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On Fractional Integro-Differential Equation with Multiple Time Delays

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Abstract. This article is devoted to investigate the anlysis for the solution for a class of linear and nonlinear Caputo fractional Fredholm-Volterra integro-differential equations with multiple delays. The convergence and stability analysis for these equations are investigated.

Keywords: Multiple delay, Fractional derivative and integrals, Stability.

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1. Introduction

Recently, investigations into fractional calculus showed that it can used in many physical systems more accurately formulation of fractional derivatives [1]. It appears that many physical processes exhibit a fractal order behavior that may be different according to space or time. Many scientists and mathematicians are attracted to studying the stability of fractional equations, as well as the convergence of the solutions to these equations in some different methods [2].

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Finite delay depicts the delay of response of re-action and appears in many engineering applications such as chemical control systems, laser models aircraft, biology, medicine and internet [3].

Soliman et. al. [4, 5, 6] considered the fractional integro-differential equations with time constant and variable delays. The existence and uniqueness of solutions of the model system and also the stability of equilibrium points are shown. The motivation behind delay fractional order system are discussed in [8].

In this manuscript our main aim is to show the existence of the solutions of the fractional multiple delay integro-differential equations

$$D^{\beta}y(t)=f\Big(t,y(t-\tau(t)),y(t-\tau_1),y(\frac{t}{\tau_2}),\int_a^t G(t,x,y(x))dx,\int_a^b H(t,x,y(x))\ dx\Big),$$
$$\beta\in(0,1),t\in[t_0-\tau,t_0] \tag{1.1}$$

with the initial conditions

$$y(t_0)=\Phi(y), \tag{1.2}$$

where $D^{\beta}y$ refers to the $\beta-$ th fractional derivative of the anonymous function $y(t)\in Y=C([a,b],\mathbb{R}^7)$ which characterized by the Caputo operator, $\tau(t)$ is the time varying delay (continuous delay function), τ_1, τ_2 are the constant and propotional time delays respectively, $\tau_2>0$, $f:[a,b]\times Y\times Y\times Y\to Y$ is a continuous function, $G,H:[a,b]^2\times Y\to Y$ are nonlinear Lipschitz continuous functions of $y(t)$ and $\Phi:Y\to\mathbb{R}^+$ is a continuous function.

Let us suppose the following conditions.
There exists constants $C_f>0$, $C_G>0$, $C_\tau>0$, $C_{\tau_1}>0$, $C_{\tau_2}>0$, $C_H>0$. $C_\Phi>0$

(1) For each $y_1,y_2,z_1,z_2,w_1,w_2,h_1,h_2,g_1,g_2,f_1,f_2\in Y$

$$|f(t,y_1,z_1,w_1,h_1,g_1,f_1)-f(t,y_2,z_2,w_2,h_2,g_2,f_2)|\le C_f\Big[|y_1-y_2|+|z_1-z_2| + |w_1-w_2|+|h_1-h_2|+|g_1-g_2|+|f_1-f_2|\Big] \tag{1.3}$$

(2)

$$|\int_a^t G(t,x,y(x))dx-\int_a^t G(t,x,z(x))dx|\le C_G|y-z| \tag{1.4}$$

(3)

$$|\int_a^b H(t,x,y(x))\ dx-\int_a^b H(t,x,z(x))\ dx|\le C_H|y-z| \tag{1.5}$$

(4)

$$|y(t-\tau(t))-z(t-\tau(t))|\le C_\tau|y-z| \tag{1.6}$$

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(5)

$$|y(t-\tau_1)-z(t-\tau_1)| \leq C_{\tau_1}|y-z| \tag{1.7}$$

(6)

$$|y(\frac{t}{-\tau_2})-z(\frac{t}{\tau_2})| \leq C_{\tau}|y-z| \tag{1.8}$$

(7)

$$|\Phi(y)-\Phi(z)| \leq C_{\Phi}|y-z|. \tag{1.9}$$

Theorem 1.1. *The equation (1.1) is equivalent to*

$$y(t) = I^{\beta} f\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx\Big). \tag{1.10}$$

Proof. Now, we integrate two both sides of equation (1.1) to obtain

$$I\ D^{\beta} y(t) = If\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx\Big).$$

Thus, we get

$$I^{1-\beta} y(t) - \vartheta = If\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx\Big).$$

On employing I^{β}, we have

$$Iy(t) = I^{\beta+1} f\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx\Big) + \frac{\vartheta}{\Gamma(\beta)} \int_0^t \frac{ds}{(t-s)^{1-\beta}}. \tag{1.11}$$

So that,

$$Iy(t) = I^{\beta+1} f\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx\Big) + \frac{\vartheta\ t^{\beta}}{\Gamma(\beta+1)}. \tag{1.12}$$

With the use of differentiation, we get

$$y(t) = I^{\beta} f\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx\Big) + \frac{\vartheta\ t^{\beta-1}}{\Gamma(\beta)}, \tag{1.13}$$

where ϑ is a constant, then at $t = a$ we deduce that (1.1) is equivalent to (1.10). □

Lemma 1.2. *A function $y \in Y$ is a solution of the problem (1.1) if and only if it is a solution of the delay integro-differential equation.*

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2. Methodology

We managed to present existence and uniqueness of the solution for the equation (1.10). Also, we investigate the stability of much more complicated fractional multiple delay integro-differential equations. Let $\rho : Y \to Y$, for any $y \in Y$. Now, we establish the following theorem for the fixed point ρ.

Theorem 2.1. *The operator ρ maps Y into itself and it is also continuous on [a,b].*

Proof. From Cauchy Schwartz inequality,

$$\begin{aligned} \|\rho y(t)\| = &\ \|\Phi(y) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}[f\Big(s, y(s-\tau(s)), \int_a^s G(s,x,u(x))dx \\ & , y(s-\tau_1), y(\frac{s}{\tau_2}), \int_a^s G(s,x,y(x))dx, \int_a^b H(s,x,y(x))\ dx\Big)ds\| \\ & \leq c\|y\| + \frac{f_{\max}}{\Gamma(\beta+1)} t^{\beta}\|y(s-\tau(s))\|C_G\|y\|C_H\|y\| \\ & \leq c_1\|y\| \leq c_2. \end{aligned} \tag{2.1}$$

□

Thus, ρ maps Y into itself. Also, A becomes uniformly bounded. Suppose a sufficiently small number $n > 0$,

$$\begin{aligned} \|\rho y(t+n) - \rho y(t)\| = &\ \frac{1}{\Gamma(\alpha)}[\| \int_0^t (t-s)^{\beta-1} f\Big(s, y((s+n)-\tau((s+n))) \\ & , y((s+n)-\tau_1), y(\frac{(s+n)}{\tau_2}) \\ & , \int_a^b H((s+n),x,y(x))\ dx, \int_a^s G((s+n),x,y(x))dx\Big)ds\| \\ & - \|\int_0^t (t-s)^{\beta-1} f\Big(s, y(s-\tau((s+n))), \int_a^s G(s,x,y(x))dx \\ & , y((s)-\tau_1), y(\frac{(s)}{\tau_2}), \int_a^b H(s,x,y(x))\ dx\Big)ds\|]. \end{aligned} \tag{2.2}$$

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Clearly,

$$\begin{aligned}\|\rho y(t+n)-\rho y(t)\| \leq\ & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|y((s+n)-\tau(s+n)))-y(s-\tau(s))\| \\ & +\|y((s+n)-\tau_1)-y(s-\tau_1)\|+\|y(\frac{s+n}{\tau_2})-y(\frac{s}{\tau_2})\| \\ & +\|\int_a^s G((s+n),x,y(x))-G(s,x,y(x))dx\| \\ & +\|\int_a^b H((s+n),x,y(x))-H(s,x,y(x))\ dx\|ds \\ \leq\ & \frac{C_f t^\beta}{\Gamma(\beta+1)}\Big[\|y((s+n)-\tau((s+n)))-y(s-\tau(s))\| \\ & +C_G\|y(s+n)-y(s))\|+C_H\|y(s+n)-y(s))\|\Big]. \end{aligned} \tag{2.3}$$

Consequently, we thus conclude that

$$\begin{aligned}\|\rho y(t+n)-\rho y(t)\| \leq\ & \frac{C_f t^\beta}{\Gamma(\beta+1)}\Big[C_\tau+C_{\tau_1}+C_{\tau_2}+C_G+C_H\Big]\|y(s+n)-y(s))\| \\ \leq\ & D\|y(s+n)-y(s))\|, \end{aligned} \tag{2.4}$$

where $t\in[a,b]$, $D=\max\{\frac{C_f t^\alpha}{\Gamma(\beta+1)}\Big[C_\tau+C_{\tau_1}+C_{\tau_2}+C_G+C_H\Big]\},\ 0<D<1.$
It follows that,

$$\|\rho y(t+n)-\rho y(t)\| \to 0 \text{ as } n\to\infty.$$

Then, $\rho y(t)$ is continuous on [a,b]. Our approach for proving that ρ is continuous, we assume that y_n converge to y, $\forall n\in\mathbb{N}$. Then

$$\begin{aligned}\|\rho y_n(t)-\rho y(t)\| \quad\leq\quad & \|\Phi(y_n)-\Phi(y)\|+I^\beta f\Big(t,y_n(t-\tau(t)),y_n(t-\tau(t)),\int_a^t G(t,x,y_n(x))dx \\ , \quad & y_n(\frac{t}{\tau_2}),\int_a^b H(t,x,y_n(x))\ dx\Big)-I^\beta f\Big(t,y(t-\tau(t)),\int_a^t G(t,x,y(x))dx \\ , \quad & y(\frac{t}{\tau_2})-y(\frac{t}{\tau_2}),\int_a^b H(t,x,y(x))\ dx\Big),\end{aligned}$$

if we follow the conditions (1)- (7), we arrive at

$$\begin{aligned}\|\rho y_n(t)-\rho y(t)\| \quad\leq\quad & C_\Phi\|y_n-y\|+\frac{C_f}{\Gamma(\beta)}\Big[(C_\tau+C_{\tau_1}+C_{\tau_2}+C_{\tau_1}+C_{\tau_2})\|y_n-y\| \\ + \quad & C_H\|y_n-u\|+C_G\|y_n-y\|\Big].\end{aligned}$$

This is equivalent to

$$\|\rho y_n(t)-\rho y(t)\| \quad\leq \Big[C_\Phi+\frac{C_f}{\Gamma(\beta)}[C_\tau+C_{\tau_1}+C_{\tau_2}+C_H+C_G]\Big]\|y_n-y\|.$$

Hence, we have $\rho y_n(t)\to\rho y(t)$.

2.1. **Existence and uniqueness of the solution.** Here, we will check the existence and uniqueness of solution for the fractional integro-differential equation with multiple delays (constant, proportional and variable).

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Theorem 2.2. *Assume that the conditions (1)-(5) hold, then the non-linear fractional delay integro-differential equation (1.1) has at least a unique solution* $y \in Y$.

Proof. By analogous proof to the continuity of ρ operator.

$$\begin{aligned}\|\rho y(t)-\rho z(t)\| = & \frac{1}{\Gamma(\beta)}[\| \int_0^t (t-s)^{\beta-1} f\Bigg(s, y((s)-\tau(s)) \\ & , y(s-\tau_1), y(\frac{s}{\tau_2}), \int_a^b H(s,x,y(x))\ dx, \int_a^s G((s),x,y(x))dx\Bigg) ds\| \\ & - \| \int_0^t (t-s)^{\alpha-1} f\Bigg(s, z(s-\tau(s)), \int_a^s G(s,x,z(x))dx \\ & , z(s-\tau_1), z(\frac{s}{\tau_2}), \int_a^b H(s,x,z(x))\ dx\Bigg)\|]ds + \|\Phi(y)-\Phi(z)\|.\end{aligned} \tag{2.5}$$

For short,

$$\begin{aligned}\|\rho y(t)-\rho z(t)\| \leq & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [\|y((s)-\tau(s)) - z(s-\tau(s))\| \\ & + \| \int_a^s G(s,x,y(x)) - G(s,x,z(x))dx\| \\ & + \| \int_a^b H(s,x,y(x)) - H(s,x,z(x))\ dx\|]ds \\ & + \|\Phi(y)-\Phi(z)\| + \|y(s-\tau_1) - z(s-\tau_1)\| + \|y(\frac{s}{\tau_2}) - z(\frac{s}{\tau_2})\| \\ \leq & \frac{C_f t^\beta}{\Gamma(\beta+1)} \Bigg[\|y(s-\tau(s) - z(s-\tau(s))\| \\ & + C_G\|y(s)-z(s))\| + C_H\|y(s)-z(s)\|\Bigg] + C_\Phi\|y-z\|.\end{aligned} \tag{2.6}$$

As a result, we obtain

$$\begin{aligned}\|\rho y(t)-\rho z(t)\| \leq & \frac{C_f t^\beta}{\Gamma(\beta+1)} \Bigg[C_\tau + C_G + C_H\Bigg]\|y(s)-z(s)\| \\ & + C_{\ Phi}\|y-z\| \\ \leq & \ U\|y(s)-z(s)\|,\end{aligned} \tag{2.7}$$

where $t \in [a,b]$, $U = \max\{\frac{C_f b^\beta}{\Gamma(\beta+1)}\Bigg[C_\tau + C_{\tau_1} + C_{\tau_2} + C_G + C_H\Bigg] + C_\Phi\}$, provided that $0 < U < 1$. This means that ρ is Lipschitz on Y with Lipschitz constant U. Also, ρ is a well known fixed point as a consequence of fixed point theorem. i.e., ρ is a contraction mapping. So, Eq.(1.1) has at least a unique solution $y \in Y$. □

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Lemma 2.3. *Suppose that $\{y(t)\}$ is a continuous function on [a, b], it satisfies*

$$\begin{cases} D^{\beta}y(t) = f\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx\Big), \\ \\ y(a) = \Phi(y).\ \alpha \in (0,1), \end{cases}$$

Further, $|y(t_1) - y(t_2)| \leq q$. Then $\{y(t)\}$ is equicontinuous on [a,b].

Proof. Without loss of generality, for $t_1, t_2 \in [a, b]$ such that $t_1 < t_2$, we get

$$\begin{aligned} |\rho y(t_2) - \rho y(t_1)| &= |\Phi(y(t_1)) - \Phi(y(t_2))| \\ &+ \frac{1}{\Gamma(\beta)}\int_0^{t_1}(t_1-s)^{\beta-1}\Big[|f\Big(s, y(s-\tau(s)), \int_a^x G(x,w,y(w))dw \\ &, y(t_1-\tau_1), y(\frac{t_1}{\tau_2}), \int_a^b H(x,w,y(w))\ dw|\Big]ds \\ &- \frac{1}{\Gamma(\beta)}\int_0^{t_2}(t_2-s)^{\beta-1}\Big[|f\Big(s, y(s-\tau(s)), \int_a^x G(x,w,y(w))dw \\ &, y(t_1-\tau_2), y(\frac{t_2}{\tau_2}), \int_a^b H(x,w,y(w))\ dw|\Big]ds \\ &\leq \frac{1}{\Gamma(\beta)}\int_0^{t_1}\Big[(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}\Big]|f\Big(s, u(s-\tau(s)) \\ &, \int_a^x G(x,w,y(w))dw, \int_a^b H(x,w,y(w))\ dw|ds \\ &+ \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\Big[|f\Big(s, y(s-\tau(s)), y(s-\tau_1), y(\frac{s}{\tau_2}) \\ &, \int_a^x G(x,w,y(w))dw, \int_a^b H(x,w,y(w))\ dw|\Big]ds + C_{\Phi}\|y-z\| \\ &\leq qC_{\Phi} + \frac{\|f\|_{\infty}}{\Gamma(\beta+1)}\Big[t_1^{\beta} - t_2^{\beta} + 2(t_2-t_1)^{\beta}\Big], \\ &\to 0 \end{aligned} \tag{2.8}$$

whenever $t_2 \to t_1$, $q > 0$, where

$$\|f\|_{\infty} = \sup_{t\in[a,b]} |f(t, ...)|.$$

Thus, $\rho y(t)$ is equicontinuous function in U. This means that ρ is relatively compact. Hence, ρ is compact. In view of Banach contraction mapping theorem, ρ has at least one fixed point (solution of (1.1)) in Y. □

Lemma 2.4. *If the conditions (1)-(5) satisfied, then the non-linear equation (1.1) has a unique solution provided*

$$\max\{\frac{C_f b^{\beta}}{\Gamma(\beta+1)}\Big[C_{\tau} + C_{\tau_1} + C_{\tau_2} + C_G + C_H\Big] + C_{\Phi}\} < 1. \tag{2.9}$$

Our following attention is focused on checking the stability of the solution $y(t)$ for Eq. (1.1) in the frame of Ulam-Hyers and Ulam-Hyers-Rassias.

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3. Stability of the solution for Eq. (1.1).

Theorem 3.1. *Suupose that the conditions (1)-(5) hold. Then the non-linear fractional multiple delay integro-differential equation (1.1) is Ulam-Hyers stable.*

Proof. If $y(t) \in Y$ is a solution of equation (1.1), V(s) is a continuous and non negative function such that

$$\sup(\int_0^t (t-s)^{\beta-1})[W(s)]ds) < \infty,$$

$$|D^\beta y(t) = f\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx\Big)| \le \varepsilon.$$

Now, we are going to perform the integral operator I^β on both sides of above equation, we reach

$$\begin{aligned} |y(t) \quad &- \quad \Phi(y) - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f\Big(s, y(s-\tau(s)), \int_a^s G(s,p,y(p))dp \\ &, \quad y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^b H(s,p,y(p))\ dp\Big)| \\ &\le \quad \frac{\varepsilon}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}ds. \end{aligned}$$

It is equivalent to

$$\begin{aligned} |y(t) \quad &- \quad \Phi(y) - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f\Big(s, y(s-\tau(s)), \int_a^s G(s,p,y(p))dp \\ &, \quad y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^b H(s,p,y(p))\ dp\Big)| \\ &\le \quad \frac{\varepsilon t^\beta}{\Gamma(\beta+1)} \\ &\le \quad \varepsilon E_{1,1}(t), \end{aligned} \tag{3.1}$$

for $z(t) \in Y$, it can be written as

$$\begin{aligned} z(t) \quad &= \quad \Phi(v) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big(f(s, z(s-\tau(s)), \int_a^s G(s,p,z(p))dp \\ &, \quad z(t-\tau_1), z(\frac{t}{\tau_2}), \int_a^b H(s,p,z(p))\ dp\Big)ds. \end{aligned}$$

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The difference $|y(t)-z(t)|$ is given as

$$\begin{aligned}|y(t)-z(t)| &= |y(t)-y(t)+y(t)-z(t)|\\ &\leq |y(t)-\Phi(u)-\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big(f(s,y(s-\tau(s))\\ &, \int_a^s G(s,p,y(p))dp,\ y(t-\tau_1),y(\frac{t}{\tau_2}),\ \int_a^b H(s,p,y(p))\ dp\Big)ds|\\ &+ |z(t)-\Phi(z)-\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big(f(s,z(s-\tau(s))\\ &, \int_a^s G(s,p,z(p))dp,\ z(t-\tau_1),z(\frac{t}{\tau_2}),\int_a^b H(s,p,v(p))\ dp\Big)ds|\end{aligned}$$

or equivalently

$$\begin{aligned}|y(t)-z(t)| &\leq \frac{\varepsilon b^\beta}{\Gamma(\beta+1)}+|\Phi(y)-\Phi(z)|\\ &+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big[\int_a^s (G(s,p,y(p))-G(s,p,z(p)))dp\Big]ds\\ &+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big[\int_a^b (H(s,p,y(p))-H(s,p,z(p)))dp\Big]ds\\ &+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Big[y(s-\tau_1)-z(s-\tau_1)+y(\frac{s}{\tau_2})-z(\frac{s}{\tau_2})dp\Big]ds\\ &\leq \frac{\varepsilon b^\beta}{\Gamma(\alpha+1)}+\frac{C_f}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(C_G+C_H)|y-z|ds+C_\Phi|y-z|\\ &\leq \frac{\varepsilon b^\beta}{\Gamma(\beta+1)}+\frac{R}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|y-z|ds+C_\Phi|y-z|.\end{aligned}$$

In view of Gronwall's lemma, yields

$$\begin{aligned}|y(t)-z(t)| &\leq \frac{\varepsilon b^\beta}{\Gamma(\beta+1)}\exp(\frac{R}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1})ds+C_\Phi|y-z|\\ &\leq \varepsilon E_{1,1}(b)\exp(E_{1,1}(b)R)+C_\Phi|y-z|\\ &\leq \varepsilon K, \end{aligned}\tag{3.2}$$

where $K>0$, $R=C_f(C_G+C_H)$ such that

$$|y(t)-z(t)| \leq \varepsilon K, \tag{3.3}$$

As a result, the problem (1.1) is stable in the sense of Ulam-Hyers. This completes the proof. □

Theorem 3.2. *Suupose that the conditions (1)-(7) satisfied, $P(t)\in Y$ is an increasing function and $\exists\ C_p>0$ such that $I^\beta \leq C_pP(t)$ for any $t\in[a,b]$. Then the non-linear fractional equation (1.1) is Ulam-Hyers-Rassias stable.*

Proof. Let $w\in Y$ be a solution of the following inequality

$$\|D^\alpha w(t)- \quad f\Big[t,w(t-\tau(t)),\int_a^t G(t,x,w(x))dx,\int_a^b H(t,x,w(x))\ dx\Big]\|\leq\varepsilon P(t). \tag{3.4}$$

Further, for any $t \in [a,b]$, $\varepsilon > 0$. Assume that $u \in U$ is the solution of (1.1). Now, integrate (1.1), that is

$$\begin{aligned}
|w(t) &- \Phi(w) - f\Big(s, w(s-\tau(s)), w(s-\tau_1), w(\frac{s}{\tau_2}), \int_a^s G(s,p,w(p))dp \\
&, \int_a^b H(s,p,w(p))\ dp\Big)| \\
&\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} P(t) ds \\
&\leq \varepsilon I^\beta P(t) \\
&\leq \varepsilon C_p P(t).
\end{aligned}$$

It can be easily noticed that

$$\begin{aligned}
|w(t) - y(t)| &= |w(t) - w(t) + w(t) - y(t)| \\
&\leq |w(t) - \Phi(w) - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \Big(f(s, w(s-\tau(s)) \\
&, \int_a^s G(s,p,w(p))dp, \int_a^b H(s,p,w(p))\ dp\Big) ds| \\
&+ C_\Phi \|w-y\| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|w(s-\tau(s)) - y(s-\tau(s))\| ds \\
&+ \int_0^t (t-s)^{\beta-1} \int_a^s \Big(\|G(s,p,w(p)) - G(s,p,y(p))\| dp \Big) ds \\
&+ \int_0^t (t-s)^{\beta-1} \int_a^s \Big(w(s-\tau_1) - y(s-\tau_1) + w(\frac{s}{\tau_2}) - y(\frac{t}{\tau_2}) \| dp \Big) ds \\
&+ \int_0^t (t-s)^{\beta-1} \int_a^b \Big(\|H(s,p,w(p)) - H(s,p,y(p))\| dp \Big) ds.
\end{aligned}$$

Hence,

$$\|w(t) - y(t)\| \leq C_p \varepsilon P(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} E(s) |w-y| ds.$$

It directly follows from Pachpatte's lemma that

$$\|w(t) - y(t)\| \leq C\varepsilon P(t), \tag{3.5}$$

for $C > 0$ which ends the proof. □

Let us extend our results to asymptotically stable solution. For that, we shall perform the absolute value for the solution of (1.1)

$$\begin{aligned}
|y(t)| &\leq |\Phi(y)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \Big[|f\Big(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}) \\
&, \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x))\ dx)| \Big] ds.
\end{aligned}$$

In fact, by means of Cauchy Schwartz inequality, we deduce

$$|y(t)| \leq |\Phi(u)| + \frac{1}{\Gamma(\beta)} J^{\frac{1}{2}} G^{\frac{1}{2}},$$

where

$$J = \int_0^t (t-s)^{2\beta-2} ds, \ J^{\frac{1}{2}} = \frac{t^{\beta-0.5}}{\Gamma(1-2\beta)}$$

$$G = \int_0^t |f(t, y(t-\tau(t)), y(t-\tau_1), y(\frac{t}{\tau_2}), \int_a^t G(t,x,y(x))dx, \int_a^b H(t,x,y(x)) \ dx)|^2 ds.$$

Now, we observe that $|y(t)| \to 0$ whenever $t \to \infty$. Therefore, the zero solution of (1.1) is said to be asymptotically stable.

4. An illustrative example

Here, we give example which clarifying the gained results.

Example 4.1.

$$D^\alpha u(t) - 2u(t) = u(x-1) - \frac{xu(x)}{2} + u(t-\tau_1) + 2\int_0^x u(\frac{t}{\tau_2})^2 dt + f(x) \quad (4.1)$$

where $\tau(t) = Lnt$, $\tau_1 = 1$, $\tau_2 = 2$, $f(x) = 1 + e^x(\frac{Lnx}{x} - 1 - \frac{x-1}{e})$, $u(0) = 0$. The exact solution at $\alpha = 1$ is xe^x.

Solution:

$$RG[u(t)]((\frac{s}{v})^\alpha + 1) = RG[2u(t) + u(x-1) - \frac{xu(x)}{2} + u(t-\tau_1) + 2\int_0^x u(\frac{t}{\tau_2})^2 dt + f(x)). \quad (4.2)$$

By using Ramadan Group properties, RGAD method [7] and we get the analytic solution

$$u(t) = te^t. \quad (4.3)$$

This is the required solution.

Remark 4.2. If the multiple delay becomes constant delay only in the studded problem (1.1), the problem converts as the same [4]. Also, if If the multiple delay becomes time-varying (variable) delay only in the studded problem (1.1), the problem converts as the same [5, 6].

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