

## A Necessary Condition for Zero Divisors in Complex Group Algebra of Torsion-Free Groups

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**ABSTRACT.** It is proved that if  $\sum_{g \in G} a_g g$  is a non-zero zero divisor element of the complex group algebra  $\mathbb{C}G$  of a torsion-free group  $G$  then  $2 \sum_{g \in G} |a_g|^2 < (\sum_{g \in G} |a_g|)^2$ .

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### 1. INTRODUCTION AND RESULTS

Let  $G$  be any group and  $\mathbb{C}G$  be the complex group algebra of  $G$ , i.e. the set of finitely supported complex functions on  $G$ . We may represent an element  $\alpha$  in  $\mathbb{C}G$  as a formal sum  $\sum_{g \in G} a_g g$ , where  $a_g \in \mathbb{C}$  is the value of  $\alpha$  in  $g$ . The multiplication in  $\mathbb{C}G$  is defined by

$$\alpha\beta = \sum_{g,h \in G} a_g b_h gh = \sum_{g \in G} \left( \sum_{x \in G} a_{gx^{-1}} b_x \right) x$$

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for  $\alpha = \sum_{g \in G} a_g g$  and  $\beta = \sum_{g \in G} b_g g$  in  $\mathbb{C}G$ . We shall say that  $\alpha$  is a *zero divisor* if there exists  $0 \neq \beta \in \mathbb{C}G$  such that  $\alpha\beta = 0$ . If there is a non-zero  $\beta \in \ell^2(G)$  such that  $\alpha\beta = 0$ , then we may say that  $\alpha$  is *analytical zero divisor*. If  $\alpha\beta \neq 0$  for all  $0 \neq \beta \in \mathbb{C}G$ , then we say that  $\alpha$  is *regular*. The following conjecture is called the *zero divisor conjecture*.

*Conjecture 1.1.* Let  $G$  be a torsion-free group. Then all elements in  $\mathbb{C}G$  are regular.

Conjecture 1.1 is still open; it has been proven affirmative when  $G$  belongs to special classes of groups; ordered groups ([11] and [12]), supersolvable groups [6], polycyclic-by-finite groups ([1] and [5]) and unique product groups [2]. Delzant [3] deals with group rings of word-hyperbolic groups and proves the conjecture for certain word-hyperbolic groups. Let  $\mathcal{C}$  be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Let  $G$  be a torsion-free group in  $\mathcal{C}$  then  $G$  satisfies Conjecture 1.1 [8].

The map  $\langle \cdot, \cdot \rangle : \mathbb{C}G \times \mathbb{C}G \rightarrow \mathbb{R}$  defined by

$$\langle \alpha, \beta \rangle := \sum_{g \in G} a_g \bar{b}_g \quad (\alpha, \beta \in \mathbb{C}G)$$

is an inner product on  $\mathbb{C}G$ , so  $\|\alpha\|_2 = \langle \alpha, \alpha \rangle^{\frac{1}{2}}$  becomes a norm, called 2-norm; the completion of  $\mathbb{C}G$  w.r.t. 2-norm is the Hilbert space  $\ell^2(G)$ . Indeed, we have

$$\ell^2(G) = \left\{ \alpha : G \rightarrow \mathbb{C} : \sum_{g \in G} \|\alpha(g)\|^2 < \infty \right\}.$$

In [9], Linnell formulated an analytic version of the zero divisor conjecture.

*Conjecture 1.2.* Let  $G$  be a torsion-free group. If  $0 \neq \alpha \in \mathbb{C}G$  and  $0 \neq \beta \in \ell^2(G)$ , then  $\alpha\beta \neq 0$ .

In [7], it is shown that Since  $\mathbb{C}G \subset \ell^2(G)$ , the second conjecture implies the first one. In [4], it is proved that for finitely generated amenable groups, the two conjectures are actually equivalent. We prove this is true for all amenable torsion-free groups.

The so-called 1-norm is defined on  $\mathbb{C}G$  by

$$\|\alpha\|_1 = \sum_{g \in G} |a_g|, \quad \text{for } \alpha = \sum_{g \in G} a_g g \text{ in } \mathbb{C}G.$$

The *adjoint* of an element  $\alpha = \sum_{g \in G} a_g g$  in  $\mathbb{C}G$ , denoted by  $\alpha^*$ , is  $\alpha^* = \sum_{g \in G} \bar{a}_g g^{-1}$ . We call an element  $\alpha \in \mathbb{C}G$  *self-adjoint* if  $\alpha^* = \alpha$ , and use  $(\mathbb{C}G)_s$  to denote the set of self-adjoint elements of  $\mathbb{C}G$ . It is worthy of mention that

if  $\alpha = \sum_{g \in G} a_g g$  is self-adjoint then  $a_1$  should be a real number. For  $\alpha \in \mathbb{C}G$ ,  $\beta$  and  $\gamma$  in  $\ell^2(G)$ , the following equalities hold:

$$\langle \alpha \beta, \gamma \rangle = \langle \beta, \alpha^* \gamma \rangle.$$

The goal of this paper is to give a criterion for an element in a complex group algebra to be regular:

**Theorem 1.1.** *Let  $G$  be a torsion free group. Then  $\alpha \in \mathbb{C}G$  is regular if  $2\|\alpha\|_2^2 \geq \|\alpha\|_1^2$ .*

## 2. PRELIMINARIES

In this section we provide some preliminaries needed in the following.

Let  $G$  be a group. The *support* of an element  $\alpha = \sum_{g \in G} a_g g$  in  $\mathbb{C}G$ ,  $\text{supp}(\alpha)$ , is the finite subset  $\{g \in G : a_g \neq 0\}$  of  $G$ .

Let  $H$  be a subgroup of  $G$ , and  $T$  be a right transversal for  $H$  in  $G$ . Then every element  $\alpha \in \mathbb{C}G$  (resp.  $\alpha \in \ell^2(G)$ ) can be written uniquely as a finite sum of the form  $\sum_{t \in T} \alpha_t t$  with  $\alpha_t \in \mathbb{C}H$  (resp.  $\alpha_t \in \ell^2(H)$ ).

For  $S \subset G$ , we denote by  $\langle S \rangle$ , the subgroup of  $G$  generated by  $S$ . We have the following key lemma:

**Lemma 2.1.** *Let  $G$  be a group,  $\alpha \in \mathbb{C}G$  and  $H = \langle \text{supp}(\alpha) \rangle$ . Then  $\alpha$  is regular in  $\mathbb{C}G$  iff  $\alpha$  is regular in  $\mathbb{C}H$ .*

*Proof.* Suppose that  $\alpha$  is a zero divisor. Among elements  $0 \neq \gamma$  in  $\mathbb{C}G$  which satisfy  $\alpha\gamma = 0$  consider an element  $\beta$  such that  $1 \in \text{supp}(\beta)$  and  $|\text{supp}(\beta)|$  is minimal, then one can easily show that  $\beta \in \mathbb{C}H$ , and this proves the result of the lemma.  $\square$

An immediate consequence of this lemma is:

**Corollary 2.2.** *A group  $G$  satisfies Conjecture 1.1 iff all its finitely generated subgroups satisfy the Conjecture 1.1.*

By Lemma 2.1 in hand, we can generalize the main theorem of [4]:

**Theorem 2.3.** *Let  $G$  be an amenable group. If  $0 \neq \alpha \in \mathbb{C}G$ ,  $0 \neq \beta \in \ell^2(G)$  and  $\alpha\beta = 0$ , then there exists  $0 \neq \gamma \in \mathbb{C}G$  such that  $\alpha\gamma = 0$ .*

The above theorem along with results in [10] provides another proof for [7, Theorem 2].

For a normal subgroup  $N$  of a group  $G$ , we denote the natural quotient map by  $q_N : G \rightarrow G/N$ . We continue to show that:

**Lemma 2.4.** *Let  $N$  be a normal subgroup of a group  $G$  satisfying Conjecture 1.1. Consider a non-torsion element  $q_N(t)$ ,  $t \in G$ , in the quotient group. Then  $\alpha + \beta t$  is regular, for all  $\alpha, \beta \in \mathbb{C}N \setminus \{0\}$ .*

*Proof.* Suppose that  $\alpha + \beta t$  is a zero divisor for non zero elements  $\alpha, \beta \in \mathbb{C}N$ . Applying Lemma 2.1 and multiplying by a suitable power of  $t$ , we can assume that there are non zero elements  $\gamma_k$ ,  $k = 0, 1, \dots, n$ , such that

$$(\alpha + \beta t) \sum_{k=0}^n \gamma_k t^k = 0.$$

In particular,  $0 = \beta t \gamma t^n = (\beta t \gamma_n t^{-1}) t^{n+1}$ , whence  $\beta t \gamma_n t^{-1} = 0$ , a contradiction, because  $t \gamma_n t^{-1}$  is a non zero element of  $\mathbb{C}N$ .  $\square$

**Proposition 2.5.** *Let  $N$  be an amenable normal subgroup of a group  $G$  satisfying Conjecture 1.1. Consider a non-torsion element  $q_N(t)$ ,  $t \in G$ , in the quotient group. Then there is no  $0 \neq \gamma \in \ell^2(G)$  such that  $(\alpha + \beta t)\gamma = 0$ . In particular,  $a + bt$  is an analytical zero divisor, for all non-torsion element  $g \in G$  and non zero complex numbers  $a, b$ .*

*Proof.* The group  $\langle N, t \rangle$  is amenable. Hence Lemma 2.4 together with Theorem 2.3 yields the result.  $\square$

### 3. A CONE OF REGULAR ELEMENTS

The result of the Proposition 2.5 is true if we replace  $\mathbb{C}$  by an arbitrary field  $\mathbb{F}$ . The field of complex numbers allows us to define inner product on the group algebra; with the help of inner product, we can construct new regular elements from the ones we have:

**Proposition 3.1.** *Let  $G$  be a group and  $\mathcal{F}$  be a finite non-empty subset of  $\mathbb{C}G$ . If  $\sum_{\alpha \in \mathcal{F}} \alpha^* \alpha$  is an analytical zero divisor then all elements of  $\mathcal{F}$  are analytical zero divisors. In particular,  $\alpha \in \mathbb{C}G$  is an analytical zero divisor if and only if  $\alpha^* \alpha$  is an analytical zero divisor.*

*Proof.* Let  $\tilde{\alpha} := \sum_{\alpha \in \mathcal{F}} \alpha^* \alpha$  and  $\tilde{\alpha} \beta = 0$  for some  $\beta \in \ell^2(G)$ . Then

$$0 = \langle \tilde{\alpha} \beta, \beta \rangle = \sum_{\alpha \in \mathcal{F}} \langle \alpha^* \alpha \beta, \beta \rangle = \sum_{\alpha \in \mathcal{F}} \langle \alpha \beta, \alpha \beta \rangle = \sum_{\alpha \in \mathcal{F}} \|\alpha \beta\|_2^2,$$

whence  $\alpha \beta = 0$  for all  $\alpha \in \mathcal{F}$ . This completes the proof.  $\square$

A *cone* in a vector space  $\mathfrak{X}$  is a subset  $\mathfrak{K}$  of  $\mathfrak{X}$  such that  $\mathfrak{K} + \mathfrak{K} \subset \mathfrak{K}$  and  $\mathbb{R}_+ \mathfrak{K} \subset \mathfrak{K}$ . We proceed by introducing a cone of regular elements in  $\mathbb{C}G$ . First a definition:

**Definition 3.2.** Let  $G$  be a group and  $(\mathbb{C}G)_s$  be the set of self adjoint elements  $\alpha \in \mathbb{C}G$ , we define a function  $\Upsilon : (\mathbb{C}G)_s \rightarrow \mathbb{R}$  by

$$\Upsilon(\alpha) := a_1 - \sum_{g \neq 1} |a_g|.$$

We call an element  $\alpha \in (\mathbb{C}G)_s$  **golden** if  $\Upsilon(\alpha) \geq 0$ . The set of all golden elements in  $(\mathbb{C}G)_s$  is denoted by  $(\mathbb{C}G)_{\text{gold}}$ .

What is important about golden elements is:

**Proposition 3.3.** *For a torsion free group  $G$ ,  $(\mathbb{C}G)_{\text{gold}}$  is a cone of regular elements.*

*Proof.* It is obvious that if  $\alpha$  is golden then so is  $r\alpha$  for any  $r > 0$ . The triangle inequality for  $\mathbb{C}$  shows that if  $\alpha$  and  $\gamma$  are golden then so is  $\alpha + \gamma$ . For  $\alpha \in (\mathbb{C}G)_{\text{s}}$ , we have

$$\begin{aligned}\alpha &= \frac{1}{2}(\alpha + \alpha^*) \\ &= a_1 + \frac{1}{2} \sum_{g \neq 1} (\bar{a}_g g^{-1} + a_g g) \\ &= \Upsilon(\alpha) + \frac{1}{2} \sum_{g \neq 1} (2|a_g| + \bar{a}_g g^{-1} + a_g g) \\ &= \Upsilon(\alpha) + \frac{1}{2} \sum_{g \neq 1} |a_g| \left( \frac{\bar{a}_g}{|a_g|} + g \right)^* \left( \frac{a_g}{|a_g|} + g \right)\end{aligned}$$

Hence, by Lemma 2.5 and Proposition 3.1,  $\alpha$  is regular.  $\square$

Now, we are ready to prove our main result:

*Proof of Theorem 1.1.* For  $\alpha = \sum_{g \in G} a_g g$  in  $\mathbb{C}G$ ,  $\alpha^* \alpha$  is self-adjoint, and one can easily show that

$$\Upsilon(\alpha^* \alpha) \geq 2\|\alpha\|_2^2 - \|\alpha\|_1^2.$$

Hence, by Proposition 3.3, the result of the Theorem is proved.  $\square$

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