{\partial y}\Big)$$

$$F(x,y) \cong \widehat{F}(x,y) = 0.1x - 1.27y + 1.68$$

$$f(x,y) \cong \widehat{f}(x,y) = f(0,1.33) + \Big((x-0)\frac{\partial f(0,1.33)}{\partial x} + (y-1.33)\frac{\partial f(0,1.33)}{\partial y}\Big)$$

$$f(x,y) \cong \widehat{f}(x,y) = -1.04x + 0.37y + 0.65$$

If leader and follower select 2 and -1.5 as goal values for their objectives, respectively; according to (3.2.2) we can then transform the problem to:

$$\begin{aligned} \underset{x}{\text{Min}} \quad & |F(x,y) - F^*| = |0.1x - 1.27y - 0.32| \\ \text{s.t} \quad & \\ & x + 2y \geqslant 3 \\ & 2x - y \leqslant 5 \end{aligned}$$

$$\begin{aligned} \underset{y}{\text{Min}} \quad & |f(x,y) - f^*| = |-1.04x + 0.37y + 2.15| \\ \text{s.t} \quad & \\ & -x + 3y \leqslant 4 \\ & 3x + 2y \leqslant 12 \\ & x \geqslant 0, y \geqslant 0 \end{aligned}$$

Now again according to (3.2.3) we can further transform the problem to:

$$\begin{aligned} \underset{x}{\text{Min}} \quad & F(x,y) - F^* + 2\delta_F = 0.1x - 1.27y - 0.32 + 2\delta_F \\ \text{s.t} \quad & \\ & x + 2y \geqslant 3 \\ & 2x - y \leqslant 5 \\ & -0.1x + 1.27y + 0.32 - \delta_F \leqslant 0 \\ & \delta_F \geqslant 0 \end{aligned}$$

$$\begin{aligned} \underset{y}{\text{Min}} \quad & f(x,y) - f^* + 2\delta_f = -1.04x + 0.37y + 2.15 + 2\delta_f \\ \text{s.t} \quad & \\ & -x + 3y \leqslant 4 \\ & 3x + 2y \leqslant 12 \\ & 1.04x - 0.37y - 2.15 - \delta_f \leqslant 0 \\ & \delta_f \geqslant 0 \\ & x \geqslant 0, y \geqslant 0 \end{aligned}$$

Let us rewrite all the inequalities of final problem as follows:
$g_1 = -x - 2y + 3 \leqslant 0; g_2 = 2x - y - 5 \leqslant 0; g_3 = -0.1x + 1.27y + 0.32 - \delta_F \leqslant 0; g_4 = -\delta_F \leqslant 0; g_5 = -x + 3y - 4 \leqslant 0; g_6 = 3x + 2y - 12 \leqslant 0; g_7 = 1.04x - 0.37y - 2.15 - \delta_f \leqslant 0; g_8 = -\delta_f \leqslant 0; g_9 = -x \leqslant 0; g_{10} = -y \leqslant 0.$

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By implementing the extended Kuhn-Tucker approach, the problem has the following form:

$$
\begin{aligned}
\text{Min}\quad & 0.1x - 1.27y - 0.32 + 2\delta_F \\
\text{s.t}\quad & \\
& x + 2y \geqslant 3 \\
& 2x - y \leqslant 5 \\
& 0.1x - 1.27y + \delta_F \geqslant 0.32 \\
& -x + 3y \leqslant 4 \\
& 3x + 2y \leqslant 12 \\
& 1.04x - 0.37y - \delta_f \leqslant 2.15 \\
& -2u_1 - u_2 + 1.27u_3 + 3\nu_1 + 2\nu_2 - 0.37\nu_3 - w = -0.37 \\
& -\nu_3 - \nu_4 = -2 \\
& xu_1 + 2yu_1 - 3u_1 - 2xu_2 + yu_2 + 5u_2 + 0.1xu_3 - 1.27yu_3 \\
& -0.32u_3 + \delta_F u_3 + x\nu_1 - 3y\nu_1 + 4\nu_1 - 3x\nu_2 - 2y\nu_2 + 12\nu_2 \\
& -1.04x\nu_3 + 0.37y\nu_3 + 2.15\nu_3 + \delta_f\nu_3 + \delta_f\nu_4 + wy = 0 \\
& x \geqslant 0, y \geqslant 0, \delta_F \geqslant 0, \delta_f \geqslant 0, w \geqslant 0, u_i \geqslant 0, \nu_i \geqslant 0, \\
& i = 1, 2, 3
\end{aligned}
$$

By using (3.3.11) the above problem will be converted to:

$$
\begin{aligned}
\text{Min}\quad & 0.1x - 1.27y - 0.32 + 2\delta_F + M\Big(\sum_{i=1}^{3} u_i s_i + \sum_{j=1}^{4} \nu_j r_j + wy\Big) \\
\text{s.t}\quad & \\
& s_1 = x + 2y - 3 \\
& s_2 = -2x + y + 5 \\
& s_3 = 0.1x - 1.27y + \delta_F - 0.32 \\
& r_1 = x - 3y + 4 \\
& r_2 = -3x - 2y + 12 \\
& r_3 = -1.04x + 0.37y + \delta_f + 2.15 \\
& r_4 = \delta_f \\
& -2u_1 - u_2 + 1.27u_3 + 3\nu_1 + 2\nu_2 - 0.37\nu_3 - w = -0.37 \\
& -\nu_3 - \nu_4 = -2 \\
& x \geqslant 0, y \geqslant 0, \delta_F \geqslant 0, \delta_f \geqslant 0, w \geqslant 0, u_i \geqslant 0, \nu_i \geqslant 0, \\
& s_i \geqslant 0, r_i \geqslant 0, i = 1, 2, 3, M = 1000
\end{aligned}
$$

solving the above problem by Lingo-11 the results will be obtained as

$$
\begin{aligned}
x &= 2.208, \qquad y = 0.396 \\
F &= 1.42, \qquad f = 0.37
\end{aligned}
$$

This result may be unacceptable for leader, thus he/she should select another goal for his/her objective. proceeding in this way, the follower goal may get

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unachievable; therefore they should change theirs goals in an interactive framework that satisfy both of them. Now let us consider new goals as $F^* = 1.7$ and $f^* = 0.5$ New problem is:

$$\text{Min} \quad 0.1x - 1.27y - 0.02 + 2\delta_F + M\Big(\sum_{i=1}^{3} u_i s_i + \sum_{j=1}^{4} \nu_j r_j + wy\Big)$$

s.t

$$\begin{aligned}
&s_1 = x + 2y - 3\\
&s_2 = -2x + y + 5\\
&s_3 = 0.1x - 1.27y + \delta_F - 0.02\\
&r_1 = x - 3y + 4\\
&r_2 = -3x - 2y + 12\\
&r_3 = -1.04x + 0.37y + \delta_f + 0.15\\
&r_4 = \delta_f\\
&-2u_1 - u_2 + 1.27u_3 + 3\nu_1 + 2\nu_2 - 0.37\nu_3 - w = -0.37\\
&-\nu_3 - \nu_4 = -2\\
&x \geqslant 0, y \geqslant 0, \delta_F \geqslant 0, \delta_f \geqslant 0, w \geqslant 0, u_i \geqslant 0, \nu_i \geqslant 0,\\
&s_i \geqslant 0, r_i \geqslant 0, i = 1, 2, 3, M = 1000
\end{aligned}$$

By solving the above problem by Lingo-11 we will get the results as

$$\begin{aligned}
x = 2.6, \qquad y = 0.2\\
F = 1.69, \qquad f = 0.33
\end{aligned}$$

This is closer to DMs goal value than earlier results.

5. Conclusion

Linear fractional bi-level programming problems has been solved by many investigator in the literature so far. In this article on using GP, TS, KKT conditions and PF approach, although it may seems to be abit complicated , but, each of these technique is used individually to make the problem simpler and more important from practical point of view, since the DM by choosing different goals, can get more appropriate and accurate according to his/her wish.

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