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Lower Bounds on Signed Total Double Roman k-domination in Graphs

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Abstract. A signed total double Roman k-dominating function (STDRkDF) on an isolated-free graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ has at least two neighbors assigned 2 under f or at least one neighbor w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ has at least one neighbor w with $f(w) \geq 2$ and (iii) $\sum_{u \in N(v)} f(u) \geq k$ holds for any vertex v. The weight of an STDRkDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The signed total double Roman k-domination number $\gamma^k_{stdR}(G)$ is the minimum weight among all signed total double Roman k-dominating functions on G. In this paper we present sharp lower bounds for $\gamma^2_{stdR}(G)$ and $\gamma^3_{stdR}(G)$ in terms of the order and the size of the graph G.

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1. Introduction

In this paper we only consider finite isolated free graphs without loops and multiple edges. For notation and graph theory terminology we follow [15] in general. Let $G = (V, E)$ be a simple graphs without isolated vertices with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For any set S of vertices of a graph G and any vertex $v \in V(G)$, we denoted $deg_S(v)$, for the number of neighbors of v in S. We write P_n for the *path* of order n, C_n for the *cycle* of length n and K_n for the complete graph of order n. For two disjoint subsets S and T of $V(G)$, we write $[S, T]$ for the set of edges of G joining S to T. If K is a subset of $\mathbb{Z}$ and f is a function from $V(G)$ into K, then we write $V_i = \{v \in V(G) \mid f(v) = i\}$ for each $i \in K$.

In 2016, Beeler et al. [10] defined the double Roman domination as follows. A function $f : V \to \{0, 1, 2, 3\}$ is a *double Roman dominating function* (DRDF) on a graph G if the following conditions hold.

(i) If $f(v) = 0$, then v must have at least one neighbor in V_3 or at least two neighbors in V_2.

(ii) If $f(v) = 1$, then v must have at least one neighbor in $V_2 \cup V_3$.

The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a double Roman dominating function on G. The double Roman domination has been studied by several authors [1, 2, 4, 5]. For further results on several new variations of Roman domination see [6, 7, 8, 11, 14].

Amjadi et al. [9], introduced a new variation of double Roman domination as signed double Roman k-domination number. A *signed double Roman k-dominating function* (SDRkDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(v) = \sum_{u \in N[v]} f(u) \geq k$ holds for any vertex v. The weight of an SDRkDF f is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The *signed double Roman k-domination number* $\gamma^k_{sdR}(G)$ is the minimum weight of an SDRkDF on G. For further results on signed double Roman k-domination see [7, 16].

A *signed total double Roman k-dominating function* (STDRkDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned a 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $f(v) = \sum_{u \in N(v)} f(u) \geq k$ holds for

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any vertex v. The weight of an STDRkDF f is the value $\omega(f) = \sum_{u\in V(G)} f(u)$. The *signed total double Roman k-domination number* $\gamma^k_{stdR}(G)$ is the minimum weight of an STDRkDF on G. For an STDRkDF f, let $V_i(f) = \{v \in V \mid f(v) = i\}$. In the context of a fixed STDRkDF, we suppress the argument and simply write V_{-1}, V_1, V_2 and V_3. Since this partition determines f, we can equivalently write $f = (V_{-1}, V_1, V_2, V_3)$. The concept of signed total double Roman k-domination was introduced and investigated by Shahbazi et al. [12]. The special case $k = 1$ is the usual signed total double Roman domination which has been investigated in [13]. Shahbazi et al. [13] proved that for any connected graph G of order $n \geq 3$ and size m, $\gamma^t_{sdR}(G) \geq \frac{11n-12m}{3}$.

Following the same idea, in this paper we present sharp lower bounds for $\gamma^2_{stdR}(G)$ and $\gamma^3_{stdR}(G)$ in terms of the order and the size of the graph G.

We make use of the following results in this paper.

Propsition A. [12] *For* $n \geq 2$,

$$\gamma^2_{stdR}(P_n) = \begin{cases} 4 & \text{if } n = 2, 3 \\ n & \text{if } n \equiv 0 \pmod 4 \\ n+2 & \text{if } n \equiv 1, 3 \pmod 4 \\ n+3 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Propsition B. [12] *For* $n \geq 2$,

$$\gamma^3_{stdR}(P_n) = \begin{cases} \frac{3n}{2} + 3 & \text{if } n \equiv 2 \pmod 4 \\ \lceil \frac{3n}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

Propsition C. [12] *For* $n \geq 3$,

$$\gamma^2_{stdR}(C_n) = \begin{cases} 4 & \text{if } n = 3 \\ n & \text{if } n \equiv 0 \pmod 4 \\ n+2 & \text{if } n = 6, n \equiv 1 \text{ or } 3 \pmod 4 \text{ and } n \neq 3 \\ n+4 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Propsition D. [12] *If* $n \geq 3$, *then*

$$\gamma^3_{stdR}(C_n) = \begin{cases} \lceil \frac{3n}{2} \rceil + 1 & \text{if } n \equiv 2 \pmod 4 \\ \lceil \frac{3n}{2} \rceil & \text{otherwise.} \end{cases}$$

Propsition E. [12] *For* $k \geq 2$ *and* $n \geq \lceil \frac{k}{2} \rceil + 1$, $\gamma^k_{stdR}(K_n) = k + 2$.

We close this section with two simple results.

Lemma 1.1. *If G is a connected graph of order 4 and size m, then* $\gamma^2_{stdR}(G) \geq \frac{92-24m}{5}$.

Proof. Let G be a connected graph of order 4. If $\Delta(G) = 2$, then $G \in \{P_4, C_4\}$ and the result follows from Propositions A and C. Assume that $\Delta(G) = 3$. If G is the complete graph K_4, then the result follows from Proposition E. Suppose G is not the complete graph K_4. Let $V(G) = \{v_1, v_2, v_3, v_4\}$, $\deg(v_1) = 3$ and

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f be a $\gamma^2_{stdR}(G)$-function. If v_i is a leaf for some $i \in \{2,3,4\}$, say $i=2$, then we have

$$\begin{array}{rcl} \gamma^2_{stdR}(G) & = & \omega(f) \\ & = & f(v_1)+f(N(v_1)) \\ & = & f(N(v_2))+f(N(v_1)) \\ & \geq & 4 \\ & \geq & \frac{92-24m}{5}. \end{array}$$

Hence, we assume that $\delta(G) \geq 2$. This implies that $m \geq 5$ and so

$$\begin{array}{rcl} \gamma^2_{stdR}(G) & = & \omega(f) \\ & = & f(v_1)+f(N(v_1)) \\ & \geq & f(v_1)+2 \\ & \geq & 1 \\ & > & \frac{92-24m}{5}. \end{array}$$

□

2. Lower bounds on $\gamma^2_{stdR}(G)$ and $\gamma^3_{stdR}(G)$

In this section we provide sharp bounds on $\gamma^k_{stdR}(G)$ for $k=2,3$, in terms of the order and the size of G. To this end, we introduce some notation.

If $f=(V_{-1},V_1,V_2,V_3)$ is an STDRkDF of G, then for notational convenience, we assume that $V'_{-1}=\{v \in V_{-1} \mid N(v)\cap V_3 \neq \emptyset\}$ and $V''_{-1}=V_{-1}-V'_{-1}$. Also, we let $V_{12}=V_1\cup V_2, V_{13}=V_1\cup V_3, V_{123}=V_1\cup V_2\cup V_3$, $|V_{12}|=n_{12}, |V_{13}|=n_{13}, |V_{123}|=n_{123}$, $|V_1|=n_1, |V_2|=n_2$, $|V_3|=n_3$ and $|V_{-1}|=n_{-1}$. Then $n_{123}=n_1+n_2+n_3$ and $n_{-1}=n-n_{123}$. Let $G_{123}=G[V_{123}]$ be the subgraph induced by the set V_{123} and let G_{123} have size m_{123}. For $i=1,2,3$, if $V_i \neq \emptyset$, let $G_i=G[V_i]$ be the subgraph induced by the set V_i and let G_i have size m_i. Hence, $m_{123}=m_1+m_2+m_3+|[V_1,V_2]|+|[V_1,V_3]|+|[V_2,V_3]|$.

Theorem 2.1. *Let G be a connected graph of order $n \geq 4$ and size m. Then*

$$\gamma^2_{stdR}(G) \geq \frac{23n-24m}{5}.$$

Proof. Let $f=(V_{-1},V_1,V_2,V_3)$ be a $\gamma^3_{stdR}(G)$-function such that (i) $|V_3|$ is maximized and (ii) subject to (i), $|V_3\cap L|$ is minimized where $L=\{v\in V(G) \mid \deg(v)=1\}$. The result is immediate for $n=4$ by Lemma 1.1. Assume that $n \geq 5$.

If $V_{-1}=\emptyset$, then clearly $\gamma^2_{stdR}(G) \geq n+1 \geq \frac{23n-24m}{5}$ since $n \geq 5$ and $m \geq n-1$. Henceforth, we assume $V_{-1} \neq \emptyset$. We consider the following cases.

Case 1. $V_3 \neq \emptyset$.
We distinguish the following situation.

Subcase 1.1. $V_2 \neq \emptyset$.
Since each vertex in V_{-1} is adjacent to at least one vertex in V_3 or to at least

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two vertices in V_2, we have

$$|[V_{-1},V_3]|+|[V_{-1},V_2]|\geq |V'_{-1}|+2|V''_{-1}|\geq |V'_{-1}|+|V''_{-1}|\geq n_{-1}.$$

Furthermore we have

$$2n_{-1}=2|V'_{-1}|+2|V''_{-1}|\leq 2|[V_{-1},V_3]|+|[V_{-1},V_2]|=2\sum_{v\in V_3}\deg_{V_{-1}}(v)+\sum_{u\in V_2}\deg_{V_{-1}}(u).$$

For each vertex $v\in V_2\cup V_3$, we have that $3\deg_{V_3}(v)+2\deg_{V_2}(v)+\deg_{V_1}(v)-\deg_{V_{-1}}(v)=f(N(v))\geq 2$, and so

$$\deg_{V_{-1}}(v)\leq 3\deg_{V_3}(v)+2\deg_{V_2}(v)+\deg_{V_1}(v)-2.$$

Now, we have

$$\begin{aligned}2n_{-1}&\leq 2\sum_{v\in V_3}\deg_{V_{-1}}(v)+\sum_{u\in V_2}\deg_{V_{-1}}(u)\\&\leq 2\sum_{v\in V_3}(3\deg_{V_3}(v)+2\deg_{V_2}(v)+\deg_{V_1}(v)-2)\\&+\sum_{u\in V_2}(3\deg_{V_3}(u)+2\deg_{V_2}(u)+\deg_{V_1}(u)-2)\\&=(12m_3+4|[V_2,V_3]|+2|[V_1,V_3]|-4n_3)+(3|[V_2,V_3]|+4m_2+|[V_1,V_2]|-2n_2)\\&=12m_3+4m_2+7|[V_2,V_3]|+2|[V_1,V_3]|+|[V_1,V_2]|-4n_3-2n_2\\&=12m_{123}-12m_1-8m_2-5|[V_2,V_3]|-10|[V_1,V_3]|-11|[V_1,V_2]|-4n_3-2n_2,\end{aligned}$$

and this implies that

$$m_{123}\geq \frac{1}{12}(2n_{-1}+12m_1+8m_2+5|[V_2,V_3]|+10|[V_1,V_3]|+11|[V_1,V_2]|+4n_3+2n_2).$$

Therefore,

$$\begin{aligned}m&\geq m_{123}+|[V_{-1},V_{123}]|+m_{-1}\\&\geq m_{123}+|[V_{-1},V_{123}]|\\&\geq \frac{1}{12}(2n_{-1}+12m_1+8m_2+5|[V_2,V_3]|+10|[V_1,V_3]|+11|[V_1,V_2]|+4n_3+2n_2)\\&+|[V_{-1},V_1]|+n_{-1}\\&=\frac{1}{12}(14n_{-1}+4n_{123}-4n_1-2n_2+12m_1+8m_2+5|[V_2,V_3]|+10|[V_1,V_3]|\\&+11|[V_1,V_2]|+12|[V_{-1},V_1]|)\\&=\frac{1}{12}(14n-10n_{123}-4n_1-2n_2+12m_1+8m_2+5|[V_2,V_3]|+10|[V_1,V_3]|\\&+11|[V_1,V_2]|+12|[V_{-1},V_1]|),\end{aligned}$$

and so

$$\begin{aligned}n_{123}\geq \frac{1}{10}(&-12m+14n-4n_1-2n_2+12m_1+8m_2+5|[V_2,V_3]|\\&+10|[V_1,V_3]|+11|[V_1,V_2]|+12|[V_{-1},V_1]|).\end{aligned}$$

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Now, we have

$$\begin{aligned}
\gamma^2_{stdR}(G) &= 3n_3 + 2n_2 + n_1 - n_{-1}\\
&= 4n_3 + 3n_2 + 2n_1 - n\\
&= 4n_{123} - n - n_2 - 2n_1\\
&\geq \frac{4}{10}(-12m + 14n - 4n_1 - 2n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]|\\
&+ 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|) - n - n_2 - 2n_1\\
&= \frac{2}{5}(\frac{23n}{2} - \frac{24m}{2}) + \frac{2}{5}(-9n_1 - \frac{9}{2}n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]|\\
&\quad + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|).
\end{aligned}$$

Let $\Theta = -9n_1 - \frac{9}{2}n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12|[V_{-1}, V_1]|$. We show that $\Theta \geq 0$. First let $n_1 = 0$, then $\Theta = -\frac{9}{2}n_2 + 8m_2 + 5|[V_2, V_3]|$. Let V_2^1 be the set of vertices with label 2 having a neighbor in V_3, V_2^2 be the subset of $V_2 - V_2^1$ with label 2 having a neighbor in V_2^1, V_2^3 be the subset of $V_2 - (V_2^1 \cup V_2^2)$ and etc. Since any vertex in V_2 have a neighbor in $V_3 \cup V_2$, by repeating this process we obtain a partition $V_2^1 \cup V_2^2 \cup \ldots \cup V_2^r$ of V_2 such that each vertex in V_2^i has a neighbor in V_2^{i-1} for each $2 \leq i \leq r-1$ and that $N(x) \cap V_2 \subseteq V_2^r$ for each $x \in V_2^r$. We claim that each vertex $v \in V_2^r$ has at least two neighbors in V_2^r. Suppose, to the contrary, that there exists a vertex $v \in V_2^r$ having exactly one neighbor u in V_2^r. Then $\deg_G(v) = 1$ and since G is connected and f is an STDR2DF of G we deduce that u has a neighbor in $V_2^r - \{v\}$. But then the function g defined on G by $g(v) = 1, g(u) = 3$ and $g(x) = f(x)$ otherwise, is a $\gamma^2_{stdR}(G)$-function which contradicts the choice of f. Hence each vertex $v \in V_2^r$ has at least two neighbors in V_2^r. Then

$$\begin{aligned}
\Theta &= -\frac{9}{2}n_2 + 8m_2 + 5|[V_2, V_3]|\\
&\geq -\frac{9}{2}n_2 + 5|[V_2^1, V_3]| + 8(\sum_{i=2}^{r-1}|[V_2^i, V_2^{i-1}]|) + 8|E(G[V_2^r])|\\
&\geq -\frac{9}{2}n_2 + 5|V_2^1| + 8(\sum_{i=2}^{r-1}|V_2^i|) + 8|V_2^r|\\
&\geq -\frac{9}{2}n_2 + 5|V_2|\\
&> 0.
\end{aligned}$$

Therefore $\gamma^2_{stdR}(G) > \frac{23n-24m}{5}$. Suppose now that $n_1 \geq 1$. Let V_1^1 be the set of vertices with label 1 having a neighbor in V_3 and V_2^1 be the set of vertices with label 2 having a neighbor in $V_3 \cup V_1^1$. Suppose V_1^2 is the subset of $V_1 - V_1^1$ having a neighbor in $V_1^1 \cup V_2^1$ and V_2^2 is the subset of $V_2 - V_2^1$ having a neighbor in $V_1^2 \cup V_2^1$. Since V_1 and V_2 are finite sets, by repeating this process we obtain disjoint subsets $V_1^1 \cup V_1^2 \cup \ldots \cup V_1^r$ of V_1 (possibly some of V_1^i are empty) such

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that each vertex in V_1^i has a neighbor in $V_1^{i-1} \cup V_2^{i-1}$ for each $2 \leq i \leq r$, and disjoint subsets $V_2^1 \cup V_2^2 \cup \ldots \cup V_2^r$ of V_2 (possibly some of V_2^i are empty) so that every vertex in V_2^i has a neighbor in $V_1^i \cup V_2^{i-1}$ for each $2 \leq i \leq r$ and that $V_1^r = V_2^r = \emptyset$. Let $V_1^{r+1} = V_1 - (\cup_{i=1}^r V_1^r)$ and $V_2^{r+1} = V_2 - (\cup_{i=1}^r V_2^r)$. Clearly, $V_1^1 \cup V_1^2 \cup \ldots \cup V_1^{r+1}$ is a weak partition of V_1 and $V_2^1 \cup V_2^2 \cup \ldots \cup V_2^{r+1}$ is a weak partition of V_2. Note that $N(x) \subseteq V_1^{r+1} \cup V_2^{r+1} \cup V_{-1}$ for each $x \in V_1^{r+1} \cup V_2^{r+1}$. Assume that $H_1, \ldots, H_t$ be the components of $G[V_1^{r+1} \cup V_2^{r+1}]$. Since G is connected and f is a STDR2DF of G, we must have $|V(H_i)| \geq 3$ for each $1 \leq i \leq t$, if $V_1^{r+1} \cup V_2^{r+1} \neq \emptyset$. Then

$$\begin{aligned}
\Theta &= -9n_1 - \frac{9}{2}n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| + 12[|V_{-1}, V_1|] \\
&\geq \left(-9|V_1^1| + 10|[V_1^1, V_3]|\right) + \sum_{i=2}^{r} \left(-9|V_1^i| + 12|[V_1^i, V_1^{i-1}]| + 11|[V_1^i, V_2^{i-1}]|\right) + \\
&\quad \left(-\frac{9}{2}|V_2^1| + 5|[V_2^1, V_3]| + 11|[V_1^1, V_2^1]|\right) \\
&\quad + \sum_{i=2}^{r} \left(-\frac{9}{2}|V_2^i| + 8|[V_2^i, V_2^{i-1}]| + 11|[V_1^i, V_2^i]|\right) + \sum_{i=1}^{t} \left(-\frac{9}{2}n(H_i) + 8m(H_i)\right) \\
&\geq \sum_{i=1}^{t} \left(-\frac{9}{2}n(H_i) + 8(n(H_i) - 1)\right) \\
&\geq \sum_{i=1}^{t} \left(\frac{7}{2}n(H_i) - 8\right) \\
&> 0.
\end{aligned}$$

Therefore $\gamma_{stdR}^2(G) > \frac{23n-24m}{5}$.

Subcase 1.2. $V_2 = \emptyset$.

By definition of STDR2DF, each vertex in V_{-1} is adjacent to one vertex in V_3, and so

$$\sum_{v \in V_3} \deg_{V_{-1}}(v) = |[V_{-1}, V_3]| \geq |V_{-1}| = n_{-1}.$$

As in Subcase 1.1, for each $v \in V_3$ we have $3\deg_{V_3}(v) + \deg_{V_1}(v) - \deg_{V_{-1}}(v) = f(N(v)) \geq 2$, and so $\deg_{V_{-1}}(v) \leq 3\deg_{V_3}(v) + \deg_{V_1}(v) - 2$. Now, we have

$$\begin{aligned}
n_{-1} &\leq \sum_{v \in V_3} \deg_{V_{-1}}(v) \\
&\leq \sum_{v \in V_3} (3\deg_{V_3}(v) + \deg_{V_1}(v) - 2) \\
&= 6m_3 + |[V_1, V_3]| - 2n_3 \\
&= 6m_{13} - 6m_1 - 5|[V_1, V_3]| - 2n_3,
\end{aligned}$$

which implies that $m_{13} \geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_3]| + 2n_3)$. Hence,

$$\begin{aligned} m &= m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| + m_{-1} \\ &\geq m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| \\ &\geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_2]| + 2n_3) + n_{-1} + |[V_{-1}, V_1]| \\ &= \frac{1}{6}(7n_{-1} + 2n_3 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\ &= \frac{1}{6}(7n_{-1} + 2n_{13} - 2n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\ &= \frac{1}{6}(7n - 5n_{13} - 2n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|), \end{aligned}$$

and so

$$n_{13} \geq \frac{1}{5}(-6m + 7n - 2n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|).$$

Now, we have

$$\begin{aligned} \gamma^2_{stdR}(G) &= 3n_3 + n_1 - n_{-1} \\ &= 4n_3 + 2n_1 - n \\ &= 4n_{13} - n - 2n_1 \\ &\geq \frac{4}{5}(-6m + 7n - 2n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) - n - 2n_1 \\ &= \frac{4}{5}(-6m + 7n - \frac{5}{4}n - 2n_1 - \frac{5}{2}n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) \\ &= \frac{4}{5}(\frac{23}{4}n - 6m) + \frac{4}{5}(-\frac{9}{2}n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|). \end{aligned}$$

Let $\Theta = -\frac{9}{2}n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|$. We show that $\Theta \geq 0$. If $n_1 = 0$, then $\Theta = 0$. Suppose that $n_1 \geq 1$. Since each vertex of V_1 is adjacent to a vertex of V_3, we have $|[V_1, V_3]| \geq n_1$. It follows that

$$\begin{aligned} \Theta &= -\frac{9}{2}n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]| \\ &\geq -\frac{9}{2}n_1 + 6m_1 + 5n_1 + 6|[V_{-1}, V_1]| \\ &> 0. \end{aligned}$$

Therefore $\gamma^2_{stdR}(G) > \frac{23n-24m}{5}$.

Case 2. $V_3 = \emptyset$.

Since $V_{-1} \neq \emptyset$, we conclude that $V_2 \neq \emptyset$. By definition of STDR2DF, each vertex in V_{-1} is adjacent to at least two vertices in V_2, and so

$$\sum_{v \in V_2} \deg_{V_{-1}}(v) = |[V_{-1}, V_2]| \geq 2|V_{-1}| = 2n_{-1}.$$

As in Subcase 1.1, for each $v \in V_2$ we have that $2\deg_{V_2}(v) + \deg_{V_1}(v) - \deg_{V_{-1}}(v) = f(N(v)) \geq 2$, and so $\deg_{V_{-1}}(v) \leq 2\deg_{V_2}(v) + \deg_{V_1}(v) - 2$.

Now, we have

$$\begin{aligned}2n_{-1} &\leq \sum_{v\in V_2}\deg_{V_{-1}}(v)\\ &\leq \sum_{v\in V_2}(2\deg_{V_2}(v)+\deg_{V_1}(v)-2)\\ &= 4m_2+|[V_1,V_2]|-2n_2\\ &= 4m_{12}-4m_1-3|[V_1,V_2]|-2n_2,\end{aligned}$$

which implies that

$$m_{12}\geq \frac{1}{4}(2n_{-1}+4m_1+3|[V_1,V_2]|+2n_2).$$

Hence,

$$\begin{aligned}m &= m_{12}+|[V_{-1},V_{12}]|+m_{-1}\\ &\geq m_{12}+|[V_{-1},V_{12}]|\\ &\geq \frac{1}{4}(2n_{-1}+4m_1+3|[V_1,V_2]|+2n_2)+2n_{-1}+|[V_1,V_{-1}]|\\ &= \frac{1}{4}(10n_{-1}+2n_{12}-2n_1+4m_1+3|[V_1,V_2]|+4|[V_1,V_{-1}]|)\\ &= \frac{1}{4}(10n-8n_{12}-2n_1+4m_1+3|[V_1,V_2]|+4|[V_1,V_{-1}]|)\end{aligned}$$

and so $n_{12}\geq \frac{1}{8}(-4m+10n-2n_1+4m_1+3|[V_1,V_2]|+4|[V_1,V_{-1}]|)$. Now, we have

$$\begin{aligned}\gamma^2_{stdR}(G) &= 2n_2+n_1-n_{-1}\\ &= 3n_2+2n_1-n\\ &= 3n_{12}-n-n_1\\ &\geq \frac{3}{8}(-4m+10n-2n_1+4m_1+3|[V_1,V_2]|+4|[V_1,V_{-1}]|)-n-n_1\\ &= \frac{3}{8}(-4m+10n-\frac{8}{3}n)+\frac{3}{8}(-\frac{14}{3}n_1+4m_1+3|[V_1,V_2]|\\ &+4|[V_1,V_{-1}]|)\\ &\geq \frac{3}{8}(-4m+\frac{22}{3}n)-\frac{5}{8}m+\frac{5}{8}m+\frac{3}{8}(-\frac{14}{3}n_1+4m_1+3|[V_1,V_2]|\\ &+4|[V_1,V_{-1}]|)\\ &= \frac{1}{8}(-17m+22n)+\frac{3}{8}(-\frac{14}{3}n_1+4m_1+\frac{5}{3}m+3|[V_1,V_2]|\\ &+4|[V_1,V_{-1}]|).\end{aligned}$$

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Let $\Theta = -\frac{14}{3}n_1 + 4m_1 + \frac{5}{3}m + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|$. If $n_1 = 0$, then $\Theta > 0$. Suppose that $n_1 \geq 1$. Since any vertex in V_1 is adjacent to a vertex in V_2, we have

$$\begin{aligned}\Theta &= -\frac{14}{3}n_1 + 4m_1 + \frac{5}{3}m + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]| \\ &\geq -\frac{14}{3}n_1 + \frac{17}{3}m_1 + \frac{14}{3}|[V_1, V_2]|. \\ &\geq 0\end{aligned}$$

Therefore $\gamma^2_{stdR}(G) \geq \frac{1}{8}(22n - 17m) > \frac{1}{5}(23n - 24m)$. This completes the proof. □

In the next example, we present an infinite family of graphs that attain the bound of Theorem 2.1.

EXAMPLE 2.2. For any connected graph F of order $t \geq 2$, let F_t be the graph obtained from F by adding $3\deg_F(v) - 2$ pendant edges to each vertex v of F. Then

$$n(F_t) = n(F) + \sum_{v\in V(F)} (3\deg_F(v) - 2) = 6m(F) - n(F)$$

and

$$m(F_t) = m(F) + \sum_{v\in V(F)} (3\deg_F(v) - 2) = 7m(F) - 2n(F).$$

Assigning a 3 to every vertex in $V(F)$ and a -1 to every vertex in $V(F_t) - V(F)$ produces an STDR2DF of weight

$$3n(F) - \sum_{v\in V(F)} (3\deg_F(v) - 2) = 5n(F) - 6m(F) = \frac{23n(F_t) - 24m(F_t)}{5},$$

and so $\gamma^2_{stdR}(F_t) \leq \frac{23n(F_t)-24m(F_t)}{5}$. Applying Theorem 2.1, we have $\gamma^2_{stdR}(F_t) = \frac{23n(F_t)-24m(F_t)}{5}$.

Next we present a sharp lower bound on $\gamma^3_{stdR}(G)$.

Theorem 2.3. *Let G be a connected graph of order $n \geq 5$ and size m. Then*

$$\gamma^3_{stdR}(G) \geq 6n - 6m.$$

Furthermore, this bound is sharp.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^2_{stdR}(G)$-function such that (i) $|V_3|$ is maximized and (ii) subject to (i), $|V_3 \cap L|$ is minimized where $L = \{v \in V(G) \mid \deg(v) = 1\}$. If $V_{-1} = \emptyset$, then $\gamma^3_{stdR}(G) \geq n + 1 > 6n - 6m$. Suppose that $V_{-1} \neq \emptyset$. Consider the following cases.

Case 1. $V_3 \neq \emptyset$.

First let $V_2 \neq \emptyset$. As in the proof of Theorem 2.1, we have

$$\sum_{v\in V_3} \deg_{V_{-1}}(v) + \frac{1}{2}\sum_{u\in V_2} \deg_{V_{-1}}(u) = |[V_{-1}, V_3]| + \frac{1}{2}|[V_{-1}, V_2]| \geq |V'_{-1}| + |V''_{-1}| = n_{-1},$$

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and for each vertex $v \in V_2 \cup V_3$, $\deg_{V_{-1}}(v) \leq 3\deg_{V_3}(v) + 2\deg_{V_2}(v) + \deg_{V_1}(v) - 3$. Now, we have

$$\begin{aligned} 3n_{-1} &\leq 3\sum_{v\in V_3}\deg_{V_{-1}}(v) + \frac{3}{2}\sum_{u\in V_2}\deg_{V_{-1}}(u) \\ &\leq 3\sum_{v\in V_3}(3\deg_{V_3}(v) + 2\deg_{V_2}(v) + \deg_{V_1}(v) - 3) \\ &+ \frac{3}{2}\sum_{u\in V_2}(3\deg_{V_3}(u) + 2\deg_{V_2}(u) + \deg_{V_1}(u) - 3) \\ &= (18m_3 + 6|[V_2,V_3]| + 3|[V_1,V_3]| - 9n_3) + (\frac{9}{2}|[V_2,V_3]| + 6m_2 + \frac{3}{2}|[V_1,V_2] - \frac{9}{2}n_2) \\ &= 18m_3 + 6m_2 + \frac{21}{2}|[V_2,V_3]| + 3|[V_1,V_3]| + \frac{3}{2}|[V_1,V_2]| - 9n_3 - \frac{9}{2}n_2 \\ &= 18m_{123} - 18m_1 - 12m_2 - \frac{15}{2}|[V_2,V_3]| - 15|[V_1,V_3]| - \frac{33}{2}|[V_1,V_2]| - 9n_3 - \frac{9}{2}n_2, \end{aligned}$$

and so

$$m_{123} \geq \frac{1}{18}(3n_{-1} + 18m_1 + 12m_2 + \frac{15}{2}|[V_2,V_3]| + 15|[V_1,V_3]| + \frac{33}{2}|[V_1,V_2]| + 9n_3 + \frac{9}{2}n_2).$$

Using an argument similar to that described in the proof of Theorem 2.1, we obtain

$$\begin{aligned} n_{123} \geq \frac{1}{12}(&-18m + 21n - 9n_1 - \frac{9}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}|[V_2,V_3]| + 15|[V_1,V_3]| \\ &+ \frac{33}{2}|[V_1,V_2]| + 18|[V_{-1},V_1]|). \end{aligned}$$

Now, we have

$$\begin{aligned} \gamma^3_{stdR}(G) &= 3n_3 + 2n_2 + n_1 - n_{-1} \\ &= 4n_3 + 3n_2 + 2n_1 - n \\ &= 4n_{123} - n - n_2 - 2n_1 \\ &\geq \frac{4}{12}(-18m + 21n - 9n_1 - \frac{9}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}|[V_2,V_3]| \\ &\quad + 15|[V_1,V_3]| + \frac{33}{2}|[V_1,V_2]| + 18|[V_{-1},V_1]|) - n - n_2 - 2n_1 \\ &= \frac{1}{3}(-18m + 18n - 15n_1 - \frac{15}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}|[V_2,V_3]| \\ &\quad + 15|[V_1,V_3]| + \frac{33}{2}|[V_1,V_2]| + 18|[V_{-1},V_1]|) \\ &= 6n - 6m + \frac{1}{3}(-15n_1 - \frac{15}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}|[V_2,V_3]| \\ &\quad + 15|[V_1,V_3]| + \frac{33}{2}|[V_1,V_2]| + 18|[V_{-1},V_1]|). \end{aligned}$$

Let $\Theta = -15n_1 - \frac{15}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}|[V_2,V_3]| + 15|[V_1,V_3]| + \frac{33}{2}|[V_1,V_2]| + 18|[V_{-1},V_1]|$. We show that $\Theta \geq 0$. If $n_1 = 0$, then $\Theta = -\frac{15}{2}n_2 + 12m_2 + \frac{15}{2}|[V_2,V_3]|$ and as in the proof of Theorem 2.1 we can see that $\Theta > 0$ implying that $\gamma^t_{sdR}(G) > 6n - 6m$. Suppose now that $n_1 \geq 1$.

Now we use the notations defined in the proof of Theorem 2.1 (Subcase 1.1). Since G is connected and f is a STDR3DF of G, we must have $|V(H_i)| \geq 3$

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and $\delta(H_i) \geq 2$ for each $1 \leq i \leq t$. It follows that $|E(H_i)| \geq |V(H_i)|$ for each $1 \leq i \leq t$. Thus

$$\begin{aligned}
\Theta &= -15n_1 - \frac{15}{2}n_2 + 18m_1 + 12m_2 + \frac{15}{2}|[V_2, V_3]| + 15|[V_1, V_3]| \\
&+ \frac{33}{2}|[V_1, V_2]| + 18|[V_{-1}, V_1]| \\
&\geq \left(-15|V_1^1| + 15([V_1^1, V_3]|)\right) + \sum_{i=2}^{r}\left(-15|V_1^i| + 18|[V_1^i, V_1^{i-1}]| + \frac{33}{2}|[V_1^i, V_2^{i-1}]|\right) \\
&+ \left(-\frac{15}{2}|V_2^1| + \frac{15}{2}|[V_2^1, V_3]| + \frac{33}{2}|[V_1^1, V_2^1]|\right) \\
&+ \sum_{i=2}^{r+1}\left(-\frac{15}{2}|V_2^i| + 12|[V_2^i, V_2^{i-1}]| + \frac{33}{2}|[V_1^i, V_2^i]|\right) + \sum_{i=1}^{t}\left(-\frac{15}{2}n(H_i) + 12m(H_i)\right) \\
&\geq \sum_{i=1}^{t}\left(-\frac{15}{2}n(H_i) + 12n(H_i)\right) \\
&> 0.
\end{aligned}$$

Therefore $\gamma^3_{stdR}(G) \geq 6n - 6m$.

Now let $V_2 = \emptyset$. As above, we have $\sum_{v\in V_3} \deg_{V_{-1}}(v) = |[V_{-1}, V_3]| \geq n_{-1}$ and $\deg_{V_{-1}}(v) \leq 3\deg_{V_3}(v) + \deg_{V_1}(v) - 3$ for each vertex $v \in V_3$. Now, we have

$$\begin{aligned}
n_{-1} &\leq \sum_{v\in V_3} \deg_{V_{-1}}(v) \\
&\leq \sum_{v\in V_3} (3\deg_{V_3}(v) + \deg_{V_1}(v) - 3) \\
&= 6m_3 + |[V_1, V_3] - 3n_3 \\
&= 6m_{13} - 6m_1 - 5|[V_1, V_3]| - 3n_3,
\end{aligned}$$

which implies that $m_{13} \geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_3]| + 3n_3)$. Hence,

$$\begin{aligned}
m &= m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| + m_{-1} \\
&\geq m_{13} + |[V_{-1}, V_3]| + |[V_{-1}, V_1]| \\
&\geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_3]| + 3n_3) + n_{-1} + |[V_{-1}, V_1]| \\
&= \frac{1}{6}(7n_{-1} + 3n_3 + 6m_1 + 5|[V_1, V_3]| + 6|V_{-1}, V_1|) \\
&= \frac{1}{6}(7n_{-1} + 3n_{13} - 3n_1 + 6m_1 + 5|[V_1, V_3]| + 6|V_{-1}, V_1|) \\
&= \frac{1}{6}(7n - 4n_{13} - 3n_1 + 6m_1 + 5|[V_1, V_3]| + 6|V_{-1}, V_1|)
\end{aligned}$$

and this implies that

$$n_{13} \geq \frac{1}{4}(-6m + 7n - 3n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1|]).$$

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Now, we have

$$\begin{aligned}
\gamma^3_{stdR}(G) &= 3n_3 + n_1 - n_{-1}\\
&= 4n_3 + 2n_1 - n\\
&= 4n_{13} - n - 2n_1\\
&\geq \frac{4}{4}(-6m + 7n - 3n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1]|) - n - 2n_1\\
&= (-6m + 6n) + (-5n_1 + 6m_1 + 5|[V_1, V_3]| + 6|[V_{-1}, V_1|])\\
&> (-6m + 6n).
\end{aligned}$$

Case 2. $V_3 = \emptyset$.
Since f is a STDR3DF of G, we conclude that $\delta(G) \geq 2$ and so $m \geq n$. Now $V_{-1} \neq \emptyset$ implies that $V_2 \neq \emptyset$. By definition of STDR3DF, each vertex in V_{-1} is adjacent to at least two vertices in V_2, and so

$$|[V_{-1}, V_{12}]| \geq |[V_{-1}, V_2]| \geq 2|V_{-1}| = 2n_{-1}.$$

As above, we have $2n_{-1} \leq 4m_{12} - 4m_1 - 3|[V_1, V_2]| - 3n_2$ and hence

$$m_{12} \geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| + 3n_2).$$

Now we have

$$\begin{aligned}
m &= m_{12} + |[V_{-1}, V_{12}]| + m_{-1}\\
&\geq m_{12} + |[V_{-1}, V_{12}]|\\
&\geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]| + 3n_2) + 2n_{-1} + |[V_1, V_{-1}]|\\
&\geq \frac{1}{4}(10n_{-1} + 3n_{12} + 4m_1 + 3|[V_1, V_2]| - 3n_1 + 4|[V_1, V_{-1}]|)\\
&= \frac{1}{4}(10n - 7n_{12} + 4m_1 + 3|[V_1, V_2]| - 3n_1 + 4|[V_1, V_{-1}]|)
\end{aligned}$$

and so

$$n_{12} \geq \frac{1}{7}(-4m + 10n + 4m_1 + 3|[V_1, V_2]| - 3n_1 + 4|[V_1, V_{-1}]|).$$

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Thus

$$\begin{aligned}
\gamma^3_{stdR}(G) &= 2n_2 + n_1 - n_{-1} \\
&= 3n_2 + 2n_1 - n \\
&= 3n_{12} - n - n_1 \\
&\geq \frac{3}{7}(-4m + 10n + 4m_1 + 3|[V_1, V_2]| - 3n_1 + 4|[V_1, V_{-1}]|) - n - n_1 \\
&= \frac{3}{7}(-4m + \frac{23}{3}n - \frac{16}{3}n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\
&\geq \frac{3}{7}(-4m + \frac{23}{3}n) + \frac{3}{7}(-\frac{16}{3}n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\
&= \frac{-12m + 23n}{7} + \frac{3}{7}(-\frac{16}{3}n_1 + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\
&\geq \frac{-12m + 23n}{7} \\
&> -6m + 6n.
\end{aligned}$$

To prove the sharpness, let H_t $(t \geq 2)$ be the graph obtained from a connected graph H of order t by adding $3\deg_H(v) - 3$ pendant edges to each vertex v of H. Then

$$n(H_t) = n(H) + \sum_{v \in V(H)} (3\deg_H(v) - 3) = 6m(H) - 2n(H)$$

and

$$m(H_t) = m(H) + \sum_{v \in V(H)} (3\deg_H(v) - 3) = 7m(H) - 3n(H).$$

Assigning a 3 to every vertex in $V(H)$ and a -1 to every vertex in $V(H_t) - V(H)$ produces an STDR3DF f of weight

$$\omega(f) = 3n(H) - \sum_{v \in V(H)} (3\deg_H(v) - 3) = 6n(H) - 6m(H) = 6n(H_t) - 6m(H_t),$$

and hence $\gamma^3_{stdR}(H_t) \leq 6n(H_t) - 6m(H_t)$. Thus $\gamma^3_{stdR}(H_t) = 6n(H_t) - 6m(H_t)$ and the proof is complete. □

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