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Some Mathematical Logical Proofs for the Radon-Nikodym Theorem and the Stone Representation Theorem for Measure Algebras

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Abstract. The celebrated Radon-Nikodym theorem and Stone representation theorem for measure algebras are two important classical results in analysis. This paper pursues two main goals. One is to give new proofs for these theorems by using ideas from logic and application of an important theorem, namely, the logical compactness theorem. The second and even more important goal is to try to reveal more the power of logical methods in analysis in particular measure theory, and make stronger connections between two fields of analysis and logic. Through the paper, we use a logical setting called "integration logic" which is a framework for studying measure and probability structures through logical means. The paper is mostly written for general mathematicians, in particular the people active in logic or analysis as the main audiences. It is self-contained and does not require advanced prerequisite knowledge from logic or analysis.

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1. Introduction

The general theme of this paper belongs to the area of applications of model theory (a subfield of mathematical logic) in mathematics. The Radon-Nikodym theorem and Stone representation theorem for measure algebras are two important classical existence result in analysis. These theorems have been widely used in the literature. On the other hand, one of the missions of the mathematical logic is to study mathematical objects by logical means. Indeed, there are numerous applications of ideas and techniques from mathematical logic in analysis, probability theory, dynamical systems, etc (see for example [3] or [5] where logic gets involved with probability theory and dynamical systems). Probability logics, such as the variants introduced in [2], [3] and [8], are among various classical logical frameworks designed to deal with probability and measure structures. These frameworks have been investigated from different perspectives in particular model theory (see [1] and [3]). Integration logic is one of the forms of probability logics which was investigated in [3], and then was further developed in [1]. This setting enables one to use integral operation as a logical quantifier. It was used for giving new proofs for the classical Daniell-Stone theorem and Riesz representation theorem in [6]. It is worth mentioning that in [7], integration logic is represented as a specific example of a more abstract and general framework with a viewpoint close to functional analysis.

During the course of our investigation in this paper, we mainly pursue two main goals. One is to provide new proofs with logical flavor in the setting of integration logic for Radon-Nikodym theorem and also the Stone representation theorem for measure algebras. The main logical ingredient of our proofs is the logical compactness theorem. The second and even more important goal is to elaborate, highlight and emphasize the power of the logical methods in the realm of analysis, in particular measure theory, and make stronger connections between these two fields. Indeed, our proofs, beside the fact that are some new proofs for two classical theorems which might be of interest of its own right, indicates the strength of the logical methods in analysis and measure theory. Although the methods of the proofs involves techniques from mathematical logic, the reader does not need to have any significant amount of prerequisite knowledge from logic to follow the proofs or even be possibly able to apply its ideas for proving similar results. These proofs can be considered as some applications of the setting of the integration logic. Employing suitable logical frameworks can sometimes enables one to provide uniform proofs with similar techniques for seemingly different theorems. It is the case in this paper and both theorems are proved by using the technique of the logical compactness theorem. There are many facts in analysis that can be considered in this way. In fact, the method and strategy of the proof here seems to be more general than the

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result of this paper and is possibly applicable, to some degree, in more results in measure theory. Logical compactness theorem is easy to be used in practice for a general mathematician. In a suitable logical setting, it roughly states that if we have a family of properties formally stated in that setting and every finite subset of them is satisfied in some structure, then there exists a structure which satisfies all of them together. There are many interesting notions, properties or claims in measure theory that can be expressed by an infinite number of simple statements. Then, by applying the compactness theorem, the truth of such notions or claims is reduced to the satisfiability of every finite number of those statements. The paper is self-contained and we mention all prerequisites from logic and measure theory in it. It is mostly written for general mathematicians, in particular those active in logic or analysis as the main audiences.

Presentation of the rest of the paper is as follows. In Section 2, we briefly review basic measure theoretic concepts and give a concise introduction to the integration logic. We also introduce some technical notions and prove some lemmas which we need later in the paper. Section 3 contains the the main results of the paper namely the proofs of the Radon-Nikodym theorem and the Stone representation theorem for measure algebras by using logical tools.

2. Preliminaries

A measure on a σ-algebra $\mathcal{B}$ of subsets of a set M is a real-valued function $\mu : \mathcal{B} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and for any countable sequence $A_k \in \mathcal{B}$ of disjoint sets, we have $\mu(\bigcup_k A_k) = \sum_k \mu(A_k)$. As usual, the notion of "almost everywhere" (usually abbreviated by "*a.e.*") in a given measure space means everywhere in that space except on a subset with measure zero. We recall the definition of a subspace measures. For any measure space $(N, \mathcal{B}, \mu)$, the outer measure μ^* on N is defined by $\mu^*(X) := \inf\{\mu(A)|\ X \subseteq A \in \mathcal{B}\}$ for every $X \subseteq N$. If $M \subseteq N$, then $\mathcal{B}_M := \{A \cap M|\ A \in \mathcal{B}\}$ forms a σ-algebra of subsets of M and the restriction of μ^* to it, denoted by μ_M, is a measure. Indeed, elements of $\mathcal{B}_M$ are $\mu^*|_M$-measurable. μ_M is called the *subspace measure* on M. If $f : N \to \mathbb{R}$ is a measurable function, then by $\int_M f|_M$ we mean $\int_M (f|_M) d\mu_M$ where $f|_M$ is the notation for the restriction of f to M.

Proposition 2.1. (see [4], Subsection 214) *Assume that $(N, \mathcal{B}, \mu)$ is a measure space, $M \subseteq N$ and f an integrable function on N. Then $f|_M$ is μ_M-integrable. Moreover, If $\mu^*(M) = \mu(N)$ or f is equal to 0 almost everywhere on $N - M$, then $\int_M f|_M = \int_N f$.*

As usual, for every two real-valued functions f and g, we denote $\max(f, g)$ and $\min(f, g)$ by $f \vee g$ and $f \wedge g$ respectively. In a measure space and for any two measurable subsets A and B of it, by the notation $A \overset{a.e}{\subseteq} B$ we mean that $A \setminus B$ has measure zero with the respected measure.

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Now we consider logic and briefly review the framework of the "integration logic". This logical framework was investigated in [1], [2] and [3] for studying measure and probability structures by logical methods. We use the terminology of [1]. Using this framework enables us to formalize and express certain measure theoretic properties of spaces, functions on them, etc, in a unified way. We first briefly review some essential concepts and then formally define some notions.

By a *simple relational structure* (or simply, a structure), intuitively we mean a measure space we wish to study equipped with a family of *relations* where a relation is a real-valued measurable function on (some power of) the measure space. Also there might be a family of elements of the domain set of the measure space which are required to be considered as distinguished elements. We usually assign a symbol corresponding to each of such relations and distinguished elements and call them *relation symbols* and *constant symbols* respectively. We also call the family of such symbols a (relational) *language* and usually denote it by the notation $\mathcal{L}$.

We call any structure in which the symbols in $\mathcal{L}$ are interpreted, a $\mathcal{L}$-*structure* (it formally will be defined in Definition 2.2 below). Indeed, in order for us to systematically study one or a family of structures by logical means, we usually first choose a suitable language $\mathcal{L}$ consisting of the symbols corresponding to all relations and distinguished elements we intend to investigate in our structure(s). Now those structure(s) can be viewed as $\mathcal{L}$-structure(s). Then, we can use symbols in $\mathcal{L}$ as well as *variable symbols* (which will be defined below) and *logical symbols* (namely, connectives and quantifiers as explained below) to write formal logical expressions (called formulas, statements and sentences) describing our $\mathcal{L}$-structure(s) logically. This enables us to study mathematical properties of the structure(s) in hand through formal logical tools and syntactical methods.

It should be mentioned that in most of the structures in this paper, the real-valued functions on the spaces play the main role. Therefore, it would be sufficient for us to only work with relational structures and avoid introducing or working with function symbols in a fundamental way.

We always assume that any language $\mathcal{L}$ contains a distinguished binary relation symbol $\mathbf{e}$ for equality. Moreover, we assume that to each relation symbol R a nonnegative real number $\flat_R$ is assigned which is called its *universal bound*. In particular, for the equality relation we have $\flat_{\mathbf{e}} = 1$. As will be explained more in Definition 2.3, logical symbols consist of the binary functions $+$, $\cdot$, the unary absolute value function $|\ \ |$ and a 0-ary function r for every real number r. We consider these as connectives in this logical setting on integration logic. The integration symbol $\int$ is also considered as a logical symbol and used as a quantifier. We also use an infinite list $x, y, ...$ of individual variable symbols. We call the family of all variable symbols and constant symbols, the collection of $\mathcal{L}$-*terms* in this logical setting.

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Definition 2.2. Let $\mathcal{L}$ be a relational language. By a *simple relational $\mathcal{L}$-structure*, or simply a $\mathcal{L}$-structure), we mean a a nonempty measure space $(M, \mathcal{B}, \mu)$, in which every singleton is measurable and $\mu(M) = 1$, and is equipped with the following:

(1) For each constant symbol $c \in \mathcal{L}$ (if there is any), there is an element $c^M \in M$.
(2) For each n-ary relation symbol $R \in \mathcal{L}$ (if there is any), there is a measurable function $R^M : M^n \to \mathbb{R}$ with $|R^M(\bar{a})| \leqslant \flat_R$ for each $\bar{a} \in M$.

We call R^M and c^M the *interpretations* of the relation and constant symbols R and c in M.

Regarding Definition 2.2, it worth to emphasize that μ or $\mathcal{B}$ are not logical symbols or formulas or elements of the syntax of the logic. Instead, they are just part of the notion of structure which satisfy the conditions mentioned in the definition.

Note that in every structure, the binary equality relation $\mathbf{e}(x, y)$ is interpreted as a two variable function taking value 1 if $x = y$ and 0 otherwise.

Definition 2.3. For a language $\mathcal{L}$, the family of $\mathcal{L}$-formulas is inductively defined as follows.

(1) If R is a n-ary relation symbol in $\mathcal{L}$ and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $R(t_1, ..., t_n)$ is a formula. In particular, $\mathbf{e}(x, y)$ is a formula.
(2) For any $r \in \mathbb{R}$, r is a formula.
(3) If ϕ and ψ are formulas then $|\phi|, \phi + \psi$, $\phi \vee \psi$, $\phi \wedge \psi$ and $\phi \cdot \psi$ are all formulas too.
(4) If $\phi(\bar{x}, y)$ is a formula, then $\int \phi(\bar{x}, y)dy$ is a formula.

Free variables of formulas are easily defined (inductively) as the variables which are not bounded by the quantifies $\int$. For example, in the formula $\int(2x + y)\, dy + |3z|$, the variables x and z are free while y is bounded by the quantifier $\int$. One writes $\phi(x_1, ..., x_n)$ to indicate that all free variables of the formula ϕ are among $x_1, ..., x_n$. By a *closed* formula we mean a formula without any free variable. If $\phi(\bar{x})$ is a formula and $\bar{a} \in M^{|\bar{x}|}$, then the value of $\phi(\bar{a})$ in M, denoted by $\phi^M(\bar{a})$, is defined inductively in the natural way. For example

$$(\phi + \psi)^M(\bar{a}) = \phi^M(\bar{a}) + \psi^M(\bar{a}), \qquad (\phi \vee \psi)^M(\bar{a}) = \phi^M(\bar{a}) \vee \psi^M(\bar{a}),$$

$$\Big(\int \phi(\bar{x}, y)dy\Big)^M(\bar{a}) = \int_M \phi^M(\bar{a}, y)dy.$$

Thus, $\phi(\bar{x})$ gives rise to a real-valued function on $M^{|x|}$, which is called the *interpretation* of the formula ϕ and is denoted by ϕ^M. Note that, in particular, if ϕ is a closed formula, then for any $\mathcal{L}$-structure M, ϕ^M is a uniquely determined constant function with a fixed real number as its value. For example if

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$\phi = \int \psi(y)dy$ where $\psi(y)$ is a formula, then we have

$$\phi^M = \int_M \psi^M(y)dy.$$

Definition 2.4. By a *statement*, we mean an expression of the form $\phi(x) \geqslant r$ or $\phi(x) = r$ for some formula $\phi(x)$ and some $r \in \mathbb{R}$. If ϕ is a closed formula, then we call the statement a *closed statement* (or *sentence*). We call any set of closed statements a *theory*.

Obviously expressions such as $\phi(x) \leqslant r$, $\phi(x) \geqslant \psi(x) + r$ or $\phi(x) = \psi(x) + r$, where $\phi(x)$ and $\psi(x)$ are formulas, are statements since they can be written in the form $-\phi(x) \geqslant -r$, $\phi(x) - \psi(x) \geqslant r$ or $\phi(x) - \psi(x) = r$ while we know that $-\phi(x)$ and $\phi(x) - \psi(x)$ are again formulas. A closed statement $\phi = r$ or $\phi \geqslant r$ is called *satisfied* in a simple $\mathcal{L}$-structure M, denoted by $M \vDash$ "$\phi = r$" and $M \vDash$ "$\phi \geqslant r$", if we have $\phi^M = r$ and $\phi^M \geqslant r$ respectively. We call a simple $\mathcal{L}$-structure M a *model* of a theory T, denoted by $M \vDash T$, if each statement in T is satisfied in M. A theory is called *satisfiable* if it has a model. Also a theory is called *finitely satisfiable* if every finite subset of it has a model.

The main logical tool used in this paper is the following theorem. It is basically Theorem 4.7 of [1].

Theorem 2.5. (The logical compactness theorem) *Every finitely satisfiable theory is satisfiable.*

In the following remark, we mention some basic measure theoretic properties expressible in the logical setting of integration logic.

Remark 2.6. For any formulas $\phi(x)$ and $\psi(x)$ with the same free variables x (in a relevant language in the integration logic), the expressions "$\phi(x) = 0$ almost everywhere" and "$\phi(x) = \psi(x)$ almost everywhere" can be expressed by the closed statements $\int\ |\phi(x)|\ dx = 0$ and $\int\ |\phi(x) - \psi(x)|\ dx = 0$ respectively in integration logic, where we recall that the interpretations of formulas ϕ and ψ are measurable functions on our measure space. If $A := \{r_1, \dots, r_n\} \subseteq \mathbb{R}$, then the expression "$\psi$ takes its values in A almost everywhere" (or equivalently speaking, "range(ψ) $\subseteq A$ (a.e)"), is expressible by the closed statement $\int |(\phi(x) - r_1).(\phi(x) - r_2) \dots (\phi(x) - r_n)| dx = 0$. In particular, the expression $\phi(x) \overset{a.e}{=} 0$ or 1 is expressible by the statement $\int |(\phi(x) - 0).(\phi(x) - 1)| dx = 0$.

2.1. **Some new notions and facts.** In this subsection, we introduce some notions and prove some lemma which will be used later in the proof of our main results. We first recall the definition of "absolute continuity" in the measure theory. Let $(M, \mathcal{B}, \mu)$ be a measure space of finite measure and ν be another finite measure on $(M, \mathcal{B})$. We say that ν is *absolutely continuous* with respect to μ if $\nu(B) = 0$ whenever $\mu(B) = 0$. We introduce the following notion and will use it in our proofs later.

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Definition 2.7. Let $(M, \mathcal{B}, \mu)$ be a measure space of finite measure. Also let ν be a finite measure on $(M, \mathcal{B})$. For each $A \in \mathcal{B}$ with $\mu(A) > 0$ define $r(A) =: \frac{\nu(A)}{\mu(A)}$ and call it the (ν, μ)-*ratio* of A (or simply the *ratio* of A). Also if $\mu(A) = 0$ let $r(A) := 0$. For a non-negative $\alpha \in \mathbb{R}$ and any $M' \subseteq M$, we say that M' is α-*dominated* if for each measurable subset $U \subseteq M'$, we have $r(U) \leqslant \alpha$.

Let $(M, \mathcal{B}, \mu)$ be a measure space of finite measure and ν be another finite measure on $(M, \mathcal{B})$. For each $n \geqslant 2$, define $\mathcal{U}_n$ to be the family of all $A \in \mathcal{B}$ with ratio $\geqslant n$. Also define $\mathcal{H}_n$ to be the family of the members of $\mathcal{B}$ with μ-positive measure which are n-dominated. For every $A_1, A_2 \in \mathcal{B}$, define $A_1 \sim A_2$ if $\mu(A_1 \triangle A_2) = 0$. Obviously, $\sim$ is an equivalence relation on $\mathcal{B}$ (and also every $\mathcal{U}_n$ and $\mathcal{H}_n$). For every $A \in \mathcal{B}$, denote $A/\sim$ (the class of the element A in this equivalence relation) by $\tilde{A}$. Also denote the quotient space $\mathcal{B}/\sim$ by $\tilde{\mathcal{B}}$. For every $A_1, A_2 \in \mathcal{B}_n$ define $\tilde{A}_1 \leqslant_\mu \tilde{A}_2$ if $A_1 \subseteq A_2$ (*a.e.* with respect to μ). By $\tilde{A}_1 <_\mu \tilde{A}_2$ we mean $\tilde{A}_1 \leqslant_\mu \tilde{A}_2$ but $\tilde{A}_1 \neq \tilde{A}_2$. It is easy to see that $(\tilde{\mathcal{B}}, \leqslant_\mu)$ is a poset. Let $\tilde{\mathcal{U}}_n := \mathcal{U}_n/\sim$ and $\tilde{\mathcal{H}}_n := \mathcal{H}_n/\sim$ be the quotient spaces. Obviously, $\tilde{\mathcal{U}}_n$ and $\tilde{\mathcal{H}}_n$ are posets too when equipped with the order defined by $\leqslant_\mu$, and in fact are sub-posets of $(\tilde{\mathcal{B}}, \leqslant_\mu)$.

In the proof of the Radon-Nikodym theorem we will need the following technical statements.

Lemma 2.8. *Let μ and ν are finite measures on the same space and σ-algebra $(M, \mathcal{B})$. Moreover, assume that ν is absolutely continuous with respect to μ. Then, for every $n > r(M)$, $\tilde{\mathcal{H}}_n \neq \emptyset$ and has maximal element as a $\leqslant_\mu$-poset.*

Proof. We use the notations defined before. If $\mathcal{U}_{n_0}$ is empty (for some $n_0 \geqslant 2$), then M would be n_0-dominated and so for every $n \geqslant n_0$, $\tilde{\mathcal{H}}_n \neq \emptyset$ (since for example $\tilde{M}$ would belong to $\tilde{\mathcal{H}}_n$) and also it is easily seen that $\tilde{M}$ itself would be the maximal element of $\tilde{\mathcal{H}}_n$. So, from now on, we may assume that $\mathcal{U}_n$'s are all nonempty. We first show (by Zorn's lemma) that $\tilde{\mathcal{U}}_n$ has maximal element with respect to $\leqslant_\mu$ order. Let $P = \{\tilde{A}_i\}_{i<\alpha}$ be a $\leqslant_\mu$-increasing proper chain in $\tilde{\mathcal{U}}_n$ where α is an ordinal, $A_i \in \mathcal{U}_n$ for each $i < \alpha$ and by proper we mean that the members of the chain are distinct. Since P is proper, $\mu(A_{i+1} \setminus A_i) > 0$ for each $i < \alpha$. So, we must have $\alpha < \omega_1$ (which means that α is a countable ordinal) since otherwise $\mu(M)$ would be infinite. Therefore, $A := \bigcup_{i<\alpha} A_i$ is measurable and one can verify that has ratio $\geqslant n$. So $\tilde{A} \in \tilde{\mathcal{U}}_n$ and is an upper bound for the chain P. Now, by Zorn's lemma, $\tilde{\mathcal{U}}_n$ has a maximal element say S_n where $S_n := \tilde{B}_n$ for some $B_n \in \mathcal{U}_n$. We claim that $\mu(B_n^c) > 0$. The reason is that if $\mu(B_n^c) = 0$, then $\mu(B_n) = \mu(M)$ and by the absolute continuity assumption, we would have $\nu(B_n^c) = 0$ which would follow that $\nu(B_n) = \nu(M)$. Hence, we would have

$$r(M) = \frac{\nu(M)}{\mu(M)} = \frac{\nu(B_n)}{\mu(B_n)} = r(B_n) \geqslant n > r(M)$$

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which would be a contradiction. So $\mu(B_n^c) > 0$. Also B_n^c is n-dominated since otherwise there would exist $C \subseteq B_n^c$ with $\mu(C) > 0$ such that $r(C) > n$. This would imply that $\tilde{T}_n \in \tilde{\mathcal{U}}_n$ where $T_n := B_n \cup C$. The reason is that

$$r(T_n) = \frac{\nu(T_n)}{\mu(T_n)} = \frac{\nu(B_n) + \nu(C)}{\mu(B_n) + \mu(C)} = \frac{r(B_n)\mu(B_n) + r(C)\mu(C)}{\mu(B_n) + \mu(C)}$$

$$\geqslant \frac{n\mu(B_n) + n\mu(C)}{\mu(B_n) + \mu(C)} = n.$$

But it would contradict the maximality of $\tilde{B}_n$ in $\tilde{\mathcal{U}}_n$ since $\tilde{B}_n <_\mu \tilde{T}_n$. So, we conclude that B_n^c is n-dominated and also as was shown above $\mu(B_n^c) > 0$. Thus, $B_n^c \in \mathcal{H}_n$ and $\mathcal{H}_n \neq \emptyset$. It follows that $\tilde{\mathcal{H}}_n \neq \emptyset$.

Now we will show (again, by using Zorn's lemma) that $\tilde{\mathcal{H}}_n$ has maximal element. Let $P = \{\tilde{A}_i\}_{i<\alpha}$ be a $\leqslant_\mu$-increasing proper chain in $\tilde{\mathcal{H}}_n$ where α is an ordinal and $A_i \in \mathcal{H}_n$ for each $i < \alpha$. Similar to above, we have $\alpha < \omega_1$ and $A := \bigcup_{i<\alpha} A_i$ is measurable. We show that A is n-dominated. Assume not. So there is $L \subseteq A$ such that $\frac{\nu(L)}{\mu(L)} > n$. Choose $\epsilon > 0$ small enough such that $\frac{\nu(L)-\epsilon}{\mu(L)} > n$. Since $A = \bigcup_{i<\alpha} A_i$, there is index $i_0 < \alpha$ such that $\nu(A \triangle A_{i_0}) < \epsilon$. Let $L' := L \cap A_{i_0}$. So $\nu(L \triangle L') < \epsilon$. Thus,

$$\nu(L) - \nu(L') < \epsilon.$$

Hence,

$$r(L') = \frac{\nu(L')}{\mu(L')} \geqslant \frac{\nu(L) - \epsilon}{\mu(L)} > n$$

which contradicts the fact that $A_{i_0} \in \mathcal{H}_n$. It follows that A is n-dominated or equivalently speaking, $A \in \mathcal{H}_n$. So, $\tilde{A}$ is an upper bound for the chain P in $\tilde{\mathcal{H}}_n$. Now, by Zorn's lemma, $(\tilde{\mathcal{H}}_n, \leqslant_\mu)$ has maximal element. $\square$

Lemma 2.9. *Let $(M, \mathcal{B}, \mu)$ be a measure space of finite measure and ν be another finite measure on $(M, \mathcal{B})$. Also assume that ν is absolutely continuous with respect to μ. Then, there is a countable measurable partition $\{M_n\}_n$ of M such that each M_n is α_n-dominated for some $\alpha_n \in \mathbb{R}$.*

Proof. We use the notations defined before. If $\mathcal{U}_n$ is empty (for some $n \geqslant 2$), then M would be n-dominated and we would be done. So, from now on, we may assume that $\mathcal{U}_n$'s are all nonempty. Let n_0 be an integer larger than $r(M)$. By Lemma 2.8, for every integer $n \geqslant n_0$, $\tilde{\mathcal{H}}_n$ has a maximal element say $\tilde{D}_n$ for some $D_n \in \mathcal{H}_n$. One can observe that $D_n \subseteq D_{n+1}$ (a.e. with respect to μ) for each $n \geqslant n_0$ since otherwise, if we let $O := D_n \cup D_{n+1}$, then, using the $(n+1)$-domination, it would not be very hard to verify that $O \in \mathcal{H}_{n+1}$, $\tilde{O} \in \tilde{\mathcal{H}}_{n+1}$ and $\tilde{D}_{n+1} <_\mu \tilde{O}$ which contradicts the maximality of $\tilde{D}_{n+1}$ in $\tilde{\mathcal{H}}_{n+1}$. Now define $M_n := D_n \setminus D_{n-1}$ for each $n > n_0$ and $M_{n_0} := D_{n_0}$. Also define $N := M \setminus \bigcup_{n \geqslant n_0} M_n$. Note that each M_n is n-dominated since $M_n \subseteq D_n$. If we

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are able to show that $\mu(N) = 0$, then $\{N\} \cup \{M_n\}_{n \geqslant n_0}$ would be a countable partitioning of M as desired and we would be done.

Assume for contradiction that $\mu(N) > 0$. Let m be an integer larger than both $r(N)$ and n_0. Applying Lemma 2.8 on the measure space $(N, \mathcal{B}', \mu')$ and measure ν', one can find a m-dominated subset E of N with $\mu(E) > 0$ where $\mathcal{B}', \mu'$ and ν' are the restrictions of $\mathcal{B}, \mu$ and ν on N. Note that $E \cap D_m = \emptyset$ since $E \subseteq N$ and $N \cap D_m = \emptyset$. Now $V := D_m \cup E$ is m-dominated and $\tilde{D}_m <_\mu \tilde{V}$. But this contradicts the maximality of $\tilde{D}_m$ in $\tilde{\mathcal{H}}_m$. It follows that $\mu(N) = 0$. Now the proof is complete. $\square$

3. Main results

In this section, we give new proofs for two celebrated classical theorems in analysis, namely, Radon-Nikodym theorem and the Stone representation theorem for measure algebras, both by using ideas from logic and by application of an important theorem namely the logical compactness theorem in the setting of the integration logic. These proofs can be viewed as some instances of how the logical techniques, in particular logical compactness theorem, can be employed in measure theory in a systematic way for proving certain measure theoretic results.

For those readers who are not familiar with logic, we give a general picture of how logic and the logical compactness theorem comes to the picture and plays role in our proofs and also roughly explain the steps of the proofs. In the first step, we express (by suitable logical expressions in the integration logic) some properties of a measure structure which will help to prove the existence of the object we are looking for. These expressions are close to the ordinary ways in mathematics to express the properties of a measure space or a measurable function and form a possibly infinite list of expressions which, as usual, is called a theory T. Then, in the second step, we prove the finitely satisfiability of T. It means that for every finite subset of T, say T', we find a model of T', where by a model, as defined in preliminaries section, we mean a measure structure satisfying all expression in T'. In the third step, we use the logical compactness theorem in integration logic to conclude (from finitely satisfiability) that T itself has a model. It means that there exists a measure structure satisfying all expressions in T. Finally, this model helps us to quickly find the object (function or measure) we were looking for at the beginning.

3.1. **Radon-Nikodym theorem.** In this part, we give a new logical proof for the Radon-Nikodym theorem which is an important classical result in analysis.

Assume that $(X, \mathcal{B}, P)$ is a probability space and $\mathcal{A}$ a sub-σ-algebra of $\mathcal{B}$. We remind that the conditional expectation of a random variable $f : X \to \mathbb{R}$ with respect to $\mathcal{A}$ is the unique $\mathcal{A}$-measurable random variable $E(f|\mathcal{A})$ such

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that

$$\int_A f \, dP = \int_A E(f|\mathcal{A}) \, dP$$

for every $A \in \mathcal{A}$.

Theorem 3.1. (Radon-Nikodym theorem) *Let $(M, \mathcal{B}, \mu)$ be a measure space of finite measure and ν be a finite measure on $(M, \mathcal{B})$ which is absolutely continuous with respect to μ. Then, there exists a measurable function $h \geqslant 0$, called the Radon-Nikodym derivative, such that for every $B \in \mathcal{B}$, $\nu(B) = \int_B h \, d\mu$.*

Proof. We may assume $\mu(M) = \nu(M) = 1$. First, suppose there is an α such that M is α-dominated (see Definition 2.7).

Step 1: Choosing a suitable language and formalizing the properties of the required space and function

Let $\mathcal{L}$ be the language (see definitions of Section 2) consisting of a unary relation symbol f, a constant symbol c_a for each $a \in M$ and a unary relation symbol R_A for each $A \in \mathcal{B}$. We let $\flat_f = \alpha$ and $\flat_{R_A} = 1$ for each A. Let T be a $\mathcal{L}$-theory consisting of the following expressions (axioms) which can be carefully written as some closed statements in integration logic in the language $\mathcal{L}$ (one can get help from Remark 2.6 for stating them).

(1) For every distinct $a, b \in M$, add the closed statement "$\mathbf{e}(c_a, c_b) = 0$" to T.
(2) For each $a \in M$ and $A \in \mathcal{B}$, write the expression $R_A(c_a) = \chi_A(a)$ in the form of a closed statement and add it to T.
(3) For each $A \in \mathcal{B}$, write the expression $R_A(x) = 0$ or 1 $(a.e.)$ in the form of a closed statement and add it to T.
(4) For each $A, B \in \mathcal{B}$, write the expression $R_{A\cap B} = R_A \wedge R_B$ $(a.e.)$ in the form of a closed statement and add it to T.
(5) For each $A \in \mathcal{B}$, write the expression $R_{A^c} = 1 - R_A$ $(a.e.)$ in the form of a closed statement and add it to T.
(6) For each $A \in \mathcal{B}$ write the expression $\int R_A \, dx = \mu(A)$ in the form of a closed statement and add it to T.
(7) Write the expression $f(x) \geqslant 0$ $\ a.e.$ in the form of a closed statement and add it to T.
(8) For each $A \in \mathcal{B}$, write the expression $\int f \cdot R_A \, dx = \nu(A)$ in the form of a closed statement and add it to T.

We first briefly talk about the intuition behind the above axioms. Assume that a $\mathcal{L}$-structure (with a domain set, say N) satisfies all the above axioms. Then, for example axiom (1) above intuitively states that the interpretations of c_a's will be distinct elements of N. So, by using that, M can be seen a subset of N by identifying every $a \in M$ with the interpretation of c_a in N. Also axiom (2) says that the interpretation of every relation symbol R_A (which would be

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a function on N) will be the characteristic function of A when we restrict it to M. Similarly, axiom (3) guaranties that the interpretation of every R_A would be a $\{0,1\}$-valued function almost everywhere with respect to the measure on our $\mathcal{L}$-structure. Axiom (8) guaranties that the interpretation of the function symbol f will have a property close to the Radon-Nikodym derivative. But that interpretation would be a function on N not M (and indeed, some part of the rest of the proof will be for finding the Radon-Nikodym derivative on M by restricting that interpretation of f from N to M).

Step 2: Proving the finitely satisfiability of T

Let's show that T is finitely satisfiable. Obviously, M is itself an $\mathcal{L}$-structure satisfying the first six axioms of T (with interpreting R_A (for each $A \in \mathcal{B}$) in M with the characteristic function of A). So, the satisfiability of any finite part of T reduces to the satisfiability of any finite number of the statements of the form

$$f(x) \geqslant 0 \;\; a.e, \qquad \int f \cdot R_{A_i} \; d\mu = \nu(A_i) \qquad i = 1, ..., k$$

in M where we may assume (by using the axioms) that A_i's are pairwise disjoint. So we need to interpret the function symbol f in M in a suitable way. In this situation, for each $x \in M$ set

$$f^M(x) := \begin{cases} \frac{\nu(A_i)}{\mu(A_i)} & x \in A_i \;\&\;\; \mu(A_i) \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then, with this interpretation of f on M, the above mentioned finite number of statements are satisfied in M. Also due to our assumption in the beginning of the proof that M is α-dominated, we have $\frac{\nu(A)}{\mu(A)} \leqslant \alpha$ for every $A \in \mathcal{B}$ which follows that f^M is bounded by $\alpha (= \flat_f)$. So with this way of defining and interpreting symbols in M, M itself would be a model for every finite subset of the set of the axioms T. It follows that T is finitely satisfiable.

Step 3: Applying logical compactness theorem, constructing a suitable measure structure and pushing down the interpretation of f to find the Radon-Nikodym derivative

Since T was proven in previous step to be finitely satisfiable, by applying the logical compactness theorem (Theorem 2.5) which is the main logical tool we use in our proof, we conclude that T is satisfiable and has model. Let $(N, \mathcal{C}_0, \rho_0)$ be a model of T. Let $\mathcal{C} \subseteq \mathcal{C}_0$ be the minimal σ-algebra that makes the interpretations of all formulas measurable. Also let ρ be the restriction of ρ_0 on $\mathcal{C}$. Notice that $(N, \mathcal{C}, \rho)$ is also a model of T and from now on, we work with this model. By identifying every $a \in M$ with c_a^N (the interpretation of c_a in N) and also using axiom (1), we can view M as a subset of N. Using this identification and also by using axiom (2), one can see that for every $A \in \mathcal{B}$, the equality $R_A^N|_M = \chi_A$ holds where we remind that R_A^N is the interpretations of the relation symbol R_A on N and χ_A is the characteristic function of A in M respectively. For each $A \in \mathcal{B}$ define

$$A^N := \{x \in N : \ R_A^N(x) = 1\}.$$

So, $A^N \cap M = A$. By using the axioms, specially axiom (5), for every $A \in \mathcal{B}$, we have $(A^N)^c = (A^c)^N$ (ρ-a.e.) where by ρ-a.e. we mean almost everywhere with respect to the measure ρ. Also by the axioms (3) and (6), it is easy to see that $\mu(A) = \int_N R_A^N \ d\rho = \rho(A^N)$. Moreover, if $A, B \in \mathcal{B}$ and $A \subseteq B$, then $A^N \subseteq B^N$ (ρ-a.e.) where the reason is that by using axiom (4), we have $R_A^N = R_{A\cap B}^N = R_A^N \wedge R_B^N$ (ρ-a.e.) which follows that $A^N \subseteq B^N$ (ρ-a.e.). If $\{A_n\}_n$ is a countable family of pairwise disjoint members of $\mathcal{B}$, then for each index n_0 we have $A_{n_0} \subseteq \bigcup_n A_n$ and so by what just above mentioned, $A_{n_0}^N \subseteq (\bigcup_n A_n)^N$ (ρ-a.e.). Therefore, $\bigcup_n A_n^N \subseteq (\bigcup_n A_n)^N$ (ρ-a.e.). Also, again by using some axioms in particular axiom (4), every two of such A_n^N's are almost disjoint (which means that the ρ-measure of their intersection is zero). So we have

$$\rho(\bigcup_n A_n^N) = \sum_n \rho(A_n^N) = \sum_n \mu(A_n) = \mu(\bigcup_n A_n) = \rho((\bigcup_n A_n)^N).$$

Putting the above facts together, we have $\bigcup_n A_n^N = (\bigcup_n A_n)^N$ (ρ-a.e.). By some more efforts and using the axioms, if A_n's in above are not necessarily disjoint, then still $\bigcup_n A_n^N = (\bigcup_n A_n)^N$ (ρ-a.e.) holds.

Now using the above facts, the family of the sets of the form $A^N \cup E$ where $A \in \mathcal{B}$, $E \in \mathcal{C}$ and $\rho(E) = 0$ forms a sub-σ-algebra of $\mathcal{C}$ which we denote by $\mathcal{G}$. We can assume that this sub-σ-algebra $\mathcal{G}$ is equal to $\mathcal{C}$ itself since otherwise, we can replace $\mathcal{C}$ with this $\mathcal{G}$ in N and also replace f^N with the conditional expectation of f^N with respect to $\mathcal{G}$, and then by these changes, it would be easy to observe that we can obtain a new model of T with $\mathcal{G}$ as its σ-algebra and we can work with that model instead of $(N, \mathcal{C}, \rho)$ since then.

It is worth to mention that the existence of the conditional expectation can be itself deduced from the Radon-Nikodym theorem. Also it can be directly proved without using this theorem, for example in [9](page 136) which is obtained without using Radon-Nikodym theorem. If one wants to not use the assumption of the existence of the conditional expectation here in our proof, it would be enough to embed the argument of [9] for that existence in this part of the proof. So, from now on, we may assume that $\mathcal{C}$ is the σ-algebra of the sets of the form $A^N \cup E$ mentioned above and also f^N is $\mathcal{C}$-measurable.

Now, every set in $\mathcal{C}$ with positive ρ-measure have nonempty intersection with M since such set should be of the form $A^N \cup E$ for some A^N with positive measure and, as mentioned above, we have $A^N \cap M = A$. It follows that every member of $\mathcal{C}$ containing M has full ρ-measure. By the definition of the outer measure, it implies that our initial measure μ on M is exactly the same as the subspace measure induced by ρ from N and that M has full outer measure in

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N (which means that $\rho^*(M) = \rho(N)$) where recall that we are viewing M as a subset of N. Let h be the restriction of f^N to M. Then, by axiom (8) and Proposition 2.1, for each A we have that

$$\nu(A) = \int_N f^N \cdot R_A^N \; d\rho = \int_M (f^N \cdot R_A^N)|_M d\mu = \int_M h \cdot \chi_A d\mu = \int_A h \; d\mu.$$

It proves the result of the theorem but with assuming the extra assumption we added in the beginning, namely, that there is an α such that M is α-dominated. Now we give an argument for general case without that extra assumption. If such α in the assumption does not exist, then still by Lemma 2.9, there exists a countable measurable partitioning $\{M_n\}_{n<\omega}$ of M such that each M_n is α_n-dominated for some $\alpha_n \in \mathbb{R}$. Then, it is sufficient to apply the whole above argument for each M_n and find a Radon-Nikodym derivative h_n on that M_n. Then, the union of h_n's on those disjoint domains M_n's gives rise to a Radon-Nikodym derivative h on whole M. It completes the proof. $\square$

3.2. **Stone representation theorem for measure algebras.** In this part, we give a new logical proof for the Stone representation theorem for measure algebras which is an important classical result in analysis. We use the framework of the integration logic and the logical compactness theorem (Theorem 2.5) holding in it to prove this theorem.

We first review some notions from the theory of measure algebras in analysis. Recall that a Boolean algebra is σ-complete if every countable nonempty subset $a_1, a_2, \dots$ of it has a least upper bound $\vee_i a_i$ (or $\sup_{i<\omega} a_i$) and a greatest lower bound $\wedge_i a_i$ (or $\inf_{i<\omega} a_i$). A *measure algebra* is a σ-complete Boolean algebra $(B, \wedge, \vee, \,', \mathbf{0}, \mathbf{1})$ equipped with a map $\mu : B \to [0, \infty]$ such that (i) $\mu(a) = 0$ if and only if $a = \mathbf{0}$, and (ii) if $a_1, a_2, \dots$ are pairwise disjoint (i.e. $a_i \wedge a_j = \mathbf{0}$ for every distinct i and j), then $\mu(\vee_i a_i) = \sum_i \mu(a_i)$. Note that the notations $\wedge$, $\vee$ and $'$ in here stand for their corresponding operations in the Boolean algebra B and shouldn't be confused with the $\wedge$ and $\vee$ defined above which stood for the "max" and "min" of two functions. If $\mu(\mathbf{1}) = 1$, the measure algebra is called a *probability algebra*. A σ-order-continuous isomorphism (or sequentially order-continuous isomorphism) between measure algebras B_1, B_2 is a measure preserving Boolean isomorphism $\phi : B_1 \to B_2$ such that $\phi(\vee_i a_i) = \vee_i \phi(a_i)$ for every increasing sequence $a_1, a_2, ...$ in B_1. We recall that in any Boolean algebra, a partial order relation $\leqslant$ is naturally defined by $a \leqslant b$ if and only if $a \wedge b = a$.

To every measure space $(M, \mathcal{A}, \bar{\mu})$, a measure algebra is associated as follows. Say $X_1, X_2 \in \mathcal{A}$ are equivalent if their symmetric difference is measure zero with the respected measure. The equivalence class of X is denoted by $[X]$. Then the set of equivalence classes forms a Boolean algebra in the natural way and $\mu([X]) = \bar{\mu}(X)$ makes of it a measure algebra.

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Theorem 3.2. (The Stone representation theorem for measure algebras) *Let (B, μ) be a measure algebra such that μ is a bounded function. Then, there is a measure space $(M, \mathcal{B}, \bar{\mu})$ whose associated measure algebra is σ-order-continuous isomorphic to (B, μ).*

Proof. We may assume that $\mu(\mathbf{1}) = 1$. It is easy to see that if we prove the theorem with this assumption, then the general case also would be easily concluded. We present the proof in the following steps.

Step 1: Choosing a suitable language and formalizing the properties of the required space

We start by introducng a suitable language (see defnitions of Section 2) to work with. Let $\mathcal{L}$ be a language consisting of a unary relation symbol R_a with universal bound 1 for each $a \in B$. Let T be a $\mathcal{L}$-theory consisting of the following expressions (axioms) which can be carefully written as some closed statements in integration logic in the language $\mathcal{L}$ (one can get help from Remark 2.6 for stating them).

(1) For each $a \in B$, write the expression "$R_a(x) = 0$ or 1 (a.e.)" in the form of a closed statement and add it to T.
(2) For each $a \in B$, write the expression "$\int R_a(x)dx = \mu(a)$" in the form of a closed statement and add it to T.
(3) For each $a, b \in B$, write the expression "$R_{a \vee b}(x) = R_a(x) \vee R_b(x)$ (a.e.)" in the form of a closed statement and add it to T.
(4) For each $a \in B$, write the expression "$R_{a'}(x) = 1 - R_a(x)$ (a.e.)" in the form of a closed statement and add it to T.

Note that in axiom (3) above, the notation $\vee$ in the left side of the equality addresses the Boolean algebra operation while in the right side refers to the logical connective "max" between two formulas (as defined after Definition 2.3).

Step 2: Proving the finitely satisfiability of T

We will show that T is finitely satisfiable. Let T_0 be a finite subset of axioms of T. We must show that T_0 is satisfiable and for that we need to show it has a model. Let B_0 be a finite sub measure algebra of B containing every $a \in B$ for which R_a appears in axioms in T_0. Also let $M := \{a_1, ..., a_k\}$ be the atomic elements of B_0, where $a \in B_0$ is called an atom of B_0 if given any $b \in B_0$ such that $b \leqslant a$, either $b = 0$ or $b = a$. Then, μ naturally induces a probability measure ν on the finite space $(M, P(M))$ (indeed, μ induces a weighting on a_i's and by that easily forms the probability measure ν on $P(M)$).

Now to prove the satisfiability of T_0, we make a model of it over the underlying finite measure space $\mathcal{M} = (M, P(M), \nu)$ by interpreting relation symbols R_a's ($a \in B$) in $\mathcal{M}$. For each $a \in B_0$, interpret R_a with the function R_a^M

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defined by

$$R_a^M(a_i) := \begin{cases} 1 & a_i \leqslant a; \\ 0 & \text{otherwise} \end{cases}$$

for any $a_i \in M$. Also for any $a \in B \setminus B_0$, interpret R_a by any arbitrary $\{0,1\}$-valued function on M. Then, the resulting $\mathcal{L}$-structure is a model of T_0. This shows that T is finitely satisfiable.

Step 3: Applying logical compactness theorem and finding a measure space satisfying all required properties

Now in here we use the essential tool from logic, namely, the logical compactness theorem (Theorem 2.5). By this theorem and since T was proven in previous step to be finitely satisfiable, T has a model, say $(M, \mathcal{C}, \bar{\mu}; \{R_a^M\}_{a\in B})$, where each R_a^M is the interpretation of the relation symbol R_a in this model. Note that by definition of a model, $\bar{\mu}(M) = 1$. Also each R_a^M is a measurable function on M with respect to the σ-algebra $\mathcal{C}$.

Let $\mathcal{B} \subseteq \mathcal{C}$ be the smallest σ-algebra making every R_a^M measurable. Also restrict $\bar{\mu}$ to $\mathcal{B}$ and still denote the restricted measure by $\bar{\mu}$. We claim that $(M, \mathcal{B}, \bar{\mu})$ is the desired measure space whose associated measure algebra is σ-order-continuous isomorphic to the initial measure algebra B.

Notice that by axiom (1), each R_a^M is a characteristic function (up to a null set). Let $X_a := \{x \in M : R_a^M(x) = 1\}$ for every $a \in B$. Obviously, every X_a belongs to $\mathcal{B}$. For every $A \in \mathcal{B}$, let $[A]$ to be the equivalence class of A in D, where we define D to be the associated measure algebra to the measure space $(M, \mathcal{B}, \bar{\mu})$.

Define $\phi : B \to D$ by $\phi(a) := [X_a]$. We claim that ϕ is a measure algebra σ-order-continuous isomorphism. We first check the injectivity of ϕ. Assume that $\phi(a) = \phi(b)$ for some $a, b \in B$. So $[X_a] = [X_b]$ which follows that $X_a \overset{a.e}{=} X_b$ with respect to the measure $\bar{\mu}$. Then $X_a \triangle X_b$ is null. One can use the axioms to show that $X_a \triangle X_b \overset{a.e}{=} X_{a\underline{\triangle} b}$ where by $a\underline{\triangle} b$ in B we mean the element $(a \wedge b') \vee (a' \wedge b)$. So $\bar{\mu}(X_{a\underline{\triangle} b}) = 0$. Hence, by axiom (2), we have $\mu(a\underline{\triangle} b) = \int R^M_{a\underline{\triangle} b} = \bar{\mu}(X_{a\underline{\triangle} b}) = 0$. Now, by definition of a measure algebra, we have $a\underline{\triangle} b = \mathbf{0}$ which follows that $a = b$. Therefore, ϕ is injective.

Claim. Let $(b_i)_{i<\omega}$ be a sequence of elements of B. Then, $\phi(\bigvee_{i<\omega} b_i) = \bigvee_{i<\omega} \phi(b_i)$ and $\phi(\bigwedge_{i<\omega} b_i) = \bigwedge_{i<\omega} \phi(b_i)$.

Proof of Claim. First assume that $(b_i)_{i<\omega}$ is an increasing sequence of elements of B and let $b := \sup_{i<\omega} b_i$. It is easy to see that $X_{b_i} \overset{a.e}{\subseteq} X_{b_{i+1}}$ for each i and also $X_{b_i} \overset{a.e}{\subseteq} X_b$. So, $\bigcup_{i<\omega} X_{b_i} \overset{a.e}{\subseteq} X_b$. On the other hand, again by axioms, we have $\bar{\mu}(X_b) = \int R_b^M = \mu(b)$ and similarly, $\bar{\mu}(X_{b_i}) = \mu(b_i)$ for each i. Since $(b_i)_{i<\omega}$ is an increasing sequence in the measure algebra B, by a known fact we have $\mu(\sup_{i<\omega}(b_i)) = \lim_{i\to\infty} \mu(b_i) = \sup_{i<\omega} \mu(b_i)$. So we have

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$$\bar{\mu}(\bigcup_{i<\omega} X_{b_i}) = \sup_{i<\omega} \bar{\mu}(X_{b_i}) = \sup_{i<\omega} \mu(b_i) = \mu(\sup_{i<\omega}(b_i)) = \mu(b) = \bar{\mu}(X_b).$$

Combination of the above facts follows that $X_b \overset{a.e}{=} \bigcup_{i<\omega} X_{b_i}$. Thus, $[X_b] = [\bigcup_{i<\omega} X_{b_i}]$. Moreover, we have

$$\phi(\bigvee_{i<\omega} b_i) = \phi(b) = [X_b] = [\bigcup_{i<\omega} X_{b_i}] = \bigvee_{i<\omega} [X_{b_i}] = \bigvee_{i<\omega} \phi(b_i). \qquad (1)$$

It is easy to see that $\phi(a') = \phi(a)'$ for every $a \in B$. Now assume that $(b_i)_{i<\omega}$ is an arbitrary (not necessarily increasing) sequence of elements of B and let $b := \sup_{i<\omega} b_i$. Let $c_i := \bigvee_{j=1}^{i} b_j$. Now $(c_i)_{i<\omega}$ is an increasing sequence and by (1), $\phi(\bigvee_{i<\omega} c_i) = \bigvee_{i<\omega} [X_{c_i}]$. So

$$\phi(\bigvee_{i<\omega} b_i) = \phi(\bigvee_{i<\omega} c_i) = \bigvee_{i<\omega} [X_{c_i}] = \bigvee_{i<\omega} [X_{\bigvee_{j=1}^{i} b_j}]$$

$$= \bigvee_{i<\omega} (\bigvee_{j=1}^{i} [X_{b_j}]) = \bigvee_{i<\omega} [X_{b_i}] = \bigvee_{i<\omega} \phi(b_i).$$

Moreover, by using this, we also have

$$\phi(\bigwedge_{i<\omega} b_i) = \phi((\bigvee_{i<\omega} b_i')') = (\phi(\bigvee_{i<\omega} b_i'))' = (\bigvee_{i<\omega} \phi(b_i'))' = (\bigwedge_{i<\omega} (\phi(b_i'))') = \bigwedge_{i<\omega} \phi(b_i).$$

It completes the proof of the claim. □

Now we prove the surjectivity of ϕ. Note that since each R_a^M is a characteristic function (up to a null set) of the subset X_a, it is easy to see that the measure algebra D is the same as the measure algebra associated to the measure space $(M, \mathcal{B}', \bar{\mu}|_{\mathcal{B}'})$, where $\mathcal{B}'$ is the sub σ-algebra generated by X_a's. But by the definition of a generated σ-algebra, $\mathcal{B}'$ is the closure of the family of basic sets X_a's under the operations "countable unions", "countable intersections" and "complement". So, notice that in order to show that ϕ is surjective, it would be enough to prove that for any sequence $(b_i)_{i<\omega}$ of elements of B, $\bigvee_{i<\omega}[X_{b_i}]$ and $\bigwedge_{i<\omega}[X_{b_i}]$ are in the image of ϕ. But by the above claim, we have $\bigvee_{i<\omega}[X_{b_i}] = \bigvee_{i<\omega} \phi(b_i) = \phi(\bigvee_{i<\omega} b_i) \in \phi(B)$ and $\bigwedge_{i<\omega}[X_{b_i}] = \bigwedge_{i<\omega} \phi(b_i) = \phi(\bigwedge_{i<\omega} b_i) \in \phi(B)$. It follows that ϕ is surjective. Similarly, using the above claim and arguments, ϕ is a σ-order-continuous and measure-preserving Boolean isomorphism. □

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