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Oscillation Behavior for an n-Dimensional System of Coupled van der Pol Oscillators with Delays

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Abstract. In this paper, the oscillatory behavior of the solutions for an n-dimensional system of coupled van der Pol oscillators with delays is investigated. We extend the results in the literature from a mathematical point of view. Some sufficient conditions to guarantee the oscillation of the solutions are provided, and computer simulations are given to support the present criteria.

Keywords: Coupled van der Pol oscillator, Delay, Instability, Oscillation.

2000 Mathematics subject classification: 34K11.

1. Introduction

The van der Pol oscillator is known to be a primary model for describing self-excited oscillations observed in various biological, physical, and engineering systems [1-11]. Recently, Sathiyadevi et al. have investigated the following coupled van der Pol oscillators [12]:

$$\begin{cases} x_i'(t) = y_i(t) + \varepsilon_1[\bar{x}(t) - x_i(t)], i = 1, 2, \cdots, n, \\ y_i'(t) = \alpha(1 - x_i^2(t))y_i(t) - x_i(t) - \varepsilon_2[\bar{y}(t) - y_i(t)], \end{cases} \tag{1.1}$$

where $\bar{x}(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t), \bar{y}(t) = \frac{1}{n}\sum_{i=1}^{n} y_i(t)$. For $n = 2$ and $n = 3$, the authors have demonstrated transient chaos and multistability, the competing attractive and repulsive coupling plays a crucial role. Bukh and Anishchenko have studied numerically auto-wave structures in a two-dimensional lattice of

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van der Pol oscillators with 100×100 elements for the cases of local and nonlocal coupling between the individual oscillators [13]:

$$\begin{cases} x'_{i,j}(t) = y_{i,j}(t) + \frac{\sigma}{B_{i,j}} \sum_{m,n}(x_{m,n} - x_{i,j}), \\ y'_{i,j}(t) = \varepsilon(1 - x^2_{i,j})y_{i,j} - w^2 x_{i,j}, \end{cases} \tag{1.2}$$

where $m, n \in \{1, 2, \cdots, 100\}$. The classical spiral and target waves that were observed in modeling and analyzing the dynamics of the heart muscle can be realized in the network. Therefore, the obtained results can be useful in cardiology.

On the other hand, time delays widely exist in various systems in engineering, biology, chemistry, physics, ecology, and so on. Those delays can cause instabilities in original systems and make system analysis harder. For example, time delay comes from the feedback and the reaction time of the actuators in magnetic bearing systems [14], in a control system [15], in an air pollution system [16], and an economic system [17]. Until now, many researchers have investigated the effects of time-delayed van der Pol oscillators [18-29]. Zhang and Gu considered the existence of the Hopf bifurcation for the following system [18]:

$$\begin{cases} x''_1(t) + \varepsilon(x^2_1(t) - 1)x'_1(t) + (1 + \alpha)x_1(t) = \alpha y'_1(t - \tau_2), \\ y''_1(t) + \varepsilon(y^2_1(t) - 1)y'_1(t) + (1 + \alpha)y_1(t) = \alpha x'_1(t - \tau_1). \end{cases} \tag{1.3}$$

The stability and direction of the Hopf bifurcation were determined by using the normal form theory and the center manifold theorem. Zhang et al. discussed the dynamic behavior of the following symmetric system [19]:

$$\begin{cases} x''_1(t) + x_1(t) - \varepsilon(1 - x^2_1(t))x'_1(t) = k[x'_2(t - \tau) - x'_1(t - \tau)] + k[x'_3(t - \tau) - x'_1(t - \tau)], \\ x''_2(t) + x_2(t) - \varepsilon(1 - x^2_2(t))x'_2(t) = k[x'_3(t - \tau) - x'_2(t - \tau)] + k[x'_1(t - \tau) - x'_2(t - \tau)], \\ x''_3(t) + x_3(t) - \varepsilon(1 - x^2_3(t))x'_3(t) = k[x'_1(t - \tau) - x'_3(t - \tau)] + k[x'_2(t - \tau) - x'_3(t - \tau)]. \end{cases} \tag{1.4}$$

The dynamic behavior of the system (1.4) was carried out using a symmetric Hopf bifurcation result. Some important results about the spontaneous bifurcations of multiple branches of periodic solutions and their spatio-temporal patterns, such as mirror-reflecting waves, standing waves, and discrete waves, which describe the oscillatory mode of each oscillator, were obtained. Motivated by the above models, in this paper we extend model (1.1) to the following time-delay system:

$$\begin{cases} x'_i(t) = b_i y_i(t) + \sum_{j=1, j\neq i}^{n} k_{i,j}[x_j(t - \tau_j) - x_i(t - \tau_i)], i = 1, 2, \cdots, n, \\ y'_i(t) = \alpha_i(1 - x^2_i(t))y_i(t) - c_i x_i(t) - \sum_{j=1, j\neq i}^{n} r_{i,j}[y_j(t - \theta_j) - y_i(t - \theta_i)], \end{cases} \tag{1.5}$$

where $b_i, c_i, \alpha_i, k_{ij}$, and r_{ij} are parameters, time delays $\tau_j \geq 0, \theta_j \geq 0 (j = 1, 2, \cdots, n)$. Noting that there are $2n$ time delays in the system (1.5), the general bifurcation method is hard to deal with such systems. In the present paper, we will use the mathematical analysis method to investigate the existence of oscillatory solutions for the system (1.5).

2. Preliminaries

The system (1.5) can be expressed in the following matrix form:

$$z'(t) = Az(t) + Bz(t-\tau) + f(z(t)) \tag{2.1}$$

where $z(t) = [x_1(t), y_1(t), x_2(t), y_2(t), \cdots, x_n(t), y_n(t)]^T$, $z(t-\tau) = [x_1(t-\tau_1), y_1(t-\theta_1), x_2(t-\tau_2), y_2(t-\theta_2), \cdots, x_n(t-\tau_n), y_n(t-\theta_n)]^T$, A and B both are $2n \times 2n$ matrices, and $f(z(t))$ is a $2n$ by one vector:

$$A = (a_{ij})_{2n\times 2n} = \begin{pmatrix} 0 & b_1 & 0 & 0 & ... & 0 & 0 \\ -c_1 & \alpha_1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & b_2 & ... & 0 & 0 \\ 0 & 0 & -c_2 & \alpha_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 0 & b_n \\ 0 & 0 & 0 & 0 & ... & -c_n & \alpha_n \end{pmatrix},$$

$$B = (b_{ij})_{2n\times 2n} = \begin{pmatrix} \rho_1 & 0 & k_{12} & 0 & ... & k_{1n} & 0 \\ 0 & \rho_2 & 0 & -r_{12} & ... & 0 & -r_{1n} \\ k_{21} & 0 & \rho_3 & 0 & ... & k_{2n} & 0 \\ 0 & -r_{21} & 0 & \rho_4 & ... & 0 & -r_{2n} \\ ... & ... & ... & ... & ... & ... & ... \\ k_{n1} & 0 & k_{n2} & 0 & ... & \rho_{2n-1} & 0 \\ 0 & -r_{n1} & 0 & -r_{n2} & ... & 0 & \rho_{2n} \end{pmatrix},$$

where $\rho_1 = -\sum_{j=2}^{n} k_{1j}, \rho_2 = -\sum_{j=2}^{n} r_{1j}, \cdots, \rho_2 n = -\sum_{j=2}^{n} r_{nj}$. The linearized system of (2.1) is

$$z'(t) = Az(t) + Bz(t-\tau). \tag{2.2}$$

Lemma 2.1. *If matrix $S(= A+B)$ is a nonsingular matrix for selected parameters, then there exists a unique equilibrium point for system (1.5) (or (2.1)).*

Proof. Assume that $z^* = (x_1^*, y_1^*, x_2^*, y_2^*, \cdots, x_n^*, y_n^*)^T$ is an equilibrium point of system (1.5), then we have the following algebraic equations:

$$\begin{cases} b_1 y_1^* + \sum_{j=2}^{n} k_{1j}(x_j^* - x_1^*] = 0, \\ \alpha_1[1-(x_1^*)^2]y_1^* - c_1 x_1^* - \sum_{j=2}^{n} r_{1j}(y_j^* - y_1^*) = 0, \\ b_2 y_2^* + \sum_{j=1, j\neq 2}^{n} k_{2j}(x_j^* - x_2^*] = 0, \\ \alpha_2[1-(x_2^*)^2]y_2^* - c_2 x_2^* - \sum_{j=1, j\neq 2}^{n} r_{2j}(y_j^* - y_2^*) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_n y_n^* + \sum_{j=1}^{n-1} k_{nj}(x_j^* - x_n^*] = 0, \\ \alpha_n[1-(x_n^*)^2]y_n^* - c_n x_n^* - \sum_{j=1}^{n-1} r_{nj}(y_j^* - y_n^*) = 0. \end{cases} \tag{2.3}$$

From system (2.3), we get

$$\begin{cases} b_1y_1^* + \sum_{j=2}^n k_{1j}(x_j^* - x_1^*] = 0, \\ \alpha_1 y_1^* - c_1x_1^* - \sum_{j=2}^n r_{1j}(y_j^* - y_1^*) = \alpha_1(x_1^*)^2y_1^*, \\ b_2y_2^* + \sum_{j=1,j\neq 2}^n k_{2j}(x_j^* - x_2^*] = 0, \\ \alpha_2 y_2^* - c_2x_2^* - \sum_{j=1,j\neq 2}^n r_{2j}(y_j^* - y_2^*) = \alpha_2(x_2^*)^2y_2^*, \\ \cdots\cdots \\ b_ny_n^* + \sum_{j=1}^{n-1} k_{nj}(x_j^* - x_n^*] = 0, \\ \alpha_n y_n^* - c_nx_n^* - \sum_{j=1}^{n-1} r_{nj}(y_j^* - y_n^*) = \alpha_n(x_n^*)^2y_n^*. \end{cases} \tag{2.4}$$

Now we first consider the corresponding homogeneous system of (2.4) as follows:

$$\begin{cases} b_1y_1^* + \sum_{j=2}^n k_{1j}(x_j^* - x_1^*] = 0, \\ \alpha_1 y_1^* - c_1x_1^* - \sum_{j=2}^n r_{1j}(y_j^* - y_1^*) = 0, \\ b_2y_2^* + \sum_{j=1,j\neq 2}^n k_{2j}(x_j^* - x_2^*] = 0, \\ \alpha_2 y_2^* - c_2x_2^* - \sum_{j=1,j\neq 2}^n r_{2j}(y_j^* - y_2^*) = 0, \\ \cdots\cdots \\ b_ny_n^* + \sum_{j=1}^{n-1} k_{nj}(x_j^* - x_n^*] = 0, \\ \alpha_n y_n^* - c_nx_n^* - \sum_{j=1}^{n-1} r_{nj}(y_j^* - y_n^*) = 0. \end{cases} \tag{2.5}$$

The matrix form of system (2.5) is

$$(A+B)z^* = Sz^* = 0. \tag{2.6}$$

According to basic linear algebra knowledge, system (2.6) (or (2.5)) has a unique trivial solution since $S = A+B$ is a nonsingular matrix. Note that systems (2.5) and the nonlinear system (2.4) have the same n-equations. The right-hand side of the system (2.4) is $(0, \alpha_1(x_1^*)^2y_1^*, 0, \alpha_2(x_2^*)^2y_2^*, \cdots, 0, \alpha_n(x_n^*)^2y_n^*)^T$. It is easy to see that system (2.4) also has a unique trivial solution. The proof is complete. □

Lemma 2.2. *All solutions of system (1.5) (or (2.1)) are bounded; assume that* $\alpha_i > 0. i = 1, 2, \cdots, n$.

Proof. To prove the boundedness of the solutions in the system (1.5), we construct a Lyapunov function $V(t) = \sum_{i=1}^n \frac{1}{2}[x_i^2(t) + y_i^2(t)]$. Calculating the derivative of $V(t)$ through system (1.5) we have

$$\begin{aligned} V'(t)|_{(1.5)} &= \sum_{i=1}^n [x_i'(t)x_i(t) + y_i'(t)y_i(t)] \\ = &\sum_{i=1}^n (b_i - c_i)x_iy_i + \sum_{i=1}^n \sum_{j=1,j\neq i}^n k_{ij}[x_j(t-\tau_j) - x_i(t-\tau_i)]x_i + \sum_{i=1}^n \alpha_i y_i^2 \\ &- \sum_{i=1}^n \sum_{j=1,j\neq i}^n r_{ij}[y_j(t-\theta_j) - y_i(t-\theta_i)]y_i - \sum_{i=1}^n \alpha_i x_i^2 y_i^2. \end{aligned} \tag{2.7}$$

Noting that $\alpha_i (i = 1, 2, \cdots, n)$ are positive constants. Obviously, when $x_i \to +\infty, y_i \to +\infty (1 \leq i \leq n,)$ $x_i^2 y_i^2$ are higher order infinity than x_i^2, y_i^2, and $x_i x_j$, respectively. Therefore, there exists suitably large $K > 0$ such that $V'(t)|_{(1.5)} < 0$ as $|x_i| > K, |y_i| > K$. This means that all solutions of system (1.5) are bounded. □

3. The existence of oscillatory solutions

Theorem 3.1. *Assume that zero is the unique equilibrium point of the system (1.5) for selecting parameter values. Let $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ and $\beta_1, \beta_2, \cdots, \beta_{2n}$ are characteristic values of matrix A and matrix B, respectively. If the real part of each α_i and $\beta_i (i = 1, 2, \cdots, n)$ are negative, then the trivial solution of system (1.5) is stable. If there exists a characteristic value, say α_k, and $Re(\alpha_k) > 0$ with $Re(\alpha_k) > |Re(\beta_k)| + |Im(\beta_k)|$; or $Re(\alpha_k) < 0$ with $|Re(\alpha_k)| < Re(\beta_k)$, then the unique trivial solution of system (1.5) is unstable, implying that there exists an oscillatory solution in system (1.5).*

Proof. According to the time-delay basic differential equation theory, if the real part of each α_i and $\beta_i (i = 1, 2, \cdots, 2n)$ are negative, then the trivial solution of system (2.2) is stable [30]. Noting that the nonlinear term $f(z)$ of system (2.1) is a higher-order infinitesimal as $x_i \to 0$ and $y_i \to 0$. Therefore, the stability of the trivial solution of system (2.2) ensures the stability of the trivial solution of system (1.5) (or (2.1)). If the trivial solution of system (2.2) is unstable, then the trivial solution of system (1.5) (or (2.1)) is also unstable. Therefore, to discuss the instability of the trivial solution of system (1.5) (or (2.1)) we only need to deal with the instability of the trivial solution of the system (2.2). Firstly, consider an auxiliary system of (2.2) as follows:

$$z'(t) = Az(t) + Bz(t - \tau_*) \tag{3.1}$$

where $\tau_* = \min_{1 \leq i \leq n}\{\tau_i, \theta_i\}$ and $z(t - \tau_*) = [x_1(t - \tau_*), y_1(t - \tau_*), x_2(t - \tau_*), y_2(t - \tau_*), \cdots, x_n(t - \tau_*), y_n(t - \tau_*)]^T$. Since $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ and $\beta_1, \beta_2, \cdots,$ β_{2n} are characteristic values of matrix A and matrix B, respectively, then the characteristic equations corresponding to the system (3.1) are the following:

$$\Pi_{i=1}^{2n} \lambda - \alpha_i - \beta_i e^{-\lambda \tau_*} = 0. \tag{3.2}$$

Thus, we are led to an investigation of the nature of the roots for some $k, k \in \{1, 2, \cdots, 2n\}$

$$\lambda - \alpha_k - \beta_k e^{-\lambda \tau_*} = 0. \tag{3.3}$$

Equation (3.3) is a transcendental equation that is hard to find all solutions for the equation. However, we show that there exists a positive real part eigenvalue of equation (3.3) under the assumption of Theorem 1. Indeed, if $Re(\alpha_k) > 0$ with $Re(\alpha_k) > |Re(\beta_k)| + |Im(\beta_k)|$. Let $\lambda = \sigma + i\gamma, \alpha_k = \alpha_{k1} + i\alpha_{k2}, \beta_k = \beta_{k1} + i\beta_{k2}$, where $\sigma = Re(\lambda), \alpha_{k1} = Re(\alpha_k), \beta_{k1} = Re(\beta_k)$, and

$\gamma = Im(\lambda), \alpha_{k2} = Im(\alpha_k), \beta_{k2} = Im(\beta_k)$, respectively. Separating the real part and the imaginary part of equation (3.3), we get

$$\sigma = \alpha_{k1} + \beta_{k1} e^{-\sigma\tau_*} \cos(\gamma\tau_*) - \beta_{k2} e^{\sigma\tau_*} \sin(\gamma\tau_*) \tag{3.4}$$

$$\gamma = \alpha_{k2} - \beta_{k1} e^{-\sigma\tau_*} \sin(\gamma\tau_*) + \beta_{k2} e^{\sigma\tau_*} \cos(\gamma\tau_*). \tag{3.5}$$

We show that equation (3.3) has a positive real part root. Let

$$\Psi(\sigma) = \sigma - \alpha_{k1} - \beta_{k1} e^{-\sigma\tau_*} \cos(\gamma\tau_*) + \beta_{k2} e^{\sigma\tau_*} \sin(\gamma\tau_*). \tag{3.6}$$

Obviously, $\Psi(\sigma)$ is a continuous function of σ. Noting that $\alpha_{k1} > |\beta_{k1}| + |\beta_{k2}|$, then $\Psi(0) = -\alpha_{k1} - \beta_{k1}\cos(\gamma\tau_*) + \beta_{k2}\sin(\gamma\tau_*) \leq -\alpha_{k1} + |\beta_{k1}| + |\beta_{k2}| < 0$. Since $\lim_{\sigma\to+\infty} e^{-\sigma\tau_*} = 0$, so there exists a suitably large σ, say $\sigma_1 (> 0)$ such that $\Psi(\sigma_1) = \sigma_1 - \alpha_{k1} - \beta_{k1} e^{-\sigma_1\tau_*}\cos(\gamma\tau_*) + \beta_{k2} e^{\sigma_1\tau_*}\sin(\gamma\tau_*) > 0$. By the Intermediate Value Theorem, there exists a σ, say $\sigma_0 \in (0, \sigma_1)$ such that $\Psi(\sigma_0) = 0$, implying that there is a positive real part characteristic value of equation (3.3). This means that the trivial solution of system (3.1) is unstable. It is known that if the solution of a delayed equation is unstable for a small delay, then the instability of the solution will be maintained as the delays increase [30]. Therefore, the instability of the trivial solution of the system (3.1) implies the instability of the trivial solution of the system (2.2). This suggests that the unique positive equilibrium point $(x_1^*, y_1^*, x_2^*, y_2^*, \cdots, x_n^*, y_n^*)^T$ of system (1.5) is unstable. This instability of the unique positive equilibrium point, together with the boundedness of the solutions, will force system (1.5) to generate an oscillatory solution [31, 32]. For the case $Re(\alpha_k) < 0$ with $|Re(\alpha_k)| < Re(\beta_k)$, the equation (3.3) will have a positive real part characteristic root. The proof is similar, and we omit it. □

For simplify, setting $\| B \| = \max_{1\leq j\leq 2n} \sum_{i=1}^{2n} |b_{ij}|, p = \max_{1\leq i\leq n}\{-c_i, b_i + \alpha_i\}$. Then we have

Theorem 3.2. *Assume that the conditions of Lemma 1 and Lemma 2 hold. If the following inequality is satisfied*

$$7|p|\tau_*^2 \| B \| > 4e^{|p|\tau_*}. \tag{3.7}$$

Then the trivial solution of system (3.1) is unstable, implying that the system (1.5) has an oscillatory solution.

Proof. To prove the instability of the trivial solution of system (3.1), let $w(t) = \sum_{i=1}^{n}(|x_i(t)| + |y_i(t)|)$. Therefore, $w(t) > 0$ and

$$w'(t) \leq pw(t) + \| B \| w(t - \tau_*). \tag{3.8}$$

Specifically, consider the equation

$$v'(t) = pv(t) + \| B \| v(t - \tau_*). \tag{3.9}$$

Obviously, $w(t) \leq v(t)$. If the trivial solution of equation (3.9) is unstable, then the trivial solution of (3.8) is still unstable. The characteristic equation associated with equation (3.9) is given by

$$\lambda = p + \parallel B \parallel e^{-\lambda\tau_*}. \tag{3.10}$$

If the trivial solution of equation (3.10) is stable, then equation (3.10) must have a real negative root, say λ_*, and we have from (3.10)

$$|\lambda_*| > \parallel B \parallel e^{-\lambda_*\tau_*} - |p|. \tag{3.11}$$

One can prove that $e^x \geq \frac{7}{4}x^2 (x > 0)$. So we have

$$1 \geq \frac{\parallel B \parallel e^{|\lambda_*|\tau_*}}{|\lambda_*| + |p|} = \frac{\parallel B \parallel \tau_* e^{(|\lambda_*|+|p|)\tau_*}}{(|\lambda_*| + |p|)\tau_* \cdot e^{|p|\tau_*}} \geq \frac{7\tau_*^2 \parallel B \parallel (|\lambda_*| + |p|)}{4e^{|p|\tau_*}} > \frac{7|p|\tau_*^2 \parallel B \parallel}{4e^{|p|\tau_*}}. \tag{3.12}$$

A contradiction with equation (3.7). Implying that the trivial solution of equation (3.9) is unstable. It suggests that the trivial solution of equation (3.8) is unstable. Thus, for all $\{\tau_i, \theta_i\} \geq \tau_* (i = 1, 2, \cdots, n)$, the trivial solution of system (3.1) is unstable, implying that the equilibrium point of system (1.5) is unstable. Similar to Theorem 1, system (1.5) generates an oscillatory solution. The proof is complete. □

4. Simulation result

We first consider $n = 4$ in system (1.5) as follows:

$$\begin{cases} x_1' = b_1 y_1 + \sum_{j=2}^{4} k_{1j}[x_j(t-\tau_j) - x_1(t-\tau_1)], \\ y_1' = \alpha_1(1 - x_1^2)y_1 - c_1 x_1 - \sum_{j=2}^{4} r_{1j}[y_j(t-\theta_j) - y_1(t-\theta_1)], \\ x_2' = b_2 y_2 + \sum_{j=1,j\neq 2}^{4} k_{2j}[x_j(t-\tau_j) - x_2(t-\tau_2)], \\ y_2' = \alpha_2(1 - x_2^2)y_2 - c_2 x_2 - \sum_{j=1,j\neq 2}^{4} r_{2j}[y_j(t-\theta_j) - y_2(t-\theta_2)], \\ x_3' = b_3 y_3 + \sum_{j=1,j\neq 3}^{4} k_{3j}[x_j(t-\tau_j) - x_3(t-\tau_3)], \\ y_3' = \alpha_3(1 - x_3^2)y_3 - c_3 x_3 - \sum_{j=1,j\neq 3}^{4} r_{3j}[y_j(t-\theta_j) - y_3(t-\theta_3)], \\ x_4' = b_4 y_4 + \sum_{j=1}^{3} k_{4j}[x_j(t-\tau_j) - x_4(t-\tau_4)], \\ y_4' = \alpha_4(1 - x_4^2)y_4 - c_4 x_4 - \sum_{j=1}^{3} r_{4j}[y_j(t-\theta_j) - y_4(t-\theta_4)]. \end{cases} \tag{4.1}$$

Firstly, the parameters are selected as $b_1 = 1.45, b_2 = 1.48, b_3 = 1.46, b_4 = 1.44, c_1 = 1.75, c_2 = 1.78, c_3 = 1.74, c_4 = 1.76, \alpha_1 = 0.94, \alpha_2 = 0.98, \alpha_3 = 0.96, \alpha_4 = 0.95; k_{12} = 0.125, k_{13} = 0.116, k_{14} = 0.114, k_{21} = 0.115, k_{23} = 0.120, k_{24} = 0.125, k_{31} = 0.124, k_{32} = 0.122, k_{34} = 0.125, k_{41} = 0.112, k_{42} = 0.116, k_{43} = 0.118, r_{12} = 0.64, r_{13} = 0.82, r_{14} = 0.76, r_{21} = 0.65, r_{23} = 0.72, r_{24} = 0.75, r_{31} = 0.64, r_{32} = 0.55, r_{34} = 0.68, r_{41} = 0.62, r_{42} = 0.55, r_{43} = 0.64$; The time delays as $\tau_1 = 0.87, \tau_2 = 0.90, \tau_3 = 0.88, \tau_4 = 0.86, \theta_1 = 0.86, \theta_2 = 0.92, \theta_3 = 0.94, \theta_4 = 0.92$. Then the characteristic values of matrix A and B in system (4.1) are $0.4700 \pm 1.5220i, 0.4750 \pm 1.5195i, 0.4800 \pm 1.5199i, 0.4900 \pm 1.5474i$ and $2.8627, 2.6731, 2.5161, 0.0137, -0.4512, -0.4728, -0.4928$, respectively. Since all characteristic values of matrix B are real numbers, so $Re(\beta_k) = \beta_k$, and $Re(\alpha_5) = 0.4800 > 0.0037 = \beta_5$, the conditions of

Theorem 1 are satisfied. There exists an oscillatory solution for the system (4.1) (see Fig.1). When we change time delays as $\tau_1 = 1.45, \tau_2 = 1.48, \tau_3 = 1.40, \tau_4 = 1.42, \theta_1 = 1.46, \theta_2 = 1.42, \theta_3 = 1.38, \theta_4 = 1.36$, the other parameters are the same as in Fig.1, we see that the oscillatory behavior of the solutions is still maintained (see Fig.2).

Then we consider $n = 9$ in the system (1.5) as follows:

$$\begin{cases} x_1' = b_1 y_1 + \sum_{j=2}^{9} k_{1j}[x_j(t-\tau_j) - x_1(t-\tau_1)], \\ y_1' = \alpha_1(1-x_1^2)y_1 - c_1 x_1 - \sum_{j=2}^{9} r_{1j}[y_j(t-\theta_j) - y_1(t-\theta_1)], \\ x_2' = b_2 y_2 + \sum_{j=1,j\neq 2}^{9} k_{2j}[x_j(t-\tau_j) - x_2(t-\tau_2)], \\ y_2' = \alpha_2(1-x_2^2)y_2 - c_2 x_2 - \sum_{j=1,j\neq 2}^{9} r_{2j}[y_j(t-\theta_j) - y_2(t-\theta_2)], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_8' = \alpha_8(1-x_8^2)y_8 - c_8 x_8 - \sum_{j=1,j\neq 8}^{9} r_{8j}[y_j(t-\theta_j) - y_8(t-\theta_8)], \\ x_9' = b_9 y_9 + \sum_{j=1}^{8} k_{9j}[x_j(t-\tau_j) - x_9(t-\tau_9)], \\ y_9' = \alpha_9(1-x_9^2)y_9 - c_9 x_9 - \sum_{j=1}^{8} r_{9j}[y_j(t-\theta_j) - y_9(t-\theta_9)]. \end{cases} \tag{4.2}$$

We select the parameters as $b_1 = 1.55, b_2 = 1.58, b_3 = 1.56, b_4 = 1.52, b_5 = 1.55, b_6 = 1.58, b_7 = 1.56, b_8 = 1.52, b_9 = 1.54, c_1 = 1.45, c_2 = 1.42,$ $c_3 = 1.50, c_4 = 1.48, c_5 = 1.45, c_6 = 1.44,$ $c_7 = 1.46, c_8 = 1.42, c_9 = 1.48, \alpha_1 = 0.92, \alpha_2 = 0.90, \alpha_3 = 0.98, \alpha_4 = 0.97, \alpha_5 = 0.98, \alpha_6 = 0.92, \alpha_7 = 0.92, \alpha_8 = 0.95, \alpha_9 = 0.88, k_{12} = 0.110,$ $k_{13} = 0.111, k_{14} = 0.112, k_{15} = 0.113,$ $k_{16} = 0.114, k_{17} = 0.115, k_{18} = 0.116, k_{19} = 0.118, k_{21} = 0.68, k_{23} = 0.121, k_{24} = 0.122, k_{25} = 0.123, k_{26} = 0.124, k_{27} = 0.125, k_{28} = 0.126, k_{29} = 0.128, k_{31} = 0.67,$ $k_{32} = 0.131, k_{34} = 0.132, k_{35} = 0.133, k_{36} = 0.134, k_{37} = 0.135, k_{38} = 0.136, k_{39} = 0.138, k_{41} = 0.66, k_{42} = 0.141, k_{43} = 0.142, k_{45} = 0.143,$ $k_{46} = 0.144, k_{47} = 0.145, k_{48} = 0.146, k_{49} = 0.148, k_{51} = 0.65, k_{52} = 0.121, k_{53} = 0.122, k_{54} = 0.123, k_{56} = 0.124, k_{57} = 0.125, k_{58} = 0.156, k_{59} = 0.158, k_{61} = 0.66, k_{62} = 0.113, k_{63} = 0.114, k_{64} = 0.115, k_{65} = 0.116, k_{67} = 0.117, k_{68} = 0.118, k_{69} = 0.119, k_{71} = 0.68, k_{72} = 0.133, k_{73} = 0.134, k_{74} = 0.135, k_{75} = 0.136, k_{76} = 0.137, k_{78} = 0.138, k_{79} = 0.139, k_{81} = 0.69, k_{82} = 0.093, k_{83} = 0.094, k_{84} = 0.095, k_{85} = 0.096, k_{86} = 0.097, k_{87} = 0.098, k_{89} = 0.099, k_{91} = 0.67, k_{92} = 0.123, k_{93} = 0.124, k_{94} = 0.125, k_{95} = 0.126, k_{96} = 0.127, k_{97} = 0.128, k_{98} = 0.130; r_{12} = -0.61, r_{13} = 0.62, r_{14} = -0.63, r_{15} = 0.64, r_{16} = -0.65, r_{17} = 0.66, r_{18} = -0.67, r_{19} = 0.68, r_{21} = -0.51,$ $r_{23} = 0.52, r_{24} = -0.53, r_{25} = 0.54,$ $r_{26} = -0.55, r_{27} = 0.56, r_{28} = -0.57,$ $r_{29} = 0.58, r_{31} = -0.71, r_{32} = 0.72, r_{34} = -0.73, r_{35} = 0.74, r_{36} = -0.75,$ $r_{37} = 0.76, r_{38} = -0.77, r_{39} = 0.78,$ $r_{41} = -0.64, r_{42} = 0.65, r_{43} = -0.66, r_{45} = 0.67, r_{46} = -0.68, r_{47} = 0.69, r_{48} = -0.70, r_{49} = 0.72, r_{51} = -0.72, r_{52} = 073, r_{53} = -0.74, r_{54} = 0.75, r_{56} = -0.76, r_{57} = 0.77, r_{58} = -0.78, r_{59} = 0.79, r_{61} = -0.63, r_{62} = 0.64, r_{63} = -0.65, r_{64} = 0.66, r_{65} = -0.67, r_{67} = 0.68, r_{68} = -0.69, r_{69} = 0.70, r_{71} = -0.52, r_{72} = 0.53, r_{73} = -0.54, r_{74} = 0.55, r_{75} = -0.56, r_{76} = 0.57, r_{78} = -0.58, r_{79} = 0.59, r_{81} = -0.42, r_{82} = 0.43, r_{83} = -0.44, r_{84} = 0.45, r_{85} = -0.46, r_{86} = 0.47, r_{87} = -0.48, r_{89} = 0.49, r_{91} =$

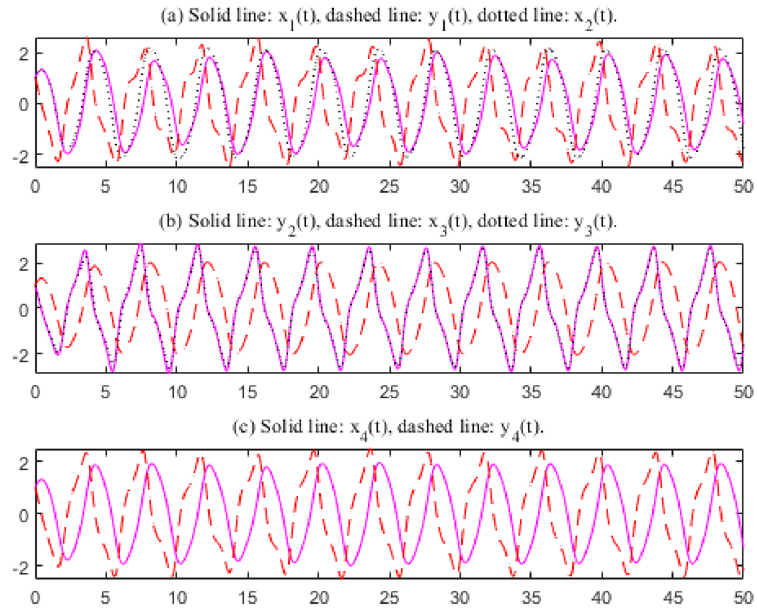


FIGURE 1. Oscillation of the solutions, delays: 0.87, 0.86, 0.90, 0.92, 0.88, 0.94, 0.86, 0.92.

$-0.65, r_{92} = 0.66, r_{93} = -0.67, r_{94} = 0.68, r_{95} = -0.69, r_{96} = 0.70, r_{97} = -0.71, r_{98} = 0.72$. We see that $p = 2.54$, and $\| B \| = 4.448$ in the system (4.2). When time delays are selected as $\tau_1 = 0.45, \tau_2 = 0.48, \tau_3 = 0.40, \tau_4 = 0.42, \tau_5 = 0.45, \tau_6 = 0.48, \tau_7 = 0.40, \tau_8 = 0.54, \tau_9 = 0.45, \theta_1 = 0.46, \theta_2 = 0.42, \theta_3 = 0.38, \theta_4 = 0.36, \theta_5 = 0.46, \theta_6 = 0.42, \theta_7 = 0.52, \theta_8 = 0.48, \theta_9 = 0.45$. then $\tau_* = 0.36$, and $7|p|\tau_*^2 \| B \| = 10.2495 > 9.9867 = 4e^{|p|\tau_*}$. The conditions of Theorem 2 are satisfied. There exists an oscillatory solution for the system (4.2) (see Fig.3). When we change time delays as $\tau_1 = 0.60, \tau_2 = 0.62, \tau_3 = 0.61, \tau_4 = 0.61, \tau_5 = 0.60, \tau_6 = 0.61, \tau_7 = 0.60, \tau_8 = 0.61, \tau_9 = 0.60, \theta_1 = 0.61, \theta_2 = 0.60, \theta_3 = 0.62, \theta_4 = 0.62, \theta_5 = 0.62, \theta_6 = 0.61, \theta_7 = 0.60, \theta_8 = 0.62, \theta_9 = 0.61$. the other parameters are the same as in Fig.3, we see that the oscillatory behavior of the solutions is still kept (see Fig.4). It is pointed out that the synchronization of the solutions appeared (see Fig.5).

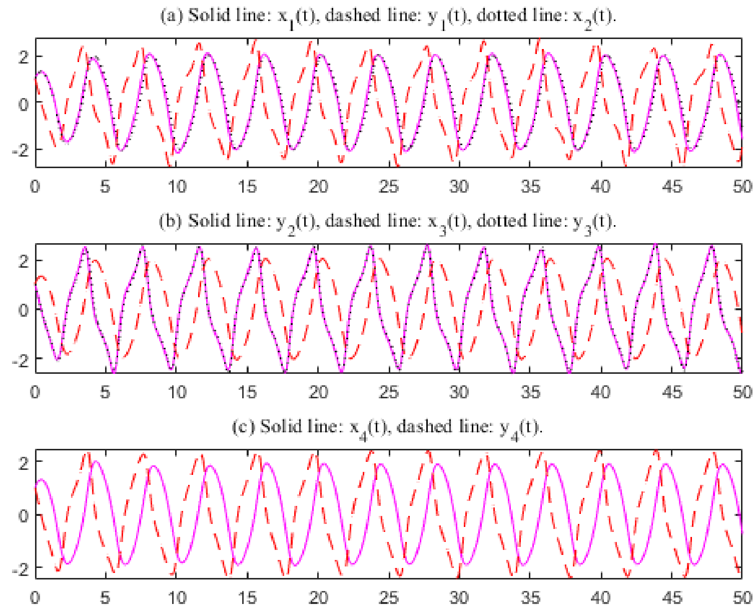
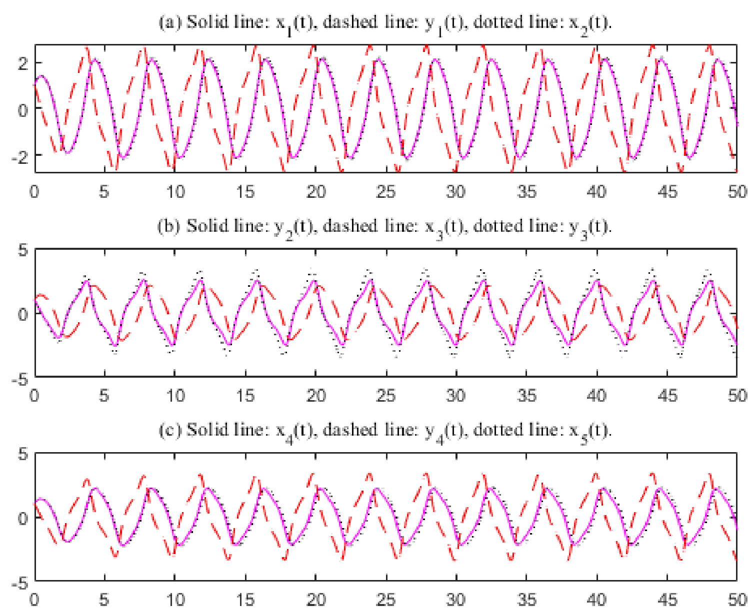


FIGURE 2. Oscillation of the solutions, delays: 1.45, 1.46, 1.48, 1.42, 1.40, 1.38, 1.42, 1.36.



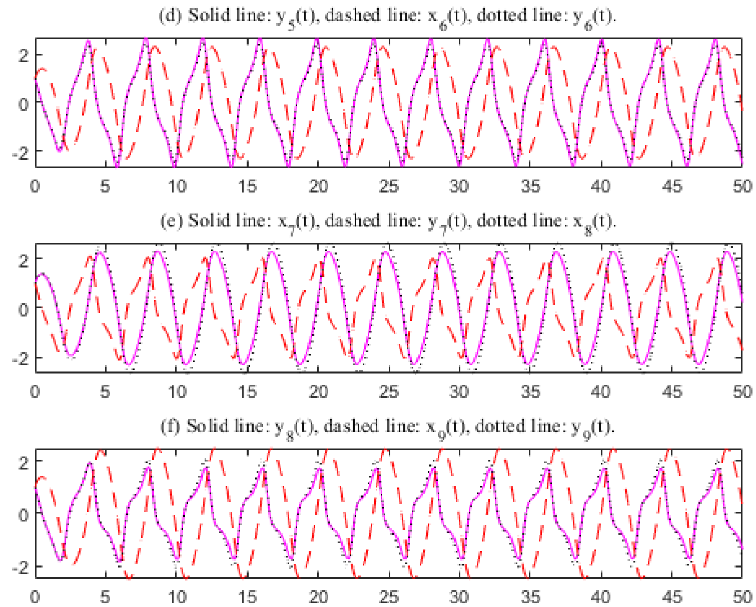
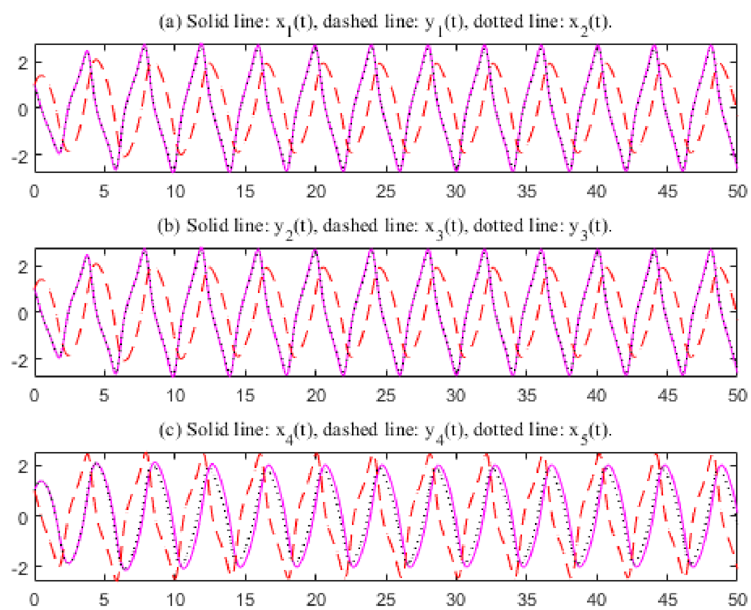


FIGURE 3. Oscillation of the solutions, minimum delay: 0.36.



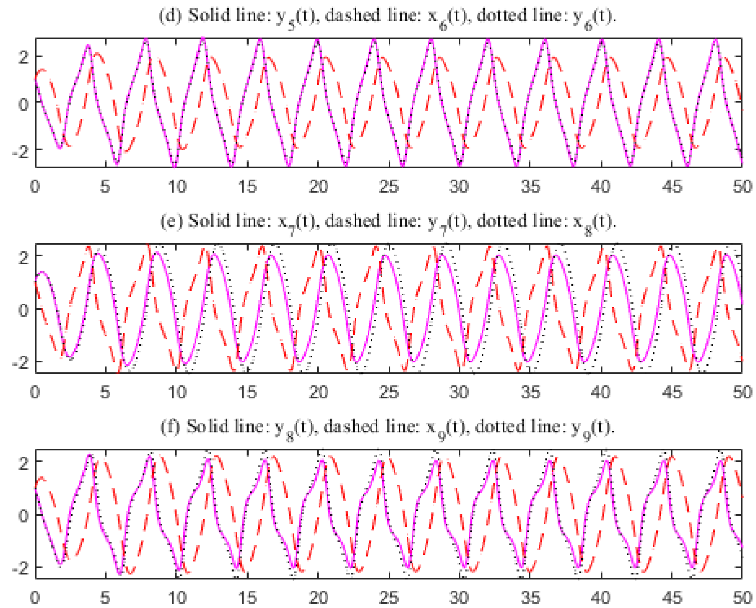
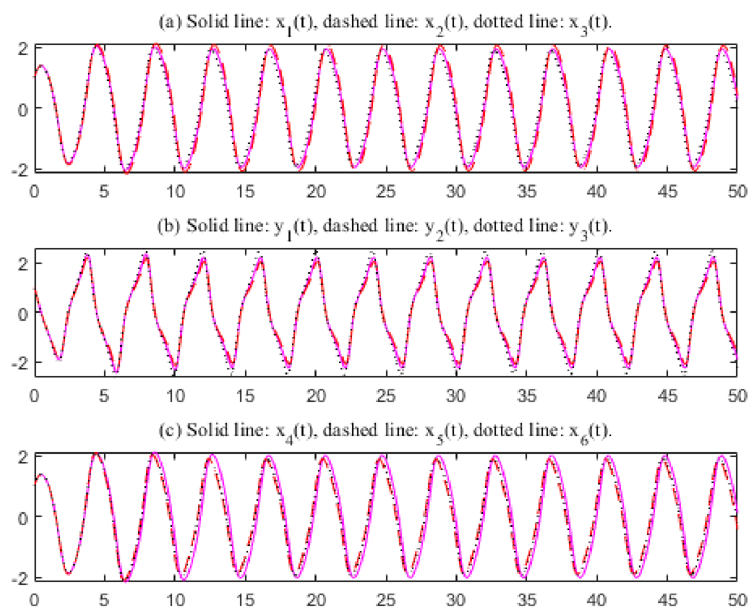


FIGURE 4. Oscillation of the solutions, minimum delay: 0.60.



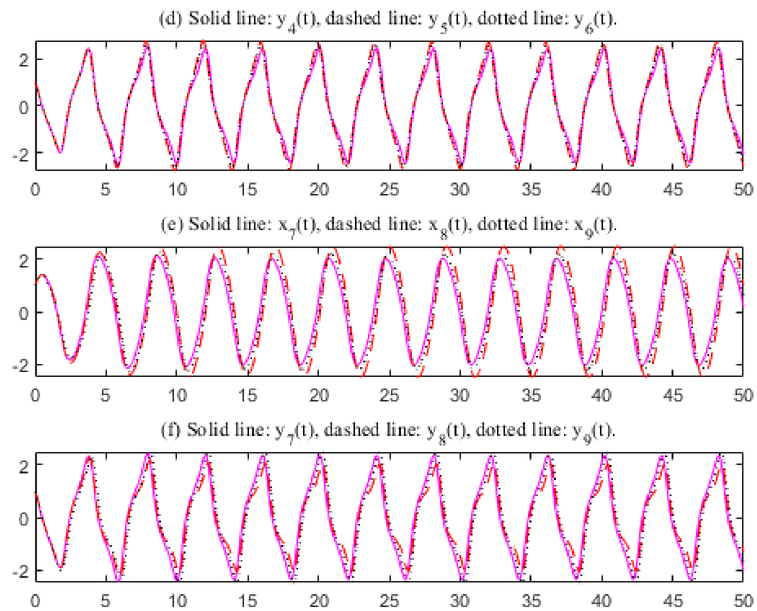


FIGURE 5. Synchronization of the solutions, minimum delay: 0.60.

5. Conclusion

In this paper, we have discussed the oscillatory behavior of the solutions for a n-dimensional system of coupled van der Pol oscillators with delays. Based on the method of mathematical analysis, we provided some sufficient conditions to guarantee the oscillation of the solutions. Some simulations are provided to indicate the effectiveness of the criteria. One can see that the synchronized behavior of the solutions appeared.

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