Iranian Journal of Mathematical Sciences and Informatics
Vol. 20, No. 2 (2025), pp 101-121
DOI: 10.61186/ijmsi.20.2.101

Multiplicative Lie Triple Higher Derivation on Unital Algebra

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Abstract. In this article, we show that under certain assumptions every multiplicative Lie triple higher derivation $\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ on $\mathfrak{U}$ is of standard form, i.e., each component L_i has the form $\mathrm{L}_i = \delta_i + \gamma_i$, where $\{\delta_i\}_{i\in\mathbb{N}}$ is an additive higher derivation on $\mathfrak{U}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$ is a sequence of mappings $\gamma_i : \mathfrak{U} \to \mathfrak{Z}(\mathfrak{U})$ vanishing at Lie triple products on $\mathfrak{U}$.

Keywords: Unital algebra, Lie triple derivation, Lie triple higher derivation.

2000 Mathematics subject classification: 16W25, 15A78.

1. Introduction

Let $\mathfrak{R}$ be a commutative ring with identity and $\mathfrak{U}$ be a unital algebra over $\mathfrak{R}$. For any $x, y \in \mathfrak{U}$, $[x, y]$ will denote the commutator $xy - yx$. A map $\mathrm{L} : \mathfrak{U} \to \mathfrak{U}$ is called a multiplicative derivation on $\mathfrak{U}$ if $\mathrm{L}(xy) = \mathrm{L}(x)y + x\mathrm{L}(y)$ holds for all $x, y \in \mathfrak{U}$. A map $\mathrm{L} : \mathfrak{U} \to \mathfrak{U}$ is called a multiplicative Lie derivation (resp. multiplicative Lie triple derivation) on $\mathfrak{U}$ if $\mathrm{L}([x, y]) = [\mathrm{L}(x), y] + [x, \mathrm{L}(y)]$ (resp. $\mathrm{L}([[x, y], z]) = [[\mathrm{L}(x), y], z] + [[x, \mathrm{L}(y)], z] + [[x, y], \mathrm{L}(z)]$) holds for all $x, y, z \in \mathfrak{U}$. The concept of derivations was extended to higher derivations. Let us recall the

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Received 22 November 2020; Accepted 14 October 2023

basic facts about higher derivations. Let $\mathbb{N}$ be the set of all non negative integers and $\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ be a family of maps $\mathrm{L}_i : \mathfrak{U} \to \mathfrak{U}$ such that $\mathrm{L}_0 = I_{\mathfrak{U}}$. Then $\mathfrak{L}$ is said to be a

(i) *multiplicative higher derivation* on $\mathfrak{U}$ if $\mathrm{L}_i(xy) = \sum_{r+s=i} \mathrm{L}_r(x)\mathrm{L}_s(y)$ for all $x, y \in \mathfrak{U}$ and for each $i \in \mathbb{N}$,

(ii) *multiplicative Lie higher derivation* on $\mathfrak{U}$ if

$$\mathrm{L}_i([x,y]) = \sum_{r+s=i} [\mathrm{L}_r(x), \mathrm{L}_s(y)]$$

for all $x, y \in \mathfrak{U}$ and for each $i \in \mathbb{N}$,

(iii) *multiplicative Lie triple higher derivation* on $\mathfrak{U}$ if

$$\mathrm{L}_i([[x,y],z]) = \sum_{r+s+t=i} [[\mathrm{L}_r(x), \mathrm{L}_s(y)], \mathrm{L}_t(z)]$$

for all $x, y, z \in \mathfrak{U}$ and for each $i \in \mathbb{N}$.

Particularly, if $\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ is a family of linear maps, then the above maps are called higher derivation, Lie higher derivation and Lie triple higher derivation on $\mathfrak{U}$ respectively. Obviously, every higher derivation is a Lie higher derivation and every Lie higher derivation is a Lie triple higher derivation. But the converse statements are not true in general (for a counterexample see [12]).

Lie triple (higher) derivation has been studied on several classes of rings and algebras [1, 4, 5, 6]. In the year 1978, Miers [11] initiated the investigation of Lie triple derivations on von Neumann algebras and proved that "if M is a von Neumann algebra with no central abelian summands then there exists an operator $A \in M$ such that $\mathrm{L}(X) = [A, X] + \lambda(X)$, where $\lambda : M \to \mathfrak{Z}(M)$ is a linear map which annihilates brackets of operators in M." Zhang et.al. [14] and Lu [9] investigated the Lie triple derivation on nest algebras. Xiao and Wei [12] obtained that every Lie triple derivation on a triangular algebra can be expressed as the sum of an additive derivation and a linear functional vanishing on all second commutators. Also, Ding and Li [3] considered the Lie n-derivation on unital algebra with a nontrivial idempotents and proved that every Lie n-derivation L on $\mathfrak{U}$ is of the form $\mathrm{L} = d + \gamma$, where d is a derivation on $\mathfrak{U}$ and γ is a linear mapping from $\mathfrak{U}$ into its centre $\mathfrak{Z}(\mathfrak{U})$ that vanishes on $[[\mathfrak{U}, \mathfrak{U}], \mathfrak{U}]$. Apart from these, Ebrahimi [4, 5] studied Lie higher derivation on $B(X)$ and Lie triple higher derivation on generalized matrix algebras respectively.

In last few decades, the multiplicative mappings on rings and algebras were studied by many authors [1, 7, 13]. Martindale [10] established a condition on a ring such that multiplicative bijective mappings on this ring are all additive. In particular, every multiplicative bijective mapping from a prime ring containing a nontrivial idempotent onto an arbitrary ring is additive. Xiao and Wei [13] considered the case of nonlinear Lie higher derivations on a triangular algebra. Let $\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ be the Lie higher derivation on a triangular algebra. Then

[DOI: 10.61882/ijmsi.20.2.101]

$\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ is of the standard form, i.e., $\mathrm{L}_i = d_i + \gamma_i$, where $\{d_i\}_{i\in\mathbb{N}}$ is an additive higher derivations and $\{\gamma_i\}_{i\in\mathbb{N}}$ is sequence of a nonlinear functional vanishing on all commutators of the triangular algebra. Furthermore, Ashraf and Jabeen in [1] showed that every nonlinear Lie triple higher derivation on the triangular algebra has standard form. Han and Wei [7] studied multiplicative Lie higher derivations of a unital algebra and obtained similar conclusion as shown by Xiao and Wei [13]. In view of cited references, the main purpose of this paper is to prove that every multiplicative Lie triple higher derivation on a unital algebra has standard form under certain assumptions.

2. Preliminaries

Throughout, this paper we shall use the following notions: Let $\mathfrak{U} = p\mathfrak{U}p + p\mathfrak{U}q + q\mathfrak{U}p + q\mathfrak{U}q$ be unital algebra with nontrivial idempotents p and $q = 1-p$ satisfying (2.2). Let $\mathrm{A} = p\mathfrak{U}p, \mathrm{M} = p\mathfrak{U}q, \mathrm{N} = q\mathfrak{U}p$ and $\mathrm{B} = q\mathfrak{U}p$. Then $\mathfrak{U} = \mathrm{A} + \mathrm{M} + \mathrm{N} + \mathrm{B}$. The center of $\mathfrak{U}$ is

$$\mathfrak{Z}(\mathfrak{U}) = \{a + b \in \mathrm{A} + \mathrm{B} \mid am = mb, na = bn \text{ for all } m \in \mathrm{M}, n \in \mathrm{N}\}.$$

Define two natural projections $\pi_{\mathrm{A}} : \mathfrak{U} \to \mathrm{A}$ and $\pi_{\mathrm{B}} : \mathfrak{U} \to \mathrm{B}$ by $\pi_{\mathrm{A}}(a+m+n+b) = a$ and $\pi_{\mathrm{B}}(a + m + n + b) = b$. Moreover, $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{U})) \subseteq \mathfrak{Z}(\mathrm{A})$ and $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{U})) \subseteq \mathfrak{Z}(\mathrm{B})$ and there exists a unique algebra isomorphism $\tau : \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{U})) \to \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{U}))$ such that $am = m\tau(a)$ and $na = \tau(a)n$ for all $a \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{U})), m \in \mathrm{M}, n \in \mathrm{N}$.

Let us assume $\mathfrak{U}$ be an algebra with a nontrivial idempotent p and let $q = 1-p$ be also an idempotent. According to the well known Peirce decomposition, $\mathfrak{U}$ can be represented in the following form:

$$\mathfrak{U} = p\mathfrak{U}p + p\mathfrak{U}q + q\mathfrak{U}p + q\mathfrak{U}q \tag{2.1}$$

where $p\mathfrak{U}p$ and $q\mathfrak{U}q$ are subalgebras with unital elements p and q, respectively, $p\mathfrak{U}q$ is an $(p\mathfrak{U}p, q\mathfrak{U}q)$-bimodule and $q\mathfrak{U}p$ is a $(q\mathfrak{U}q, p\mathfrak{U}p)$-bimodule. We will assume that $\mathfrak{U}$ satisfies

$$\begin{array}{lclcl} pxp.p\mathfrak{U}q & = & \{0\} & = q\mathfrak{U}p.pxp \quad \text{implies} \quad pxp & = & 0, \\ p\mathfrak{U}q.qxq & = & \{0\} & = qxq.q\mathfrak{U}p \quad \text{implies} \quad qxq & = & 0 \end{array} \tag{2.2}$$

for all $x \in \mathfrak{U}$. Some specific examples of unital algebras with nontrivial idempotents having the property (2.2) are triangular algebras, matrix algebras and prime (and hence in particular simple) algebras with nontrivial idempotents.

3. Multiplicative Lie triple higher derivation

Following [8, Theorem 4.2.1], in this section we study the main result of this paper. In fact we obtain this result:

Theorem 3.1. *Let $\mathfrak{U}$ be a 2-torsion free unital algebra with a nontrivial idempotent p satisfying (2.2) and $\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ be a multiplicative Lie triple higher derivation on $\mathfrak{U}$. Let us assume that*

[DOI: 10.61882/ijmsi.20.2.101]

(*i*) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{U})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{U})) = \mathfrak{Z}(\mathrm{B})$,
(*ii*) *either* A *or* B *does not contain nonzero central ideals,*
(*iii*) $[x, \mathfrak{U}] \in \mathfrak{Z}(\mathfrak{U})$ *implies that* $x \in \mathfrak{Z}(\mathfrak{U})$ *for all* $x \in \mathfrak{U}$.
(*iv*) *For each* $n \in \mathrm{N}$, *the condition* $n\mathrm{M} = 0$ *or* $\mathrm{M}n = 0$ *implies* $n = 0$;
(*v*) *For each* $m \in \mathrm{M}$, *the condition* $m\mathrm{N} = 0$ *or* $\mathrm{N}m = 0$ *implies* $m = 0$;
(*vi*) *For each* $i \in \mathbb{N}$, $eL_i(e)f = 0$ *and* $fL_i(e)e = 0$,

Then every multiplicative Lie triple higher derivation $\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ *is of the standard form, i.e., each component* L_i *has the form* $\mathrm{L}_i = \delta_i + \gamma_i$, *where* $\{\delta_i\}_{i\in\mathbb{N}}$ *is an additive higher derivation on* $\mathfrak{U}$ *and* $\{\gamma_i\}_{i\in\mathbb{N}}$ *is a sequence of mappings* $\gamma_i : \mathfrak{U} \to \mathfrak{Z}(\mathfrak{U})$ *vanishing at Lie triple products in* $\mathfrak{U}$, *i.e.,* $\gamma_i([[x, y], z]) = 0$ *for all* $x, y, z \in \mathfrak{U}$.

In order to prove the theorem we will use the method of induction for the component index i. When $i = 1$, L_1 is a multiplicative Lie derivation on $\mathfrak{U}$. By [8, Theorem 4.2.1] it is easy to observe that there exist an additive derivation δ_1 and a map $\gamma_1 : \mathfrak{U} \to \mathfrak{Z}(\mathfrak{U})$ vanishing at Lie triple products such that $\mathrm{L}_1(x) = \delta_1(x)+\gamma_1(x)$ for all $x \in \mathfrak{U}$. Now by [8, Lemma 4.2.1], we know that $pL_1(q)q = 0 = qL_1(q)p$. Following from the proof of [8, Theorem 4.2.1], It can be easily seen that L_1 and δ_1 satisfy the following properties:

$$\begin{array}{llll} \mathrm{L}_1(0) = 0, & \mathrm{L}_1(p), \mathrm{L}_1(q) \in \mathfrak{Z}(\mathfrak{U}), & \delta_1(p) = 0, & \delta_1(q) = 0 \\ \mathrm{L}_1(\mathrm{A}) \subseteq \mathrm{A} + \mathfrak{Z}(\mathfrak{U}), & \mathrm{L}_1(\mathrm{B}) \subseteq \mathrm{B} + \mathfrak{Z}(\mathfrak{U}), & \delta_1(A) \subseteq A, & \delta_1(\mathrm{B}) \subseteq \mathrm{B} \\ \mathrm{L}_1(\mathrm{M}) \subseteq \mathrm{M}, & \mathrm{L}_1(\mathrm{N}) \subseteq \mathrm{N}, & \delta_1(\mathrm{M}) \subseteq \mathrm{M}, & \delta_1(\mathrm{N}) \subseteq \mathrm{N}. \end{array}$$

Suppose that our result holds for all $1 < r < i$. It follows that there exist an additive higher derivation $\{\delta_r\}_{r\in\mathbb{N}}$ of order r and a nonlinear mapping $\{\gamma_r\}_{r\in\mathbb{N}}$ vanishing on all Lie triple product such that $\mathrm{L}_r(x) = \delta_r(x) + \gamma_r(x)$ for all $x \in \mathfrak{U}$. It can be easily seen that L_r and δ_r satisfy the following properties:

$$\begin{array}{llll} \mathrm{L}_r(0) = 0, & \mathrm{L}_r(p), \mathrm{L}_r(q) \in \mathfrak{Z}(\mathfrak{U}), & \delta_r(p) = 0, & \delta_r(q) = 0 \\ \mathrm{L}_r(\mathrm{A}) \subseteq \mathrm{A} + \mathfrak{Z}(\mathfrak{U}), & \mathrm{L}_r(\mathrm{B}) \subseteq \mathrm{B} + \mathfrak{Z}(\mathfrak{U}), & \delta_r(\mathrm{A}) \subseteq \mathrm{A}, & \delta_r(\mathrm{B}) \subseteq \mathrm{B} \\ \mathrm{L}_r(\mathrm{M}) \subseteq \mathrm{M}, & \mathrm{L}_r(\mathrm{N}) \subseteq \mathrm{N}, & \delta_r(\mathrm{M}) \subseteq \mathrm{M}, & \delta_r(\mathrm{N}) \subseteq \mathrm{N}. \end{array}$$

To prove our main result we begin with the following lemmas:

Lemma 3.2. *For the index* $i \in \mathbb{N}$, *we have*

(*i*) $\mathrm{L}_i(0) = 0$,
(*ii*) $\mathrm{L}_i(p), \mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$,
(*iii*) $\mathrm{L}_i(\mathrm{M}) \subseteq \mathrm{M}$, *and* $\mathrm{L}_i(\mathrm{N}) \subseteq \mathrm{N}$.

Proof. (*i*) On using induction hypothesis

$$\begin{aligned} \mathrm{L}_i(0) &= [[\mathrm{L}_i(0), 0], 0] + [[0, \mathrm{L}_i(0)], 0] + [[0, 0], \mathrm{L}_i(0)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(0), \mathrm{L}_s(0)], \mathrm{L}_t(0)] = 0. \end{aligned} \tag{3.1}$$

[DOI: 10.61882/ijmsi.20.2.101]

(ii) For any $m \in \mathrm{M}$, using induction hypothesis

$$\begin{aligned} \mathrm{L}_i(m) &= \mathrm{L}_i([[m,q],q]) \\ &= [[\mathrm{L}_i(m),q],q] + [[m,\mathrm{L}_i(q)],q] + [[m,q],\mathrm{L}_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(m),\mathrm{L}_s(q)],\mathrm{L}_t(q)] \\ &= [[\mathrm{L}_i(m),q],q] + [[m,\mathrm{L}_i(q)],q] + [[m,q],\mathrm{L}_i(q)] \\ &= p\mathrm{L}_i(m)q + q\mathrm{L}_i(m)p + 2[m,\mathrm{L}_i(q)]. \end{aligned} \tag{3.2}$$

Now left multiplying by p and right multiplying by q in above the expression yield $2[m, \mathrm{L}_i(q)] = 0$ and hence $[m, \mathrm{L}_i(q)] = 0$. In a similar manner, we arrive at

$$\mathrm{L}_i(n) = p\mathrm{L}_i(n)q + q\mathrm{L}_i(n)p + 2[n,\mathrm{L}_i(q)].$$

Consequently, $2[n, \mathrm{L}_i(q)] = 0$ and hence $[n, \mathrm{L}_i(q)] = 0$. Therefore, from $[m, \mathrm{L}_i(q)] = 0$ and $[n, \mathrm{L}_i(q)] = 0$, we get $\mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$. Similarly, $\mathrm{L}_i(p) \in \mathfrak{Z}(\mathfrak{U})$.
(iii) Now from (3.2) and $\mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$, we get that $\mathrm{L}_i(q) = p\mathrm{L}_i(q)p + q\mathrm{L}_i(q)q$. For any $x \in \mathfrak{U}$ and $m_1, m_2 \in \mathrm{M}$

$$\begin{aligned} 0 &= \mathrm{L}_i([[m_1,m_2],x]) \\ &= [[\mathrm{L}_i(m_1),m_2],x] + [[m_1,\mathrm{L}_i(m_2)],x] + [[m_1,m_2],\mathrm{L}_i(x)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(m_1),\mathrm{L}_s(m_2)],\mathrm{L}_t(x)] \\ &= [[\mathrm{L}_i(m_1),m_2],x] + [[m_1,\mathrm{L}_i(m_2)],x] \end{aligned}$$

which implies that $[\mathrm{L}_i(m_1), m_2] + [m_1, \mathrm{L}_i(m_2)] \in \mathfrak{Z}(\mathfrak{U})$. Also,

$$\begin{aligned} [m_1,\mathrm{L}_i(m_2)] &= [[p,m_1],\mathrm{L}_i(m_2)] \\ &= \mathrm{L}_i([[p,m_1],m_2]) - [[\mathrm{L}_i(p),m_1],m_2] - [[p,\mathrm{L}_i(m_1)],m_2] \\ &\quad - \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p),\mathrm{L}_s(m_1)],\mathrm{L}_t(m_2)] \\ &= -[[p,\mathrm{L}_i(m_1)],m_2]. \end{aligned}$$

From (3.2), we find that

$$\begin{aligned} [\mathrm{L}_i(m_1),m_2] + [m_1,\mathrm{L}_i(m_2)] &= [q\mathrm{L}_i(m_1)p,m_2] - [[p,\mathrm{L}_i(m_1)],m_2] \\ &= 2[q\mathrm{L}_i(m_1)p,m_2]. \end{aligned}$$

Since $\mathfrak{U}$ is 2-torsion free, $[q\mathrm{L}_i(m_1)p, m_2] \in \mathfrak{Z}(\mathfrak{U})$. By definition of $\mathfrak{Z}(\mathfrak{U})$, it follows that $q\mathrm{L}_i(m_1)pm_2 \in \mathfrak{Z}(\mathrm{B})$ for all $m_2 \in \mathrm{M}$. It can be easily seen that $q\mathrm{L}_i(m_1)p\mathrm{B}q$ is an ideal of B. Now from assumption (ii), $q\mathrm{L}_i(m_1)p\mathrm{B}q = \{0\}$ which implies that $q\mathrm{L}_i(m_1)p = 0$ for $m_1 \in \mathrm{M}$. Therefore, $\mathrm{L}_i(m) = p\mathrm{L}_i(m)q \in \mathrm{M}$. Similarly, we can obtain $\mathrm{L}_i(\mathrm{N}) \subseteq \mathrm{N}$. □

Lemma 3.3. *For any $x \in \mathfrak{U}$,*

[DOI: 10.61882/ijmsi.20.2.101]

(i) $L_i(pxq) = pL_i(x)q$,
(ii) $L_i(qxp) = qL_i(x)p$.

Proof. On using $L_i(q), L_i(p) \in \mathfrak{Z}(\mathfrak{U})$ and for any $x \in \mathfrak{U}$

$$\begin{aligned} L_i(pxq) &= L_i([p,x],q]) \\ &= [[L_i(p),x],q] + [[p,L_i(x)],q] + [[p,x],L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[L_r(p),L_s(x)],L_t(q)] \\ &= [[p,L_i(x)],q] \\ &= pL_i(x)q + qL_i(x)p. \end{aligned}$$

Since $L_i(pxq) \in M$, we have $qL_i(x)p = 0$. Hence $L_i(pxq) = pL_i(x)q$ for all $x \in \mathfrak{U}$. Similarly, we can prove that $L_i(qxp) = qL_i(x)p$ for all $x \in \mathfrak{U}$. □

Lemma 3.4. $L_i(-m) = -L_i(m)$, *and* $L_i(-n) = -L_i(n)$ *for all* $n \in N$, $m \in M$.

Proof. Since $L_i(q), L_i(p) \in \mathfrak{Z}(\mathfrak{U})$ and $L_i(M) \subseteq M$, we have

$$\begin{aligned} L_i(-m) &= L_i([[m,p],q]) \\ &= [[L_i(m),p],q] + [[m,L_i(p)],q] + [[m,p],L_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[L_r(m),L_s(p)],L_t(q)] \\ &= [[L_i(m),p],q] = -L_i(m) \end{aligned}$$

for all $a \in A$ and $m \in M$. Similar proof for other case. □

Lemma 3.5. $L_i(A) \subseteq A + \mathfrak{Z}(\mathfrak{U})$ *and* $L_i(B) \subseteq B + \mathfrak{Z}(\mathfrak{U})$.

Proof. Since for all $a \in A, b \in B$ and $m \in M$, we have

$$\begin{aligned} 0 &= L_i([[a,b],m]) \\ &= [[L_i(a),b],m] + [[a,L_i(b)],m] + [[a,b],L_i(m)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[L_r(a),L_s(b)],L_t(m)] \\ &= [[L_i(a),b],m] + [[a,L_i(b)],m] \\ &= -m[qL_i(a)q,b] + [a,pL_i(b)p]m. \end{aligned}$$

Similarly, $n[qL_i(a)q,b] = [a,pL_i(b)p]n$ which leads to $[a,pL_i(b)p] + [qL_i(a)q,b] \in \mathfrak{Z}(\mathfrak{U})$ for all $a \in A$ and $b \in B$, i.e., $[a,pL_i(b)p] \in A$ and $[qL_i(a)q,b] \in B$. By assumption $[a,pL_i(b)p] = 0$ and $[qL_i(a)q,b] = 0$. This implies that $pL_i(b)p \in \mathfrak{Z}(A)$ and $qL_i(a)q \in \mathfrak{Z}(B)$.

We define a map $\phi_{i_1} : A \to \mathfrak{Z}(\mathfrak{U})$ by $\phi_{i_1}(a) = \tau^{-1}(qL_i(a)q) + qL_i(a)q$ where τ is map defined in preliminaries. Therefore, on using $L_i(pxq) = pL_i(x)q$ and $L_i(qxp) = qL_i(x)p$, we have

$$\begin{aligned} L_i(a) - \phi_{i_1}(a) &= pL_i(a)p + qL_i(a)q - \tau^{-1}(qL_i(a)q) - qL_i(a)q \\ &= pL_i(a)p - \tau^{-1}(qL_i(a)q) \in A. \end{aligned}$$

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This implies that $\mathrm{L}_i(\mathrm{A}) \subseteq \mathrm{A} + \mathfrak{Z}(\mathfrak{U})$ for all $a \in \mathrm{A}$. Similarly, we can define another map $\phi_{i_2} : \mathrm{B} \to \mathfrak{Z}(\mathfrak{U})$ by $\phi_{i_2}(b) = \tau(p\mathrm{L}_i(b)p) + p\mathrm{L}_i(b)p$ such that $\mathrm{L}_i(\mathrm{B}) \subseteq \mathrm{B} + \mathfrak{Z}(\mathfrak{U})$ for all $b \in \mathrm{B}$. □

Remark 3.6. Let us define the map $\gamma_{i_1} : \mathfrak{U} \to \mathfrak{Z}(\mathfrak{U})$ by

$$\gamma_{i_1}(x) = q\mathrm{L}_i(pxp)q + \tau^{-1}(q\mathrm{L}_i(pxp)q) + p\mathrm{L}_i(qxq)p + \tau(p\mathrm{L}_i(qxq)p).$$

Obviously, $\gamma_{i_1}(x) \in \mathfrak{Z}(\mathfrak{U})$ and $\gamma_{i_1}([[x,y],z]) = 0$ for all $x, y, z \in \mathfrak{U}$.

Define another map $\xi_i : \mathfrak{U} \to \mathfrak{U}$ as $\xi_i(x) = \mathrm{L}_i(x) - \gamma_{i_1}(x)$ for all $x \in \mathfrak{U}$. It is easy to see that

$$\begin{aligned}
\xi_i(a) &= p\mathrm{L}_i(a)p - \tau^{-1}(q\mathrm{L}_i(a)q) \in \mathrm{A},\\
\xi_i(b) &= q\mathrm{L}_i(b)q - \tau(p\mathrm{L}_i(b)p) \in \mathrm{B},\\
\xi_i(m) &= \mathrm{L}_i(m) \in \mathrm{M},\\
\xi_i(n) &= \mathrm{L}_i(n) \in \mathrm{N}
\end{aligned}$$

for all $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$ and $n \in \mathrm{N}$.

Lemma 3.7. *For any $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$ and $n \in \mathrm{N}$, we have*

(*i*) $\xi_i(am) = \xi_i(a)m + a\xi_i(m) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(a)\delta_s(m)$,

(*ii*) $\xi_i(mb) = \xi_i(m)b + m\xi_i(b) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(m)\delta_s(b)$,

(*iii*) $\xi_i(bn) = \xi_i(b)n + b\xi_i(n) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(b)\delta_s(n)$,

(*iv*) $\xi_i(na) = \xi_i(n)a + n\xi_i(a) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(n)\delta_s(a)$.

Proof. (*i*) For $a \in \mathrm{A}$ and $m \in \mathrm{M}$, we have

$$\begin{aligned}
\xi_i(am) &= \xi_i([[a,m],q])\\
&= \mathrm{L}_i([[a,m],q])\\
&= [[\mathrm{L}_i(a),m],q] + [[a,\mathrm{L}_i(m)],q] + [[a,m],\mathrm{L}_i(q)]\\
&\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\mathrm{L}_r(a),\mathrm{L}_s(m)],\mathrm{L}_t(q)]\\
&= [[\xi_i(a),m],q] + [[a,\xi_i(m)],q] + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\delta_r(a),\delta_s(m)],\delta_t(q)]\\
&= \xi_i(a)m + a\xi_i(m) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(a)\delta_s(m).
\end{aligned}$$

[DOI: 10.61882/ijmsi.20.2.101]

Similarly, we can prove (ii).
(iii) For $b \in \mathrm{B}$ and $n \in \mathrm{N}$, we have

$$\begin{aligned}\xi_i(bn) &= \xi_i([[b,n],p]) \\ &= \mathrm{L}_i([[b,n],p]) \\ &= [[\mathrm{L}_i(b),n],p] + [[b,\mathrm{L}_i(n)],p] + [[b,n],\mathrm{L}_i(p)] \\ &\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\mathrm{L}_r(b),\mathrm{L}_s(n)],\mathrm{L}_t(p)] \\ &= [[\xi_i(b),n],p] + [[b,\xi_i(n)],p] + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\delta_r(b),\delta_s(n)],\delta_t(p)] \\ &= \xi_i(b)n + b\xi_i(n) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(b)\delta_s(n).\end{aligned}$$

Similarly, we can prove (iv). □

Lemma 3.8. *For any* $a_1, a_2 \in \mathrm{A}$ *and* $b_1, b_2 \in \mathrm{B}$, *we have*

(i) $\xi_i(a_1a_2) = \xi_i(a_1)a_2 + a_1\xi_i(a_2) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(a_1)\delta_s(a_2)$,

(ii) $\xi_i(b_1b_2) = \xi_i(b_1)b_2 + b_1\xi_i(b_2) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(b_1)\delta_s(b_2)$.

Proof. For any $a_1, a_2 \in \mathrm{A}$ and $m \in \mathrm{M}$,

$$\xi_i(a_1a_2m) = \xi_i(a_1a_2)m + a_1a_2\xi_i(m) + \sum_{\substack{r+s+t=i\\ r,s,t<i}} \delta_r(a_1)\delta_s(a_2)\delta_t(m).$$

On the other hand,

$$\begin{aligned}\xi_i(a_1a_2m) &= \xi_i(a_1)a_2m + a_1\xi_i(a_2m) + \sum_{\substack{r+s+t=i\\ r,s,t<i}} \delta_r(a_1)\delta_s(a_2)\delta_t(m) \\ &= \xi_i(a_1)a_2m + a_1\xi_i(a_2)m + a_1a_2\xi_i(m) \\ &\quad + a_1\sum_{\substack{r+s=i\\ r,s<i}} \delta_r(a_2)\delta_s(m) + \sum_{\substack{r+s+t=i\\ r,s,t<i}} \delta_r(a_1)\delta_s(a_2)\delta_t(m). \qquad (3.3)\end{aligned}$$

Above relations implies that

$$\{\xi_i(a_1a_2) - \xi_i(a_1)a_2 - a_1\xi_i(a_2) - \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(a_1)\delta_s(a_2)\}m = 0. \qquad (3.4)$$

In the similar manner, we obtain

$$n\{\xi_i(a_1a_2) - \xi_i(a_1)a_2 - a_1\xi_i(a_2) - \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(a_1)\delta_s(a_2)\} = 0. \qquad (3.5)$$

[DOI: 10.61882/ijmsi.20.2.101]

Comparing (3.4) and (3.5) yield that

$$\xi_i(a_1a_2) = \xi_i(a_1)a_2 + a_1\xi_i(a_2) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(a_1)\delta_s(a_2)$$

for all $a_1, a_2 \in \mathrm{A}$. Similarly, we can show the other part. □

Lemma 3.9. *For any* $a \in \mathrm{A}, m \in \mathrm{M}$ *and* $b \in \mathrm{B}$,

(*i*) $\xi_i(a+m) - \xi_i(a) - \xi_i(m) \in \mathfrak{Z}(\mathfrak{U})$,
(*ii*) $\xi_i(m+b) - \xi_i(m) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$.

Proof. (*i*) For any $a \in \mathrm{A}, m \in \mathrm{M}$ and using $\mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$, we have

$$\begin{aligned}
\xi_i([[p,m],q]) &= \mathrm{L}_i([[p,a+m],q]) \\
&= [[p,\mathrm{L}_i(a+m)],q] + [[\mathrm{L}_i(p),a+m],q] + [[p,a+m],\mathrm{L}_i(q)] \\
&\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\mathrm{L}_r(p),\mathrm{L}_s(a+m)],\mathrm{L}_t(q)] \\
&= [[p,\xi_i(a+m)],q]. \qquad (3.6)
\end{aligned}$$

Also,

$$\begin{aligned}
\xi_i([[p,m],q]) &= [[p,\mathrm{L}_i(m)],q] + [[\mathrm{L}_i(p),m],q] + [[p,m],\mathrm{L}_i(q)] \\
&\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\mathrm{L}_r(p),\mathrm{L}_s(m)],\mathrm{L}_t(q)] \\
&= [[p,\xi_i(m)],q]. \qquad (3.7)
\end{aligned}$$

From (3.6) and (3.7) it follows that $[[\xi_i(a+m) - \xi_i(a) - \xi_i(m), q], p] = 0$ which gives $p(\xi_i(a+m)-\xi_i(a)-\xi_i(m))q + q(\xi_i(a+m)-\xi_i(a)-\xi_i(m))p = 0$. Therefore, we obtain that $\xi_i(a+m)-\xi_i(a)-\xi_i(m) \in \mathrm{A}+\mathrm{B}$. Since $[[a,m'],q] = [[a+m,m'],q]$ and $\mathrm{L}_i(M) \subseteq \mathrm{M}, \mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$, for any $a \in \mathrm{A}$ and $m, m' \in \mathrm{M}$.

$$\begin{aligned}
\xi_i(am') &= \mathrm{L}_i([[a+m,m'],q]) \\
&= [[\mathrm{L}_i(a+m),m'],q] + [[a+m,\mathrm{L}_i(m')],q] + [[a+m,m'],\mathrm{L}_i(q)] \\
&\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\mathrm{L}_r(a+m),\mathrm{L}_s(m')],\mathrm{L}_t(q)] \\
&= [[\xi_i(a+m),m'],q] + [[a+m,\xi_i(m')],q] \\
&\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\delta_r(a+m),\delta_s(m')],\delta_t(q)] \\
&= [[\xi_i(a+m),m'],q] + [[a,\xi_i(m')],q] + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(a)\delta_s(m'). \qquad (3.8)
\end{aligned}$$

[DOI: 10.61882/ijmsi.20.2.101]

On the other hand,

$$\begin{aligned} \xi_i(am') &= \mathrm{L}_i([[a, m'], q]) \\ &= [[\mathrm{L}_i(a), m'], q] + [[a, \mathrm{L}_i(m')], q] + [[a, m'], \mathrm{L}_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(a), \mathrm{L}_s(m')], \mathrm{L}_t(q)] \\ &= [[\xi_i(a), m'], q] + [[a, \xi_i(m')], q] + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\delta_r(a), \delta_s(m')], \delta_t(q)] \\ &= [[\xi_i(a), m'], q] + [[a, \xi_i(m')], q] + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a)\delta_s(m'). \end{aligned} \tag{3.9}$$

Now from (3.8) and (3.9), we have $[[\xi_i(a+m) - \xi_i(a) - \xi_i(m), m'], q] = 0$. This together with $\xi_i(a+m) - \xi_i(a) - \xi_i(m) \in \mathrm{A} + \mathrm{B}$ gives that

$$(\xi_i(a+m) - \xi_i(a) - \xi_i(m))m' = m'(\xi_i(a+m) - \xi_i(a) - \xi_i(m)) \tag{3.10}$$

for all $a \in \mathrm{A}$ and $m, m' \in \mathrm{M}$. In the similar manner, from $[[a+m, n'], q] = [[a, n'], q]$ for all $a \in \mathrm{A}, n' \in \mathrm{N}$ and $m \in \mathrm{M}$, we obtain that $[[\xi_i(a+m)-\xi_i(a)-\xi_i(m), n'], q] = 0$ which together with $\xi_i(a+m) - \xi_i(a) - \xi_i(m) \in \mathrm{A} + \mathrm{B}$ yields

$$(\xi_i(a+m) - \xi_i(a) - \xi_i(m))n' = n'(\xi_i(a+m) - \xi_i(a) - \xi_i(m)) \tag{3.11}$$

for all $a \in \mathrm{A}, n' \in \mathrm{N}$ and $m \in \mathrm{M}$. Now combining (3.10) and (3.11), we get $\xi_i(a+m) - \xi_i(a) - \xi_i(m) \in \mathfrak{Z}(\mathfrak{U})$. Similarly, we can find (ii). □

Lemma 3.10. *For any* $a \in \mathrm{A}, b \in \mathrm{B}$ *and* $n \in \mathrm{N}$

(i) $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in \mathfrak{Z}(\mathfrak{U})$,
(ii) $\xi_i(n+b) - \xi_i(n) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$.

Proof. (i) For any $a \in \mathrm{A}, n \in \mathrm{N}$ and using $\mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$, we have

$$\begin{aligned} \xi_i([[p, n], q]) &= \mathrm{L}_i([[p, a+n], q]) \\ &= [[p, \mathrm{L}_i(a+n)], q] + [[\mathrm{L}_i(p), a+n], q] + [[p, a+n], \mathrm{L}_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p), \mathrm{L}_s(a+n)], \mathrm{L}_t(q)] \\ &= [[p, \xi_i(a+n)], q]. \end{aligned} \tag{3.12}$$

Also,

$$\begin{aligned} \xi_i([[p, n], q]) &= [[p, \mathrm{L}_i(n)], q] + [[\mathrm{L}_i(p), n], q] + [[p, n], \mathrm{L}_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p), \mathrm{L}_s(n)], \mathrm{L}_t(q)] \\ &= [[p, \xi_i(n)], q]. \end{aligned} \tag{3.13}$$

[DOI: 10.61882/ijmsi.20.2.101]

From (3.12) and (3.13) it follows that $[[p, \xi_i(a+n) - \xi_i(a) - \xi_i(n)], q] = 0$ which yields that $p(\xi_i(a+n) - \xi_i(a) - \xi_i(n))q + q(\xi_i(a+n) - \xi_i(a) - \xi_i(n))p = 0$. Hence, we obtain that $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in \mathrm{A} + \mathrm{B}$. Since $[[a, n'], q] = [[a+n, n'], q]$ and $\mathrm{L}_i(\mathrm{N}) \subseteq \mathrm{N}, \mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$, for any $a \in \mathrm{A}$ and $n, n' \in \mathrm{N}$

$$\begin{aligned}
\xi_i(n'a) &= \mathrm{L}_i([[a+n, n'], q]) \\
&= [[\mathrm{L}_i(a+n), n'], q] + [[a+n, \mathrm{L}_i(n')], q] + [[a+n, n'], \mathrm{L}_i(q)] \\
&\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(a+n), \mathrm{L}_s(n')], \mathrm{L}_t(q)] \\
&= [[\xi_i(a+n), n'], q] + [[a+n, \xi_i(n')], q] \\
&\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} \delta_r(q)\delta_s(n')\delta_t(a+n) \\
&= [[\xi_i(a+n), n'], q] + [[a, \xi_i(n')], q] + \sum_{\substack{r+s+t=i \\ r,s,t<i}} \delta_r(q)\delta_s(n')\delta_t(a).
\end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned}
\xi_i(n'a) &= \mathrm{L}_i([[a, n'], q]) \\
&= [[\mathrm{L}_i(a), n'], q] + [[a, \mathrm{L}_i(n')], q] + [[a, n'], \mathrm{L}_i(q)] \\
&\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(a), \mathrm{L}_s(n')], \mathrm{L}_t(q)] \\
&= [[\xi_i(a), n'], q] + [[a, \xi_i(n')], q] + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(q)\delta_s(n')\delta_t(a).
\end{aligned} \tag{3.15}$$

Now combining (3.14) and (3.15), we have $[[\xi_i(a+n) - \xi_i(a) - \xi_i(n), n'], q] = 0$ which together with $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in \mathrm{A} + \mathrm{B}$ yields that

$$(\xi_i(a+n) - \xi_i(a) - \xi_i(n))n' = n'(\xi_i(a+n) - \xi_i(a) - \xi_i(n)) \tag{3.16}$$

for all $a \in \mathrm{A}$ and $n, n' \in \mathrm{N}$. In the similar manner, from $[[a+n, m'], q] = [[a, m'], q]$ for all $a \in \mathrm{A}, n \in \mathrm{N}$ and $m' \in \mathrm{M}$, we obtain that $[[\xi_i(a+n) - \xi_i(a) - \xi_i(n), m'], q] = 0$. combine this with $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in \mathrm{A} + \mathrm{B}$ yields that

$$(\xi_i(a+n) - \xi_i(a) - \xi_i(n))m' = m'(\xi_i(a+n) - \xi_i(a) - \xi_i(n)) \tag{3.17}$$

for all $a \in \mathrm{A}, n \in \mathrm{N}$ and $m' \in \mathrm{M}$. Now combining (3.16) and (3.17), we get $\xi_i(a+n) - \xi_i(a) - \xi_i(n) \in \mathfrak{Z}(\mathfrak{U})$. Similarly we can find (ii). □

Lemma 3.11. *ξ_i is additive on* A, M, N *and* B *respectively.*

[DOI: 10.61882/ijmsi.20.2.101]

Proof. For any $m_1, m_2 \in \mathrm{M}$, we have

$$\begin{aligned}
\xi_i(m_1+m_2) &= \mathrm{L}_i([[p+m_1, q+m_2], q]) \\
&= [[\mathrm{L}_i(p+m_1), q+m_2], q] + [[p+m_1, \mathrm{L}_i(q+m_2)], q] \\
&\quad + [[p+m_1, q+m_2], \mathrm{L}_i(q)] \\
&\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p+m_1), \mathrm{L}_s(q+m_2)], \mathrm{L}_t(q)] \\
&= [[\xi_i(p+m_1), q+m_2], q] + [[p+m_1, \xi_i(q+m_2)], q] \\
&\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\delta_r(p+m_1), \delta_s(q+m_2)], \delta_t(q)] \\
&= [[\xi_i(p)+\xi_i(m_1), q+m_2], q] + [[p+m_1, \xi_i(q)+\xi_i(m_2)], q] \\
&= [[\xi_i(m_1), q+m_2], q] + [[p+m_1, \xi_i(m_2)], q] \\
&= \xi_i(m_1)+\xi_i(m_2). \qquad (3.18)
\end{aligned}$$

which implies that ξ_i is additive on M. In similar manner, we can obtain that ξ_i is additive on N.
Now using Lemma 3.7 and (3.12), we find that

$$\begin{aligned}
\xi_i((a_1+a_2)m) &= \xi_i(a_1m)+\xi_i(a_2m) \\
&= \xi_i(a_1)m+\xi_i(a_2)m+a_1\xi_i(m)+a_2\xi_i(m) \\
&\quad + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a_1)\delta_s(m) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a_2)\delta_s(m). \qquad (3.19)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\xi_i((a_1+a_2)m) &= \xi_i(a_1+a_2)m+(a_1+a_2)\xi_i(m) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a_1+a_2)\delta_s(m) \\
&= \xi_i(a_1+a_2)m+a_1\xi_i(m)+a_2\xi_i(m) \\
&\quad + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a_1)\delta_s(m) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a_2)\delta_s(m). \qquad (3.20)
\end{aligned}$$

Above two expressions implies that

$$\xi_i(a_1+a_2)m = \xi_i(a_1)m+\xi_i(a_2)m. \qquad (3.21)$$

In the similar way,

$$n\xi_i(a_1+a_2) = n\xi_i(a_1)+n\xi_i(a_2). \qquad (3.22)$$

Last two equations together yield that $\xi_i(a_1+a_2) = \xi_i(a_1)+\xi_i(a_2)$ for all $a_1, a_2 \in \mathrm{A}$ which proves that ξ_i is additive on A. Similarly, ξ_i is additive on B. □

Lemma 3.12. *For any* $a \in \mathrm{A}, b \in \mathrm{B}, n \in \mathrm{N}$ *and* $m \in \mathrm{M}$

(*i*) $\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$,
(*ii*) $\xi_i(a+n+b)-\xi_i(a)-\xi_i(n)-\xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$.

[DOI: 10.61882/ijmsi.20.2.101]

Proof. (i) For any $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$ and using $\mathrm{L}_i(p), \mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$, we have

$$\begin{aligned}
&\xi_i([[p,m],q]) \\
&\quad = \mathrm{L}_i([[p,a+m+b],q]) \\
&\quad = [[p,\mathrm{L}_i(a+m+b)],q] + [[\mathrm{L}_i(p),a+m+b],q] \\
&\qquad +[[p,a+m+b],\mathrm{L}_i(q)] + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p),\mathrm{L}_s(a+m+b)],\mathrm{L}_t(q)] \\
&\quad = [[p,\xi_i(a+m+b)],q]. \qquad (3.23)
\end{aligned}$$

Also,

$$\begin{aligned}
\xi_i([[p,m],q]) &= [[p,\mathrm{L}_i(m)],q] + [[\mathrm{L}_i(p),m],q] + [[p,m],\mathrm{L}_i(q)] \\
&\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p),\mathrm{L}_s(m)],\mathrm{L}_t(q)] \\
&= [[p,\xi_i(m)],q]. \qquad (3.24)
\end{aligned}$$

From (3.23) and (3.24) it follows that $[[p,\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b)],q]=0$ which gives $p(\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b))q+q(\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b))p=0$. Therefore, we obtain that $\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b) \in \mathrm{A}+\mathrm{B}$. Since $am'-m'b=[[a+m+b,m'],q]$ and $\mathrm{L}_i(\mathrm{M}) \subseteq \mathrm{M}, \mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$, for any $a \in \mathrm{A}, b \in \mathrm{B}$ and $m,m' \in \mathrm{M}$

$$\begin{aligned}
&\xi_i(am'-m'b) \\
&\quad = \mathrm{L}_i([[a+m+b,m'],q]) \\
&\quad = [[\mathrm{L}_i(a+m+b),m'],q] + [[a+m+b,\mathrm{L}_i(m')],q] \\
&\qquad +[[a+m+b,m'],\mathrm{L}_i(q)] + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(a+m+b),\mathrm{L}_s(m')],\mathrm{L}_t(q)] \\
&\quad = [[\xi_i(a+m+b),m'],q] + [[a+m+b,\xi_i(m')],q] \\
&\qquad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\delta_r(a+m+b),\delta_s(m')],\delta_t(q)] \\
&\quad = [[\xi_i(a+m+b),m'],q] + [[a+b,\xi_i(m')],q] \\
&\qquad + \sum_{\substack{r+s=i \\ r,s,<i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(m')\delta_s(b).
\end{aligned}$$

[DOI: 10.61882/ijmsi.20.2.101]

On the other hand,

$$\begin{aligned}\xi_i(am'-m'b) &= \xi_i(am')-\xi_i(m'b)\\ &= \xi_i(a)m'-m'\xi_i(b)+a\xi_i(m')-\xi_i(m')b\\ &\quad+\sum_{\substack{r+s=i\\ r,s,<i}}\delta_r(a)\delta_s(m')-\sum_{\substack{r+s=i\\ r,s<i}}\delta_r(m')\delta_s(b)\\ &= [[\xi_i(a),m'],q]+[[a,\xi_i(m')],q]+[[\xi_i(b),m'],q]\\ &\quad+[[b,\xi_i(m')],q]+\sum_{\substack{r+s=i\\ r,s,<i}}\delta_r(a)\delta_s(m')-\sum_{\substack{r+s=i\\ r,s<i}}\delta_r(m')\delta_s(b).\end{aligned} \tag{3.25}$$

Now from (3.25) and (3.25), we have $[[\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b),m'],q]=0$ which together with $\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b)\in \mathrm{A}+\mathrm{B}$ gives that

$$(\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b))m' = m'(\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b)) \tag{3.26}$$

for all $a\in \mathrm{A}, b\in \mathrm{B}$ and $m,m'\in \mathrm{M}$. In the similar manner, from $[[a+m+b,n'],q]=[[a+b,n'],q]$ for all $a\in \mathrm{A}, n'\in \mathrm{N}$ and $m\in \mathrm{M}$, we obtain that $[[\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b),n'],q]=0$, which together with $\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b)\in \mathrm{A}+\mathrm{B}$ yields that

$$(\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b))n' = n'(\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b)) \tag{3.27}$$

for all $a\in \mathrm{A}, b\in \mathrm{B}, n'\in \mathrm{N}$ and $m\in \mathrm{M}$. Now combining (3.26) and (3.27), we get $\xi_i(a+m+b)-\xi_i(a)-\xi_i(m)-\xi_i(b)\in \mathfrak{Z}(\mathfrak{U})$. Similarly we can find other case. □

Lemma 3.13. *For any* $n\in \mathrm{N}$ *and* $m\in \mathrm{M}$, $\xi_i(m+n)=\xi_i(m)+\xi_i(n)$.

Proof. Using Lemmas 3.9 and 3.10,we have

$$\begin{aligned}\xi_i(m+n) &= \mathrm{L}_i([[p+m,p-n],p])\\ &= [[\mathrm{L}_i(p+m),p-n],p]+[[p+m,\mathrm{L}_i(p-n)],p]\\ &\quad+[[p+m,p-n],\mathrm{L}_i(p)]\\ &\quad+\sum_{\substack{r+s+t=i\\ r,s,t<i}}[[\mathrm{L}_r(p+m),\mathrm{L}_s(p-n)],\mathrm{L}_t(p)]\\ &= [[\xi_i(p+m),p-n],p]+[[p+m,\xi_i(p-n)],p]\\ &\quad+\sum_{\substack{r+s+t=i\\ r,s,t<i}}[[\delta_r(p+m),\delta_s(p-n)],\delta_t(p)]\\ &= [[\xi_i(p)+\xi_i(m),p-n],p]+[[p+m,\xi_i(p)-\xi_i(n)],p]\\ &= [[\xi_i(m),p-n],p]+[[p+m,-\xi_i(n)],p]\\ &= \xi_i(m)+\xi_i(n).\end{aligned}$$

□

[DOI: 10.61882/ijmsi.20.2.101]

Lemma 3.14. *For any* $a \in \mathrm{A}, b \in \mathrm{B}, n \in \mathrm{N}$ *and* $m \in \mathrm{M}$,

$$\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U}).$$

Proof. Consider

$$\begin{aligned} \xi_i(m+n) &= \mathrm{L}_i([[p, a+m+n+b], q]) \\ &= [[p, \mathrm{L}_i(a+m+n+b)], q] + [[\mathrm{L}_i(p), a+m+n+b], q] \\ &\quad + [[p, a+m+n+b], \mathrm{L}_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p), \mathrm{L}_s(a+m+n+b)], \mathrm{L}_t(q)] \\ &= [[p, \xi_i(a+m+n+b)], q]. \end{aligned} \tag{3.28}$$

On the other hand,

$$\begin{aligned} \xi_i(m+n) &= \xi_i(m) + \xi_i(n) \\ &= \xi_i([[p,a],q]) + \xi_i([[p,m],q]) + \xi_i([[p,n],q]) + \xi_i([[p,b],q]) \\ &= [[p, \mathrm{L}_i(a)], q] + [[p, \mathrm{L}_i(m)], q] + [[p, \mathrm{L}_i(n)], q] + [[p, \mathrm{L}_i(b)], q]) \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p), \mathrm{L}_s(a)], \mathrm{L}_t(q)] + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p), \mathrm{L}_s(m)], \mathrm{L}_t(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p), \mathrm{L}_s(n)], \mathrm{L}_t(q)] + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(p), \mathrm{L}_s(b)], \mathrm{L}_t(q)] \\ &= [[p, \xi_i(a) + \xi_i(m) + \xi_i(n) + \xi_i(b)], q]. \end{aligned} \tag{3.29}$$

In view of (3.28) and (3.29), we have $[[p, \xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b)], q] = 0$ which gives $p(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))q + q(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))p = 0$. Therefore, we obtain that $\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in \mathrm{A} + \mathrm{B}$. Since $am' - m'b = [[a+m+n+b, m'], q]$ and $\mathrm{L}_i(\mathrm{M}) \subseteq \mathrm{M}, \mathrm{L}_i(q) \in \mathfrak{Z}(\mathfrak{U})$, for any $a \in \mathrm{A}, b \in \mathrm{B}$ and $m, m' \in \mathrm{M}$

$$\begin{aligned} \xi_i(am' - m'b) &= \mathrm{L}_i([[a+m+n+b, m'], q]) \\ &= [[\mathrm{L}_i(a+m+n+b), m'], q] + [[a+m+n+b, \mathrm{L}_i(m')], q] \\ &\quad + [[a+m+n+b, m'], \mathrm{L}_i(q)] \\ &\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\mathrm{L}_r(a+m+n+b), \mathrm{L}_s(m')], \mathrm{L}_t(q)] \end{aligned}$$

[DOI: 10.61882/ijmsi.20.2.101]

$$
\begin{aligned}
&= [[\xi_i(a+m+n+b), m'], q] + [[a+m+n+b, \xi_i(m')], q] \\
&\quad + \sum_{\substack{r+s+t=i \\ r,s,t<i}} [[\delta_r(a+m+n+b), \delta_s(m')], \delta_t(q)] \\
&= [[\xi_i(a+m+n+b), m'], q] + [[a+b, \xi_i(m')], q] \\
&\quad + \sum_{\substack{r+s=i \\ r,s,<i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(m')\delta_s(b). \qquad (3.30)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\xi_i(am' - m'b) &= \xi_i(am') - \xi_i(m'b) \\
&= \xi_i(a)m' - m'\xi_i(b) + a\xi_i(m') - \xi_i(m')b \\
&\quad + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(m')\delta_s(b) \\
&= [[\xi_i(a), m'], q] + [[a, \xi_i(m')], q] + [[\xi_i(b), m'], q] \\
&\quad + [[b, \xi_i(m')], q] + \sum_{\substack{r+s=i \\ r,s,<i}} \delta_r(a)\delta_s(m') - \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(m')\delta_s(b).
\end{aligned}
\qquad (3.31)
$$

Now from (3.30) and (3.31), we have $[[\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b), m'], q] = 0$ which together with $\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in \mathrm{A} + \mathrm{B}$ gives that

$$
\begin{aligned}
&(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))m' \\
&= m'(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))
\end{aligned}
\qquad (3.32)
$$

for all $a \in \mathrm{A}, b \in \mathrm{B}$ and $m, m' \in \mathrm{M}$. In the similar manner, from $[[a+m+n+b, n'], q] = [[a+b, n'], q]$ for all $a \in \mathrm{A}, n' \in \mathrm{N}$ and $m \in \mathrm{M}$, we obtain that $[[\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b), n'], q] = 0$ which together with $\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in \mathrm{A} + \mathrm{B}$ gives

$$
\begin{aligned}
&(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))n' \\
&= n'(\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b))
\end{aligned}
\qquad (3.33)
$$

for all $a \in \mathrm{A}, b \in \mathrm{B}, n' \in \mathrm{N}$ and $m \in \mathrm{M}$. Now combining (3.32) and (3.33), we get $\xi_i(a+m+n+b) - \xi_i(a) - \xi_i(m) - \xi_i(n) - \xi_i(b) \in \mathfrak{Z}(\mathfrak{U})$. □

Lemma 3.15. *For any $n \in \mathrm{N}$ and $m \in \mathrm{M}$, we have*

(*i*) $\xi_i(mn) = \xi_i(m)n + m\xi_i(n) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(m)\delta_s(n)$,

(*ii*) $\xi_i(nm) = \xi_i(n)m + n\xi_i(m) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(m)\delta_s(n)$.

[DOI: 10.61882/ijmsi.20.2.101]

Proof. For any $m, m' \in \mathrm{M}$ and $n \in \mathrm{N}$, we have

$$\begin{aligned}\xi_i([[m,n],m']) &= [[\mathrm{L}_i(m),n],m'] + [[m,\mathrm{L}_i(n)],m'] + [[m,n],\mathrm{L}_i(m')] \\ &\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\mathrm{L}_r(m),\mathrm{L}_s(n)],\mathrm{L}_t(m')] \\ &= [[\xi_i(m),n],m'] + [[m,\xi_i(n)],m'] + [[m,n],\xi_i(m')] \\ &\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} [[\delta_r(m),\delta_s(n)],\delta_t(m')]. \end{aligned} \tag{3.34}$$

On the other way,

$$\begin{aligned}\xi_i([[m,n],m']) &= \xi_i(mnm' + m'nm) \\ &= \xi_i(mn)m' + mn\xi_i(m') + m'\xi_i(nm) + \xi_i(m')nm \\ &\quad + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(mn)\delta_s(m') + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(m')\delta_s(nm). \end{aligned} \tag{3.35}$$

From the above two expressions, we get

$$[\xi_i(mn) - \xi_i(nm) - [\xi_i(m),n] - [m,\xi_i(n)] - \sum_{\substack{r+s=i\\ r,s<i}} [\delta_r(m),\delta_s(n)],m'] = 0. \tag{3.36}$$

Similarly, we have

$$[\xi_i(mn) - \xi_i(nm) - [\xi_i(m),n] - [m,\xi_i(n)] - \sum_{\substack{r+s=i\\ r,s<i}} [\delta_r(m),\delta_s(n)],n'] = 0. \tag{3.37}$$

From (3.36) and (3.37) it follows that

$$\begin{aligned}&\{\xi_i(mn) - \xi_i(m)n - m\xi_i(n) - \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(m)\delta_s(n)\} \\ &+ \{-\xi_i(nm) + \xi_i(n)m + n\xi_i(m) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(n)\delta_s(m)\} \in \mathfrak{Z}(\mathfrak{U}). \end{aligned} \tag{3.38}$$

From the assumption A does not contain nonzero central ideals. Assume that

$$\omega(m,n) = \xi_i(mn) - \xi_i(m)n - m\xi_i(n) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(m)\delta_s(n).$$

[DOI: 10.61882/ijmsi.20.2.101]

Note that $\omega(m,n) \in \mathfrak{Z}(\mathrm{A})$ for all $m \in \mathrm{M}$ and $n \in \mathrm{N}$. On using Lemma 3.7, we obtain that

$$\begin{aligned}\omega(am,n) &= \xi_i(amn) - \xi_i(am)n - am\xi_i(n) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(am)\delta_s(n)\\ &= \xi_i(a)mn + a\xi_i(mn) - \xi_i(a)mn - a\xi_i(m)n - am\xi_i(n)\\ &\quad + \sum_{\substack{r+s+t=i\\ r,s,t<i}} \delta_r(a)\delta_s(m)\delta_t(n)\\ &= a\xi_i(m)n + am\xi_i(n) + a\omega(m,n) - a\xi_i(m)n - am\xi_i(n)\\ &= a\omega(m,n).\end{aligned}$$

This implies $\mathrm{A}\omega(m,n)$ is a central ideal of A. Therefore, $\omega(m,n) = 0$, that is,

$$\xi_i(mn) = \xi_i(m)n + m\xi_i(n) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(m)\delta_s(n)$$

for all $m \in \mathrm{M}$ and $n \in \mathrm{N}$. Similarly, we have

$$\xi_i(nm) = \xi_i(n)m + n\xi_i(m) + \sum_{\substack{r+s=i\\ r,s<i}} \delta_r(n)\delta_s(m)$$

for all $m \in \mathrm{M}$ and $n \in \mathrm{N}$. □

Remark 3.16. Now, in view of Lemma 3.14 we define a map $\gamma_{i_2} : \mathfrak{U} \to \mathfrak{Z}(\mathfrak{U})$ by

$$\gamma_{i_2}(x) = \xi_i(x) - \xi_i(pxp) - \xi_i(pxq) - \xi_i(qxp) - \xi_i(qxq).$$

Clearly, $\gamma_{i_2}(x) \in \mathfrak{Z}(\mathfrak{U})$ and $\gamma_{i_2}(\mathrm{A}) = \gamma_{i_2}(\mathrm{M}) = \gamma_{i_2}(\mathrm{N}) = \gamma_{i_2}(\mathrm{B}) = 0$. Now we are ready to define a map $\delta_i : \mathfrak{U} \to \mathfrak{U}$ by $\delta_i(x) = \xi_i(x) - \gamma_{i_2}(x)$. Then $\mathrm{L}_i(x) = \delta_i(x) + \gamma_{i_1}(x) + \gamma_{i_2}(x) = \delta_i(x) + \gamma_i(x)$, where $\gamma_i(x) = \gamma_{i_1}(x) + \gamma_{i_2}(x)$ is a mapping from $\mathfrak{U}$ to its center.

Lemma 3.17. $\delta_i(a+m+n+b) = \delta_i(a) + \delta_i(m) + \delta_i(n) + \delta_i(b)$ *for all* $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$ *and* $n \in \mathrm{N}$.

Proof. It is clear that

$$\begin{aligned}\delta_i(a+m+n+b) &= \xi_i(a+m+n+b) - \gamma_{i_2}(a+m+n+b)\\ &= \xi_i(a+m+n+b) - \xi_i(a+m+n+b)\\ &\quad +\xi_i(a) + \xi_i(m) + \xi_i(n) + \xi_i(b)\\ &= \xi_i(a) + \xi_i(m) + \xi_i(n) + \xi_i(b)\\ &= \delta_i(a) + \delta_i(m) + \delta_i(n) + \delta_i(b)\end{aligned}$$

for all $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$ and $n \in \mathrm{N}$. □

Now we are well equipped to prove our main theorem.

[DOI: 10.61882/ijmsi.20.2.101]

Proof of Theorem 3.1. For any $x, y \in \mathfrak{U}$. Suppose that $x = a_1 + m_1 + n_1 + b_1$ and $x = a_2 + m_2 + n_2 + b_2$ where $a_1, a_2 \in \mathrm{A}, b_1, b_2 \in \mathrm{B}, n_1, n_2 \in \mathrm{N}$ and $m_1, m_2 \in \mathrm{M}$.

$$\begin{aligned}
\delta_i(x+y) &= \delta_i(a_1 + m_1 + n_1 + b_1 + a_2 + m_2 + n_2 + b_2) \\
&= \delta_i(a_1 + a_2) + \delta_i(m_1 + m_2) + \delta_i(n_1 + n_2) + \delta_i(b_1 + b_2) \\
&= \delta_i(a_1) + \delta_i(m_1) + \delta_i(n_1) + \delta_i(b_1) \\
&\quad + \delta_i(a_2) + \delta_i(m_2) + \delta_i(n_2) + \delta_i(b_2) \\
&= \delta_i(a_1 + m_1 + n_1 + b_1) + \delta_i(a_2 + m_2 + n_2 + b_2) \\
&= \delta_i(x) + \delta_i(y).
\end{aligned}$$

Thus, δ_i is additive on $\mathfrak{U}$. Also, from Lemmas 3.7, 3.8 and 3.17

$$\begin{aligned}
\delta_i(xy) &= \delta_i(a_1a_2 + a_1m_2 + m_1n_2 + m_1b_2 + n_1a_2 + n_1m_2 + b_1n_2 + b_1b_2) \\
&= \delta_i(a_1a_2 + m_1n_2) + \delta_i(a_1m_2 + m_1b_2) \\
&\quad + \delta_i(n_1a_2 + b_1n_2) + \delta_i(n_1m_2 + b_1b_2) \\
&= \delta_i(a_1a_2) + \delta_i(a_1m_2) + \delta_i(n_1a_2) + \delta_i(n_1m_2) \\
&\quad + \delta_i(m_1n_2) + \delta_i(m_1b_2) + \delta_i(b_1n_2) + \delta_i(b_1b_2) \\
&= \delta_i(a_1)a_2 + a_1\delta_i(a_2) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a_1)\delta_s(a_2) + \delta_i(a_1)m_2 + a_1\delta_i(m_2) \\
&\quad + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(a_1)\delta_s(m_2) + \delta_i(n_1)a_2 + n_1\delta_i(a_2) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(n_1)\delta_s(a_2) \\
&\quad + \delta_i(n_1)m_2 + n_1\delta_i(m_2) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(n_1)\delta_s(m_2) + \delta_i(m_1)n_2 \\
&\quad + m_1\delta_i(n_2) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(m_1)\delta_s(n_2) + \delta_i(m_1)b_2 + m_1\delta_i(b_2) \\
&\quad + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(m_1)\delta_s(b_2) + \delta_i(b_1)n_2 + b_1\delta_i(n_2) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(b_1)\delta_s(n_2) \\
&\quad + \delta_i(b_1)b_2 + b_1\delta_i(b_2) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(b_1)\delta_s(b_2) \\
&= (\delta_i(a_1) + \delta_i(m_1) + \delta_i(n_1) + \delta_i(b_1))(a_2 + m_2 + n_2 + b_2) \\
&\quad + (a_1 + m_1 + n_1 + b_1)(\delta_i(a_2) + \delta_i(m_2) + \delta_i(n_2) + \delta_i(b_2)) \\
&\quad + \sum_{\substack{r+s=i \\ r,s<i}} \{\delta_r(a_1) + \delta_r(m_1) + \delta_r(n_1) + \delta_r(b_1)\} \\
&\quad \{\delta_s(a_2) + \delta_s(m_2) + \delta_s(n_2) + \delta_s(b_2)\} \\
&= \delta_i(x)y + x\delta_i(y) + \sum_{\substack{r+s=i \\ r,s<i}} \delta_r(x)\delta_s(y).
\end{aligned}$$

[DOI: 10.61882/ijmsi.20.2.101]

This shows that $\{\delta_i\}_{i\in\mathbb{N}}$ is an additive higher derivation. Now, we have to prove that γ_i vanishes on all Lie triple products of $\mathfrak{U}$. It is easy to see that $\gamma_i(x) \in \mathfrak{Z}(\mathfrak{U})$ for all $x \in \mathfrak{U}$. Further,

$$\begin{aligned}\gamma_i([[x,y],z]) &= \mathrm{L}_i([[x,y],z]) - \delta_i([[x,y],z]) \\ &= [[\mathrm{L}_i(x),y],z] + [[x,\mathrm{L}_i(y)],z] + [[x,y],\mathrm{L}_i(z)] \\ &\quad -[[\delta_i(x),y],z] - [[x,\delta_i(y)],z] - [[x,y],\delta_i(z)] \\ &= 0.\end{aligned}$$

for all $x, y, z \in \mathfrak{U}$. Hence, this proves the required result. □

In particular, we have the following corollary.

Corollary 3.18. [1, Theorem 3.1] *Let $\mathfrak{T}$ be a 2-torsion free triangular algebra and $\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ be a multiplicative Lie triple higher derivation on $\mathfrak{T}$. Let us assume that*

(*i*) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{B})$,

(*ii*) *either* A *or* B *does not contain nonzero central ideals,*

Then every multiplicative Lie triple higher derivation $\mathfrak{L} = \{\mathrm{L}_i\}_{i\in\mathbb{N}}$ is of the standard form, i.e., each component L_i has the form $\mathrm{L}_i = \delta_i + \gamma_i$, where $\{\delta_i\}_{i\in\mathbb{N}}$ is an additive higher derivation on $\mathfrak{T}$ and $\{\gamma_i\}_{i\in\mathbb{N}}$ is a sequence of mapping $\gamma_i : \mathfrak{T} \to \mathfrak{Z}(\mathfrak{T})$ vanishing at Lie triple products in $\mathfrak{T}$.

Acknowledgments

The authors would like to express their sincere thanks to the referees for carefully reading the manuscript and their useful suggestions.

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[DOI: 10.61882/ijmsi.20.2.101]