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The Modified MCE-product: An Efficient Approach to Treat Fuzzy Coefficients in Differential Equations

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Abstract. In this paper, first, we modify the recently introduced MCE-product to include the property of shape-preserving. This product has attractive properties. For example, it is distributive with respect to the addition and it doesn't depend on the signs of multiplied fuzzy numbers. Then, the effectiveness and applicability of the modified MCE-product are investigated in treating differential equations with fuzzy multiplications. Due to the complexity of fuzzy multiplication, differential equations with fuzzy coefficients are one of the most challenging topics in the field of fuzzy differential equations. In this paper, as an example of these equations, the first-order linear differential equation with fuzzy variable coefficients is solved by using the modified MCE-product. This equation was chosen because it has been recently solved by Zadeh extension principle-based product and cross-product and we can compare our results with them. The results show the priority of the MCE-product over the mentioned methods.

Keywords: MCE-Product, GH-differentiability, Fuzzy differential equation, Analytical solution.

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1. Introduction

Nowadays, fuzzy mathematics and fuzzy logic have many applications in various fields of science and technology [13, 15, 18, 29]. Fuzzy differential equations (FDE) are used to model uncertain engineering problems [31, 33]. There are various approaches to interpret an FDE [23]. These approaches are divided into several main categories: The Zadeh extension principle approach [12], fuzzy differential inclusion [5, 27], the fuzzy bunches of function approach [21, 22], and the approaches based on fuzzy derivatives including the Hukuhara derivative [28] and its generalizations [7, 9], granular derivative [26], and interactive derivative [17, 30]. In recent decades, FDEs have developed from different areas such as numerical solution methods [1, 16, 35], existence and uniqueness results [3, 32, 34], analytical solutions [21, 22], etc. However, in the literature associated with FDE, equations involving fuzzy multiplication have been less studied. This is due to the complexities and difficulties of the fuzzy multiplication operator.

Among the proposed methods for fuzzy multiplication, the Zadeh extension based product is one of the oldest which despite its comprehensiveness, has some disadvantages limiting its use. For instance, it is not distributive with respect to the addition and doesn't preserve the shape of multiplied fuzzy numbers, it depends on the signs of multiplied fuzzy numbers and it is computationally expensive and practically difficult to use. For all these reasons, researchers in this field made a lot of effort to find alternatives to this multiplication. Recently, a new fuzzy product has been introduced which uses middle-core-ecart representation (MCE-representation, for short) and it is called MCE-product. MCE-product is distributive, easy to use and it doesn't depend on the signs of the multiplied fuzzy numbers. But it is not shape-preserving. In this work, a slight modification is done in MCE-product to take advantage of the shape-preserving property. Hereafter we use the term MMCE-product as the modified MCE-product.

The interesting features of the MMCE-product make it efficient to solve differential equations involving fuzzy products. This class of FDEs has been rarely investigated. We can mention [2, 3, 4, 14] as a few relevant work.

In this paper, the first order FDE

$$\begin{cases} u'(x) + p(x) \circledast u(x) = q(x), \\ u(x_0) = u_0 \end{cases} \tag{1.1}$$

is solved analytically in fully fuzzy form (i.e. $p, q : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ and $u_0 \in \mathbb{R}_{\mathcal{F}}$). The symbol "$\circledast$" stands for the MMCE-product. Although various works have been done in the first order fuzzy differential equations [8, 25], most of them are about nonlinear differential equations. It is well-known that the non-fuzzy form of the equation (1.1) has an analytical solution. However, obtaining this

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analytical solution in fuzzy form is complicated. In [8, 25], the equation 1.1 has been studied for crisp $p(x)$. Also, this equation in fully fuzzy form has been solved with some conditions on the signs of $p(x)$ and $q(x)$ in [4] and [14] using the cross product and the Zadeh extension based product, respectively. In the present work, we propose a solution method independent of the signs of $p(x)$ and $q(x)$. This is done based on the MMCE-product. Furthermore, we show that, in contrast to the mentioned works, the equation would have (i)- and (ii)-solutions under the same conditions.

The crisp form of the under-study differential equation has important applications in various engineering fields, such as

- Cooling of a solid body by convective heat transfer ([10]).
- Distribution of the average temperature of a fluid flowing in a tube ([10]).
- Dynamics of a particle moving in a viscous medium ([19, 20]).

It is worthy to note that in all these applications, there are inherently uncertain parameters such as geometric dimensions, thermo-physical properties of materials, and initial conditions. Basically, these uncertainties might be modeled by fuzzy numbers and functions resulting in the fuzzy form of the differential equation under study.

This paper is organized as follows: Section 2 presents some basic concepts related to the MCE-representation and MCE-product. At the end of this section, the MMCE-product is introduced. Section 3 is devoted to the derivative and integral of fuzzy number-valued functions based on the MCE-representation. The main results are given in Section 4. In the last section, several examples are presented and compared with the previous studies to show the efficiency and advantages of the proposed method.

2. Preliminaries

Throughout this paper, the space of fuzzy numbers and the space of triangular fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$ and $\mathbb{R}_{\tau}$, respectively. The notation $u_r = [u_r^-, u_r^+]$ stands for the r-cut of the fuzzy number u.

Definition 2.1. (MCE-representation [11]) For $u \in \mathbb{R}_{\mathcal{F}}$, consider the functions $\theta_u^-, \theta_u^+ : [0,1] \to \mathbb{R}_+$ defined by

$$\begin{cases} \theta_u^-(r) = m_u - u_r^- \\ \theta_u^+(r) = u_r^+ - m_u. \end{cases}$$

Where $m_u = \frac{u_1^- + u_1^+}{2}$. Then, $u = (m_u; \theta_u^-, \theta_u^+)$ is MCE-representation of u. Note that the semicolon symbol makes this different from the well-known notation of a typical triangular fuzzy number denoted by (a, b, c). Hereafter, fuzzy numbers are assumed to be in the form of MCE-representation.

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$(m_u; \theta_u^-, \theta_u^+)$ represents a fuzzy number if and only if θ_u^-, θ_u^+ are bounded, positive, non-increasing, left-continuous on $(0, 1]$ and right-continuous at 0.

Definition 2.2. Let $u = (m_u; \theta_u^-, \theta_u^+)$ and $v = (m_v; \theta_v^-, \theta_v^+)$. MCE-product of u and v is defined as

$$u \odot v = (m_u m_v; \theta_u^- \theta_v^-, \theta_u^+ \theta_v^+).$$

The r-cuts of $u \odot v$ is $(u \odot v)_r = [w_r^-, w_r^+]$, where

$$w_r^- = m_u m_v - \theta_u^- \theta_v^-$$
$$w_r^+ = m_u m_v + \theta_u^+ \theta_v^+$$

for $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\alpha \in \mathbb{R}$, the sum and scalar multiplication are defined as

$$u + v = (m_u + m_v; \theta_u^- + \theta_v^-, \theta_u^+ + \theta_v^+),$$

$$\alpha u = \begin{cases} (\alpha m_u; \alpha\theta_u^-, \alpha\theta_u^+), & \alpha \geq 0, \\ (\alpha m_u; -\alpha\theta_u^+, -\alpha\theta_u^-), & \alpha < 0. \end{cases}$$

Theorem 2.3. *Let $u = (m_u; \theta_u^-, \theta_u^+)$, $v = (m_v; \theta_v^-, \theta_v^+)$ and $w = (m_w; \theta_w^-, \theta_w^+)$ are three fuzzy numbers. Also assume $\overline{0} = (0; 0, 0)$ and $\overline{1} = (1; 1, 1)$. Then we have the following properties:*

- **(i)** *Commutativity:* $u \odot v = v \odot u$
- **(ii)** *Associativity:* $(u \odot v) \odot w = u \odot (v \odot w)$
- **(iii)** *Distributivity:* $(u + v) \odot w = (u \odot w) + (v \odot w)$
- **(iv)** *Neutral member:* $u \odot \overline{1} = u$
- **(v)** $u \odot u = \overline{0}$ *iff* $u = \overline{0}$.

For more details about the MCE-product, see [11].

In addition to the properties provided in this theorem, the MCE-product is easy to use and independent of the signs of multiplied fuzzy numbers. These properties make it useful to solve a class of differential equations that includes fuzzy multiplication. Despite the interesting properties of the MCE-product, it does not preserve the shape of the multiplied fuzzy numbers. To remove this shortcoming, we define the modified form of the MCE-product as follows.

Let $u = (a, b, c)$ be a typical triangular fuzzy number. The MCE-representation of u is

$$u = \Big(b; \big(b - a\big)(1 - r), \big(c - b\big)(1 - r)\Big).$$

which is in the form of $\big(m_u; k_u^-(1 - r), k_u^+(1 - r)\big)$, where $k_u^-, k_u^+ \in \mathbb{R}^+$, $k_u^- = b - a$, $k_u^+ = c - b$ and $m_u = b$.

Definition 2.4. (MMCE-product) Let $u, v \in \mathbb{R}_\tau$. The MMCE-product is defined as follows

$$u \circledast v = \big(m_u m_v; k_u^- k_v^- (1 - r), k_u^+ k_v^+ (1 - r)\big).$$

Clearly $u \circledast v \in \mathbb{R}_\tau$.

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In Fig. 1, the MCE- and MMCE-products are shown for $u = \big(0; 1-r, 2(1-r)\big)$ and $v = \big(4; 1-r, 2(1-r)\big)$.

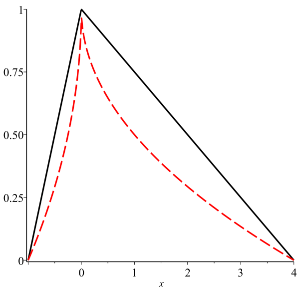

FIGURE 1. MCE-product (dash line) and MMCE-product (solid line) of $u = \big(0; 1-r, 2(1-r)\big)$ and $v = \big(4; 1-r, 2(1-r)\big)$.

The r-cuts of $u \circledast v$ is $(u \circledast v)_r = [w_r^-, w_r^+]$, where

$$w_r^- = m_u m_v - k_u^- k_v^-(1-r)$$
$$w_r^+ = m_u m_v + k_u^+ k_v^+(1-r).$$

The standard representation of $u \circledast v$ as a triangular fuzzy number is

$$u \circledast v = (m_u m_v - k_u^- k_v^-, m_u m_v, m_u m_v + k_u^+ k_v^+).$$

Under MMCE-product, the inverse of $u = \big(m_u; k_u^-(1-r), k_u^+(1-r)\big)$ is $u^{-1} = \big(\frac{1}{m_u}; \frac{1}{k_u^-}(1-r), \frac{1}{k_u^+}(1-r)\big)$.

Theorem 2.5. *Let* $u = (m_u; k_u^-(1-r), k_u^+(1-r))$, $v = (m_v; k_v^-(1-r), k_v^+(1-r))$ *and* $w = (m_w; k_w^-(1-r), k_w^+(1-r))$ *are three fuzzy numbers,* $\overline{0} = (0; 0, 0)$ *and* $\overline{1} = (1; 1-r, 1-r)$. *Then we have the following properties:*

- **(i)** *Commutativity:* $u \circledast v = v \circledast u$
- **(ii)** *Associativity:* $(u \circledast v) \circledast w = u \circledast (v \circledast w)$
- **(iii)** *Distributivity:* $(u + v) \circledast w = (u \circledast w) + (v \circledast w)$
- **(iv)** *Neutral member:* $u \circledast \overline{1} = u$
- **(v)** $u \circledast u = \overline{0}$ *iff* $u = \overline{0}$.

Proof. The proof is trivial and very similar to the Theorem 3.2 of [11] and thus we omit it here. □

3. Calculus of fuzzy number-valued functions using MCE-representation

In this section, we obtain the GH-derivative and integral of a fuzzy number-valued function based on the MCE-representation. Let us denote the MCE-representation of an arbitrary fuzzy function $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ with $f(x) = (m_f(x); \theta_f^-(x, r), \theta_f^+(x, r))$. In the following theorem, we obtain MCE-representation of (i)- and (ii)-GH-derivative of a fuzzy function.

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Theorem 3.1. *Let* $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$.

- **(i)** *If* f *is (i)-GH-differentiable, then* $f'(x) = \big(m'_f(x); (\theta_f^-)'(x,r), (\theta_f^+)'(x,r)\big)$.
- **(ii)** *If* f *is (ii)-GH-differentiable, then* $f'(x) = \big(m'_f(x); -(\theta_f^+)'(x,r), -(\theta_f^-)'(x,r)\big)$.

Here, the symbol " ′ " denotes derivative with respect to x.

Proof. If f is (i)-GH-differentiable, then $\big[f'(x)\big]_r = \big[(f_r^-)'(x), (f_r^+)'(x)\big]$. Thus, we can write

$$\begin{aligned} f'(x) &= \Big(\frac{(f_1^-)'(x) + (f_1^+)'(x)}{2}; \frac{(f_1^-)'(x) + (f_1^+)'(x)}{2} \\ &\quad - (f_r^-)'(x), (f_r^+)'(x) - \frac{(f_1^-)'(x) + (f_1^+)'(x)}{2}\Big) \\ &= \big(m'_f(x); (\theta_f^-)'(x,r), (\theta_f^+)'(x,r)\big). \end{aligned}$$

If f is (ii)-GH-differentiable, then $\big[f'(x)\big]_r = \big[(f_r^+)'(x), (f_r^-)'(x)\big]$. Thus, the MCE-representation of $f'(x)$ is

$$\begin{aligned} f'_r &= \Big(\frac{(f_1^-)' + (f_1^+)'}{2}; \frac{(f_1^-)' + (f_1^+)'}{2} - (f_r^+)', (f_r^-)' - \frac{(f_1^-)' + (f_1^+)'}{2}\Big) \\ &= \Big(\frac{(f_1^-)' + (f_1^+)'}{2}; -((f_r^+)' - \frac{(f_1^-)' + (f_1^+)'}{2}), -(\frac{(f_1^-)' + (f_1^+)'}{2} - (f_r^-)')\Big) \\ &= \big(m'_f(x); -(\theta_f^+)'(x,r), -(\theta_f^-)'(x,r)\big). \end{aligned}$$

□

The following results can be deduced from Theorem 3.1, immediately.

Corollary 3.2. *If* f *is (i)- or (ii)-GH-differentiable at* x*, then we have*

- **(a)** $f'(x) = \big(m'_f(x); \max\{-\theta_f^{+\prime}(x,r), \theta_f^{-\prime}(x,r)\}, \max\{\theta_f^{+\prime}(x,r), -\theta_f^{-\prime}(x,r)\}\big)$,
- **(b)** $\theta_f^{+\prime}(x,r)\theta_f^{-\prime}(x,r) \geq 0$.

Theorem 3.3. *Let* $f : (a,b) \to \mathbb{R}_\tau$, $f(x) = \big(m_f(x), k_f^-(x)(1-r), k_f^+(x)(1-r)\big)$ *and* $m_f, k_f^-, k_f^+ \in C(a,b)$.

- *If* $\big(k_f^-(x)\big)' > 0$ *and* $\big(k_f^+(x)\big)' > 0$*, then* f *is (i)-GH-differentiable and*
$$f'(x) = \big((m_f(x))', (k_f^-(x))'(1-r), (k_f^+)'(x)(1-r)\big).$$
- *If* $\big(k_f^-(x)\big)' < 0$ *and* $\big(k_f^+(x)\big)' < 0$*, then* f *is (ii)-GH-differentiable and*
$$f'(x) = \big((m_f(x))', -(k_f^+(x))'(1-r), -(k_f^-)'(x)(1-r)\big).$$

Proof. Let $x \in (a,b)$ be given. If $\big(k_f^-(x)\big)' > 0$ and $\big(k_f^+(x)\big)' > 0$. By using the Mean Value Theorem, for sufficiently small $h > 0$, there exist $\xi_1, \xi_2 \in (x, x+h)$ such that $\big(k_f^-(\xi_1)\big)', \big(k_f^+(\xi_2)\big)' > 0$ and

$$k_f^-(x+h) = k_f^-(x) + h\big(k_f^-(\xi_1)\big)' \tag{3.1}$$

$$k_f^+(x+h) = k_f^+(x) + h\big(k_f^+(\xi_2)\big)'. \tag{3.2}$$

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On the other hand,

$$m_f(x+h) = m_f(x) + h\big(m_f(\xi_3)\big)', \;\; \xi_3 \in (a,b). \tag{3.3}$$

From (3.1)-(3.3), one can conclude

$$\begin{aligned}\big(m_f(x+h); k_f^-(x+h)(1-r), k_f^+(x+h)(1-r)\big) &= \big(m_f(x); k_f^-(x)(1-r), k_f^+(x)(1-r)\big)\\ &+ h\Big(\big(m_f(\xi_3)\big)'; \big(k_f^-(\xi_1)\big)'(1-r), \big(k_f^+(\xi_2)\big)'(1-r)\Big)\end{aligned}$$

which means

$$f(x+h) \ominus_H f(x) = h\Big(\big(m_f(\xi_3)\big)'; \big(k_f^-(\xi_1)\big)'(1-r), \big(k_f^+(\xi_2)\big)'(1-r)\Big).$$

Thus,

$$\begin{aligned}\lim_{h\searrow 0} \frac{f(x+h)\ominus_H f(x)}{h} &= \lim_{h\searrow 0}\Big(\big(m_f(\xi_3)\big)'; \big(k_f^-(\xi_1)\big)'(1-r), \big(k_f^+(\xi_2)\big)'(1-r)\Big)\\ &= \Big(\big(m_f(x)\big)'; \big(k_f^-(x)\big)'(1-r), \big(k_f^+(x)\big)'(1-r)\Big).\end{aligned}$$

The proof of the second part is very similar to the first one. Thus we skip it. □

In the following, the integral of a fuzzy function is given by using the MCE representation. Throughout this paper, we will use the fuzzy Riemann integral for the concept of integral [6].

Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy Riemann integrable, then

$$\Big[\int_a^b f(x)dx\Big]_r = \Big[\int_a^b f_r^-(x)dx, \int_a^b f_r^+(x)dx\Big]$$

using MCE-representation, it is

$$\int_a^b f(x)dx = \Big(\int_a^b m_f(x)dx; \int_a^b \theta_f^-(x,r)dx, \int_a^b \theta_f^+(x,r)dx\Big).$$

4. The linear first order fuzzy differential equation

Here, we study the following fuzzy initial value problem

$$\begin{cases} u'(x) + p(x) \circledast u(x) = q(x), \\ u(x_0) = u_0 \end{cases} \tag{4.1}$$

where $p, q : \mathbb{R} \to \mathbb{R}_\tau$ and $u_0 \in \mathbb{R}_\tau$.

Let

$$\begin{aligned} u(x) =& \big(m_u(x); k_u^-(x)(1-r), k_u^+(x)(1-r)\big),\\ p(x) =& \big(m_p(x); k_p^-(x)(1-r), k_p^+(x)(1-r)\big),\\ q(x) =& \big(m_q(x); k_q^-(x)(1-r), k_q^+(x)(1-r)\big).\end{aligned}$$

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Definition 4.1. We say that $u : \mathbb{R} \to \mathbb{R}_\tau$ is a solution for the problem (4.1) if it is GH-differentiable and satisfies the problem (4.1) for all $x \in \mathbb{R}$. According to the type of GH-differentiability, two types of solution can be considered:

- A fuzzy solution u is called (i)-solution, if u is (i)-GH-differentiable.
- A fuzzy solution u is called (ii)-solution, if u is (ii)-GH-differentiable.

Theorem 4.2. *((i)-solution) Let*

$$\begin{aligned} F(x) &= \big(e^{\int m_p(x)dx}; e^{\int k_p^-(x)dx}(1-r), e^{\int k_p^+(x)dx}(1-r)\big),\\ F(x_0) &= \big(m_{F0}, k_{F0}^-(1-r), k_{F0}^+(1-r)\big),\\ u_0 &= \big(m_{u0}, k_{u0}^-(1-r), k_{u0}^+(1-r)\big). \end{aligned}$$

If

$$k_q^-(x) - k_{F0}^- k_{u0}^- k_p^-(x) e^{-\int k_p^-(x)dx} - k_p^-(x) e^{-\int k_p^-(x)dx} \int_{x_0}^{x} k_q^-(t) e^{\int k_p^-(t)dt} dt > 0 \tag{4.2}$$

$$k_q^+(x) - k_{F0}^+ k_{u0}^+ k_p^+(x) e^{-\int k_p^+(x)dx} - k_p^+(x) e^{-\int k_p^+(x)dx} \int_{x_0}^{x} k_q^+(t) e^{\int k_p^+(t)dt} dt > 0, \tag{4.3}$$

then

$$u(x) = F(x)^{-1} \circledast F(x_0) \circledast u_0 + F(x)^{-1} \circledast \int_{x_0}^{x} F(t) \circledast q(t)dt \tag{4.4}$$

is (i)-solution of (4.1).

Proof. Suppose that

$$u(x) = F(x)^{-1} \circledast F(x_0) \circledast u_0 + F(x)^{-1} \circledast \int_{x_0}^{x} F(t) \circledast q(t)dt.$$

By substituting $F(x)$, $F(x_0)$, $u(x_0)$ and $q(t)$ in u, we have

$$u(x) = \big(m_u(x); k_u^-(x)(1-r), k_u^+(x)(1-r)\big) \tag{4.5}$$

where

$$\begin{aligned} m_u(x) =& e^{-\int m_p(x)dx}\big[m_{F0} m_{u0} + \int_{x_0}^{x} e^{\int m_p(t)dt} m_q(t)dt\big]\\ k_u^-(x) =& e^{-\int k_p^-(x)dx}\big[k_{F0}^- k_{u0}^- + \int_{x_0}^{x} e^{\int k_p^-(t)dt} k_q^-(t)dt\big]\\ k_u^+(x) =& e^{-\int k_p^+(x)dx}\big[k_{F0}^+ k_{u0}^+ + \int_{x_0}^{x} e^{\int k_p^+(t)dt} k_q^+(t)dt\big]. \end{aligned}$$

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Since

$$\begin{aligned}\frac{d}{dx}k_u^-(x) =& k_q^-(x) - k_{F0}^- k_{u0}^- k_p^-(x) e^{-\int k_p^-(x)dx} \\ & - k_p^-(x) e^{-\int k_p^-(x)dx} \int_{x_0}^{x} k_q^-(t) e^{\int k_p^-(t)dt} dt \qquad (4.6)\\ \frac{d}{dx}k_u^-(x) =& k_q^+(x) - k_{F0}^+ k_{u0}^+ k_p^+(x) e^{-\int k_p^+(x)dx} \\ & - k_p^+(x) e^{-\int k_p^+(x)dx} \int_{x_0}^{x} k_q^+(t) e^{\int k_p^+(t)dt} dt. \qquad (4.7)\end{aligned}$$

From these equalities, (4.2), (4.3) and Theorem 3.3, u is (i)-Gh-differentiable.

Now, we show that u satisfies the problem 4.1. Clearly $u(x_0) = u_0$ and moreover, since u is (i)-Gh-differentiable,

$$u'(x) = \big(\frac{d}{dx}m_u(x); \frac{d}{dx}k_u^-(x)(1-r), \frac{d}{dx}k_u^+(x)(1-r)\big) \qquad (4.8)$$

where

$$\begin{aligned}\frac{d}{dx}m_u(x) = & m_q(x) - m_{F0} m_{u0} m_p(x) e^{\int -m_p(x)dx} \\ & - m_p(x) e^{\int -m_p(x)dx} \int_{x_0}^{x} e^{\int m_p(t)dt} m_q(t) dt. \qquad (4.9)\end{aligned}$$

By replacing $u'(x)$ and $u(x)$ in $u'(x) + p(x)u(x)$ with the right have sides of (4.8) and (4.5), respectively, we have

$$\begin{aligned}u'(x) + p(x)u(x) = & \Big(\frac{d}{dx}m_u(x) + m_p(x)m_u(x); \big(\frac{d}{dx}k_u^-(x) + k_p^- k_u^-\big)(1-r), \\ & \big(\frac{d}{dx}k_u^+(x) + k_p^+ k_u^+\big)(1-r)\Big) \qquad (4.10)\end{aligned}$$

where

$$\begin{aligned}\frac{d}{dx}m_u(x) + m_p(x)m_u(x) =& m_q(x) - m_{F0} m_{u0} m_p(x) e^{\int -m_p(x)dx} \\ & - m_p(x) e^{\int -m_p(x)dx} \int_{x_0}^{x} e^{\int m_p(t)dt} m_q(t) dt \\ & + m_p(x) e^{-\int m_p(x)dx} \big[m_{F0} m_{u0} + \int_{x_0}^{x} e^{\int m_p(t)dt} m_q(t) dt\big] \\ =& m_q(x) \qquad (4.11)\end{aligned}$$

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$$
\begin{aligned}
\frac{d}{dx}k_u^-(x)+k_p^-(x)k_u^-(x) =& k_q^-(x)-k_{F0}^-k_{u0}^-k_p^-(x)e^{-\int k_p^-(x)dx}\\
&-k_p^-(x)e^{-\int k_p^-(x)dx}\int_{x_0}^{x}k_q^-(t)e^{\int k_p^-(t)dt}dt\\
&+k_p^-(x)e^{-\int k_p^-(x)dx}\big[k_{F0}^-k_{u0}^-+\int_{x_0}^{x}e^{\int k_p^-(t)dt}k_q^-(t)dt\big]\\
=&k_q^-(x) \qquad (4.12)\\
\frac{d}{dx}k_u^+(x)+k_p^+(x)k_u^+(x) =& k_q^+(x)-k_{F0}^+k_{u0}^+k_p^+(x)e^{-\int k_p^+(x)dx}\\
&-k_p^+(x)e^{-\int k_p^+(x)dx}\int_{x_0}^{x}k_q^+(t)e^{\int k_p^+(t)dt}dt\\
&+k_p^+(x)e^{-\int k_p^+(x)dx}\big[k_{F0}^+k_{u0}^++\int_{x_0}^{x}e^{\int k_p^+(t)dt}k_q^+(t)dt\big]\\
=&k_q^+(x). \qquad (4.13)
\end{aligned}
$$

From (4.10), (4.11), (4.12) and (4.12),

$$u'(x)+p(x)u(x)=\Big(m_q(x);k_q^-(x)(1-r),k_q^+(x)(1-r)\Big)=q(x).$$

Consequently, u satisfy problem (4.1). □

(ii)-solution: We assume u is (ii)-GH-differentiable. In this case, there is not any integrating factor but the equation is still solvable. Since u is (ii)-GH-differentiable, $u'(x)=\big(m_u'(x);-(k_u^+)'(x)(1-r),-(k_u^-)'(x)(1-r)\big)$. From the definition of the MMCE-product, we obtain

$$
\begin{aligned}
(m_u';-(k_u^+)'(1-r),-(k_u^-)'(1-r))+(m_p;k_p^-(1-r),k_p^+(1-r))\odot(m_u;k_u^-(1-r),k_u^+(1-r))\\
=(m_q;k_q^-(1-r),k_q^+(1-r))\\
(m_u'+m_pm_u;-(k_u^+)'+k_p^-k_u^-(1-r),-(k_u^-)'+k_p^+k_u^+(1-r))=(m_q;k_q^-(1-r),k_q^+(1-r)).
\end{aligned}
$$

Thus, we have the following three crisp initial value problems

$$
\begin{cases}
m_u'+m_pm_u=m_q, & m_u(0)=m_0\\
-(k_u^+)'+k_p^-k_u^-=k_q^-, & k_u^-(0)=k_0^-\\
-(k_u^-)'+k_p^+k_u^+=k_q^+, & k_u^+(0)=k_0^+.
\end{cases}
$$

After solving this system of differential equations, if $k_u^-,k_u^+\geq 0$ and $\big(k_u^-\big)',\big(k_u^+\big)'<0$, the equation has a (ii)-solution, otherwse it doesn't have a (ii)-solution.

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Remark 4.3. It is worth noting that in the field of fuzzy mathematics the following initial value problems are different [8]:

$$\begin{cases} u'(x) + p(x) \circledast u(x) = q(x), \\ u(x_0) = u_0 \end{cases}$$

$$\begin{cases} u'(x) = (-p(x) \circledast u(x)) + q(x), \\ u(x_0) = u_0 \end{cases}$$

$$\begin{cases} u'(x) + (-q(x)) = (-p(x) \circledast u(x)), \\ u(x_0) = u_0. \end{cases}$$

In the current work, we explain how to calculate the solution to the first problem. The other two cases are similarly solvable (see Examples 5.2 and 5.3).

5. Examples

In this section, we provide three examples. The first example is solved using the procedure presented in Section 4. For comparing the results of the present method with the previous studies, two other examples are given from [4] and [14].

Example 5.1. Consider the following fuzzy differential equation with initial condition

$$\begin{cases} u' + \big(1; \frac{1}{x+1}(1-r), \frac{1}{x+1}(1-r)\big) \circledast u = \big(0; x(1-r), x(1-r)\big), \text{ for } x \in [1, 1.8), \\ u(1) = \big(e^{-1}; \frac{11}{12}(1-r), \frac{11}{12}(1-r)\big). \end{cases} \tag{5.1}$$

(i)-solution. By the method described in Theorem 4.2, the (i)-solution is

$$u(x) = \Big(e^{-x}; \frac{\frac{x^3}{3} + \frac{x^2}{2} + 1}{x+1}(1-r), \frac{\frac{x^3}{3} + \frac{x^2}{2} + 1}{x+1}(1-r)\Big).$$

(ii)-solution. Corresponding system is:

$$\begin{cases} m_u' + m_u = 0, & m_u(1) = -1 \\ -(k_u^-)' + \frac{1}{1+x} k_u^+ = x, & k_u^-(1) = \frac{11}{12} \\ -(k_u^+)' + \frac{1}{1+x} k_u^- = x, & k_u^+(1) = \frac{11}{12}. \end{cases}$$

By solving the above system, the (ii)-solution of (5.1) is obtained as follows

$$u(x) = \Big(e^{-x}; (1+x)(\frac{35}{24} - x + ln(\frac{1+x}{2}))(1-r), (1+x)(\frac{35}{24} - x + ln(\frac{1+x}{2}))(1-r)\Big).$$

Both of the solutions are illustrated in Figure 2.

Example 5.2. ([4, 14]) Consider the following initial value problem

$$\begin{cases} u'(x) = (2x; x(1-r), x(1-r)) \circledast u(x) + (x; \frac{x}{2}(1-r), \frac{x}{2}(1-r)), \\ u(0) = (-1; \frac{1}{2}, \frac{1}{2}). \end{cases} \tag{5.2}$$

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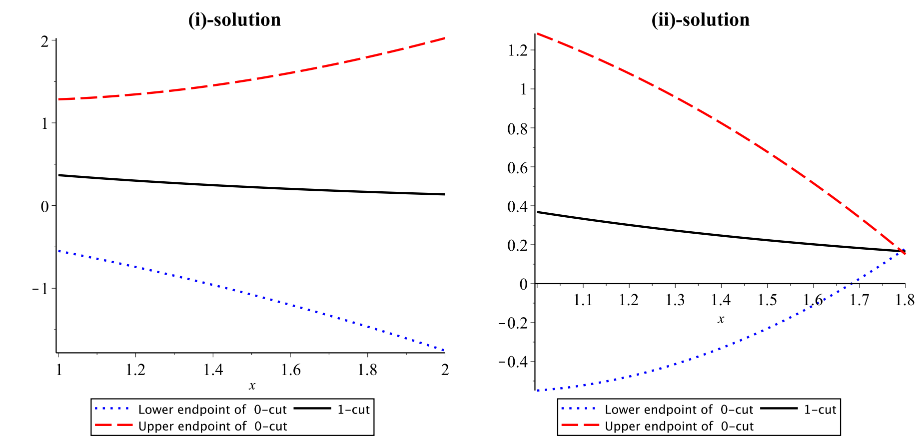

FIGURE 2. The (i)- and (ii)-solutions of Example 5.1.

This is Example 5.1 of [4] and Example 5.2 of [14].

(i)-solution. Corresponding system is:

$$\begin{cases} m_u' = 2xm_u + x, & m_u(0) = -1 \\ (k_u^-)' = xk_u^- + \frac{x}{2}, & k_u^-(0) = \frac{1}{2} \\ (k_u^+)' = xk_u^+ + \frac{x}{2}, & k_u^+(0) = \frac{1}{2}. \end{cases}$$

By solving the above systems, the (i)-solution of (5.3) is

$$u(x) = \Big(\frac{-1}{2}\big(1 + e^{x^2}\big); \big(e^{\frac{x^2}{2}} - \frac{1}{2}\big)(1 - r), \big(e^{\frac{x^2}{2}} - \frac{1}{2}\big)(1 - r)\Big)$$

with the r-cuts

$$u_r(x) = \left[\frac{-1}{2}\big(1 + e^{x^2}\big) - \big(e^{\frac{x^2}{2}} - \frac{1}{2}\big)(1 - r), \frac{-1}{2}\big(1 + e^{x^2}\big) + \big(e^{\frac{x^2}{2}} - \frac{1}{2}\big)(1 - r)\right].$$

(ii)-solution. Corresponding system is:

$$m_u' = m_u + x \quad , \quad m_u(0) = -1$$

$$(k_u^-)' = -xk_u^+ - \frac{x}{2} \quad , \quad k_u^-(0) = \frac{1}{2}$$

$$(k_u^+)' = -xk_u^- - \frac{x}{2} \quad , \quad k_u^+(0) = \frac{1}{2}.$$

By solving the above system, the (ii)-solution reads

$$y(x) = \Big(- \frac{1}{2}(1 + e^{x^2}); (- \sinh(\frac{x^2}{2}) + \cosh(\frac{x^2}{2}) - \frac{1}{2})(1 - r), (- \sinh(\frac{x^2}{2}) + \cosh(\frac{x^2}{2}) - \frac{1}{2})(1 - r)\Big).$$

In Figure 3, both of the (i)- and (ii)-solutions of this example are illustrated. The (i)-solution is compared with the corresponding solutions presented in [4]

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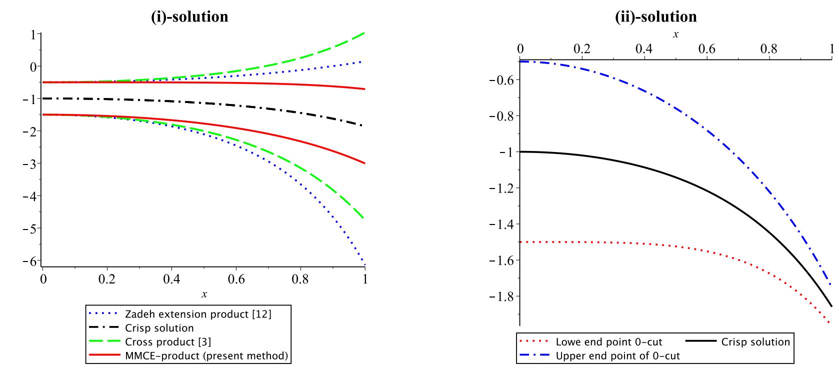

FIGURE 3. The (i)- and (ii)-solutions of Example 5.2.

and [14]. As can be seen, the growth of uncertainty in the present method is significantly less than the two other methods. Also, in contrast to the methods presented in [4] and [14], our method provides a (ii)-solution, as well.

EXAMPLE 5.3. ([4, 14]) Consider the following initial value problem

$$\begin{cases} u'(x) = (-2x; x(1-r), x(1-r)) \circledast u(x) + (x; \frac{x}{2}(1-r), \frac{x}{2}(1-r)), \\ u(0) = (-1; \frac{1}{2}, \frac{1}{2}). \end{cases} \tag{5.3}$$

This is Example 5.2 of [4] and Example 5.3 of [14].

(i)-solution. Corresponding system is:

$$\begin{cases} m_u' = -2x m_u + x, & m_u(0) = -1 \\ (k_u^-)' = x k_u^- + \frac{x}{2}, & k_u^-(0) = \frac{1}{2} \\ (k_u^+)' = x k_u^+ + \frac{x}{2}, & k_u^+(0) = \frac{1}{2}. \end{cases}$$

By solving the above system, (i)-solution of (5.3) is

$$y(x) = \Big(\frac{1}{2} - \frac{3}{2}e^{x^2}; \big(e^{\frac{x^2}{2}} - \frac{1}{2}\big)(1-r), \big(e^{\frac{x^2}{2}} - \frac{1}{2}\big)(1-r)\Big).$$

(ii)-solution. Corresponding system is:

$$m_u' = -2x m_u + x, \qquad m_u(0) = -1$$
$$(k_u^-)' = -x k_u^+ - \frac{x}{2}, \qquad k_u^-(0) = \frac{1}{2}$$
$$(k_u^+)' = -x k_u^- - \frac{x}{2}, \qquad k_u^+(0) = \frac{1}{2}.$$

By solving the above system, (ii)-solution is

$$y(x) = \Big(\frac{1}{2} - \frac{3}{2}e^{x^2}; \big(-\sinh(\frac{x^2}{2}) + \cosh(\frac{x^2}{2}) - \frac{1}{2}\big)(1-r), \big(-\sinh(\frac{x^2}{2}) + \cosh(\frac{x^2}{2}) - \frac{1}{2}\big)(1-r)\Big).$$

[DOI: 10.61882/ijmsi.20.2.1]

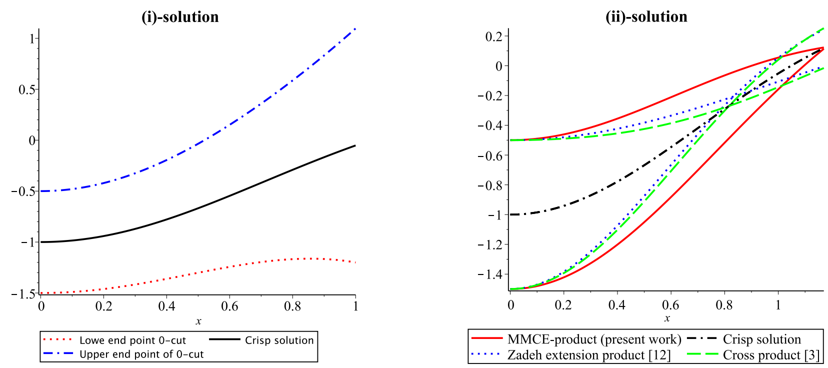

FIGURE 4. The (i)- and (ii)-solutions of Example 5.3.

In Figure 4, (i)- and (ii)-solutions are illustrated. The (ii)-solution is compared with the results of [4] and [14] (see Figure 4-right). It is clearly observed that, for the present method, the uncertainty vanishes at a slower rate than the other two methods. This makes the (ii)-solution valid over a larger interval. As seen, the validity interval of the solution in our method is $[0, 1.17]$ while, it is $[0, 0.816]$, and $[0, 0.707]$ for the methods presented in [4], and [14], respectively. Also, in contrast to the previous works, our method provides a (i)-solution (see Figure 4-left).

Conclusion and future research

The linear first order differential equation with fuzzy variable coefficients and fuzzy initial value was solved analytically. We modified the MCE-product and used it for the concept of fuzzy multiplication appeared in the equation. Depending on the type of GH-differentiability, two types of solution called (i)- and (ii)-solutions were proposed. Some examples were given that show the efficiency of the proposed method compared to the previous methods available in the literature (see examples 5.2 and 5.3). For the future work, we suggest to use modified MCE-product for obtaining the analytical solution of the higher order fuzzy differential equations.

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