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A Class of Commutative Semirings with Stable Range 2 II

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Abstract. The notion and some properties of (strongly) B-rings, in a natural way, are extended to (strongly) B- and (strongly) B_J-semirings which is somewhat similar to the notion of rings having stable range 2. Results are given showing the connection between several types of semirings whose finite sequences satisfy some stability condition, some involving the Jacobson k-radical of the semiring R. Besides some examples and other results, it is shown that $R[x]$, the semiring of polynomials over a semiring R, is not a B-semiring (consequently, not a strongly B-semiring) when R is a zerosumfree semiring. We also study some algebraic properties of the S-relative B- and B_J-semirings with respect to a nonempty subset S of R.

Keywords: (Strongly) B- and (Strongly) B_J-semirings, S-relative B- and S-relative B_J-semirings, Subtractive ideal (= k-ideal), Simple semiring, Gelfand semiring, Polynomial semiring, Stable range of a commutative semiring.

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1. Introduction

The main purpose of this paper is to continue and extend the work of the second author [12] (which is a study of some properties of *B- and B_J-semirings*) to *(strongly) B- and (strongly) B_J-semirings* ($(S)B$- and $(S)B_J$-semirings for short, respectively) in general as a natural extension of *(strongly) B-rings* [10], which is similar to the notion of rings having *stable range* 2 (see Definitions 3.1, 3.3, 4.1, and Remark 3.6(d)). Besides some other results on B- and B_J-semirings (given in [12]), we will turn our attention to the study of SB-semirings [resp. SB_J-semirings]. Our main objective here is to compare the theory of this particular subclass of B-semirings [resp. B_J-semirings] with that of B-semirings [resp. B_J-semirings].

$*$ We assume that the reader is familiar with basic notion of (commutative) semiring theory. However, in Section 1, we will write all necessary definitions and results related to (commutative) semirings that are required in this work for the sake of completeness and mainly follow Golan [5].

The concept of *stable range* was initiated by H. Bass in his investigation of *the stability properties of the general linear group in algebraic K-theory* [2]. In ring theory, stable range provides an arithmetic invariant for rings that is related to interesting issues such as cancelation, substitution, and exchange. The simplest case of stable range 1 has especially proved to be important in the study of many ring-theoretic topics.

• In this paper *a semiring (ring) R*, unless otherwise indicated, is commutative with identity $1 \neq 0$ and $0a = 0$ for all $a \in R$; and $U(R)$ denotes the set of units of R. By *a B-type semiring*, we mean *a (strongly) B-, or a (strongly) B_J-, or an S-relative B-, or an S-relative B_J-semiring* (Definition 3.9), where S is a nonempty subset of R. Also by a sequence of elements of R, we mean *a finite sequence* and will use it implicitly without any confusion in the context.

Definition 1.1. Let R be a commutative semiring (ring) and $s \geq 1$ an integer. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements of R is said to be *stable* if $(a_1, a_2, \ldots, a_s, a_{s+1}) = (a_1 + b_1 a_{s+1}, a_2 + b_2 a_{s+1}, \ldots, a_s + b_s a_{s+1})$ for some $b_1, b_2, \ldots, b_s \in R$. A sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$ of elements of R is said to be a *unimodular sequence* if 1 is in the ideal $(a_1, a_2, \ldots, a_s, a_{s+1})$.

Remark 1.2. As in [4], we use $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 1$, to denote both a sequence and the ideal generated by the elements of the sequence; but the context will always make our meaning clear. Also, we follow [4] for the term "unimodular sequence" instead of "primitive vector" as used in [10]. For a detailed study of stable range in commutative rings and (strongly) B-rings; see [4, 10, 11, 13, 14].

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We now recall the definition of a (strongly) B-ring from [10] (see also Definitions 3.3 and 4.1 for the definitions of (strongly) B- and (strongly) B_J-semirings).

• Let $J(R)$ be the *Jacobson radical* of a commutative ring R. A ring R is said to be a *B-ring* if for any unimodular sequence $(a_1, \ldots, a_{n+1})$, $n \geq 2$ with $(a_1, \ldots, a_{n-1}) \nsubseteq J(R)$, there exists an element b in R such that $(a_1, \ldots, a_n + ba_{n+1}) = R$.

• Similarly, R is defined to be a *strongly B-ring* (or *SB-ring* for short) if $d \in (a_1, \ldots, a_n, a_{n+1})$, $n \geq 2$, and $(a_1, a_2, \ldots, a_{n-1}) \nsubseteq J(R)$ implies that there exists $b \in R$ such that $d \in (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$.

In [10], Moore and Steger studied some properties of (strongly) B-rings in detail. Besides many other results regarding B- and SB-rings, they showed that *a regular ring* is an SB-ring ([10, Corollary 3.2]) and $R[X]$ is a B-ring ([10, Theorem 2.7]) [resp. an SB-ring ([10, Theorem 3.4])] if and only if R is *completely primary* (a ring consisting of units and nilpotents) [resp. a field]; and we will discuss some of these results for B-type semirings in the sequel.

$**$ The organization of this paper is as follows: In Section 1 we recall some standard definitions and results from semiring theory that will be used in the sequel. Sections 2 [resp. and 3], are devoted to (S-relative, Definition 3.9) B- and B_J-semirings [resp. SB- and SB_J-semirings]. In our study of these semirings, it suffices only to consider unimodular [resp. arbitrary] triples instead of arbitrary unimodular $(n+1)$-tuples (Remark 3.6(b) and for S-relative case (Theorem 3.12), sequences of size 3 need not be unimodular (i.e., should satisfy a special condition) [resp. $(n+1)$-tuples (Theorem 4.3)]. In these two sections, we discuss the homomorphic image of B-type semirings (Theorems 3.13 and 4.6) and also in the beginning of Section 2 recall some definitions and results from [12] (Definitions 3.1, 3.3, and Remark 3.6). In Section 3, we show that $R[x]$ (the semiring of polynomials over a semiring R) is not a B-semiring (consequently, not a strongly B-semiring) when R is a zerosumfree semiring (Theorem 4.9). Also in Theorem 4.9, we show that $R[x]$ can not be a B-semiring with respect to $I[x]$ when I is a strong proper ideal of R. In Section 4, besides some other results (Propositions 5.1, 5.3, and Example 5.4), it is shown that a plain simple yoked regular semiring is an SB-semiring (Theorem 5.2). Finally, in this section, we write a brief note on the notion of matrix completion over commutative rings and close the paper by posing a question related to the matrix completion of B- and 2-stable semirings.

[DOI: 10.61186/ijmsi.19.2.77]

2. Commutative semirings

In this section we recall some definitions and prove some results concerning semirings which will be used in the sequel. By a semiring $(R, +, \cdot)$, we will mean a nonempty set R with two binary operations of addition and multiplication defined on R such that $(R, +)$ and $(R, \cdot)$ are commutative monoids with identity elements 0 and 1, respectively, where Multiplication distributes over addition (from either side) and $0a = 0$ for all $a \in R$ and $1 \neq 0$.

• A nonempty subset I of a semiring R will be called an *ideal* if $a, b \in I$ and r in R implies $a + b$ in I and ra in I.

Note that in [5, Chapter 5], Golan defines an ideal I of a semiring R to be different from R, but we don't follow this assumption and make it clear when there is any confusion in the context. In general, by "ideal" (in contrast to Golan, we do not necessarily mean a proper ideal. We shall thus always say "proper ideal" when we mean a proper ideal.

Definition 2.1. A subtractive ideal (= k-ideal) I of a semiring S is an ideal such that if $a, a + b \in I$, then $b \in I$. An ideal I of S is said to be a strong ideal (= a strongly k-ideal) if and only if $a + b \in I$ implies that $a \in I$ and $b \in I$.

Remark 2.2. From the above definition, it is clear that (0) is a k-ideal of S. Also, every strongly k-ideal of a semiring S is a k-ideal of S. But the converse need not be true in general. For example, the set $2N$ of all nonnegative even integers is a subtractive ideal of the semiring of all nonnegative integers. But it is not a strongly k-ideal since $3 + 5 \in 2N$ while neither 3 nor 5 belong to $2N$. Note that in [5], Golan uses the term "subtractive ideal", [resp. strong] for a k-ideal [resp. strongly k-ideal] but in the literature of semirings, authors use equivalently the term "k-ideal" [resp. strongly k-ideal] as well. Throughout this work, except for some cases in this section, we mainly follow Golan in [5]. Also, for some examples of nonsubtractive ideals in a semiring, see Chapter 5 of [5].

• We define the *Jacobson k-radical* of a semiring R, denoted by $J_k(R)$ (= $\mathrm{Jac}(R)$ as used in [12]), to be the intersection of all maximal k-ideals of R. Notice that by [15, Corollary 2.2], the Jacobson k-radical of R always exists and it can easily be seen that it is a k-ideal since the intersection of any number of k-ideals is a k-ideal.
We now follow Golan [5, Chapter 8, p. 92] to define a morphism of semirings as follows.

Definition 2.3. If R and S are semirings then a function $f : R \to S$ is a morphism of semirings if and only if:

[DOI: 10.61186/ijmsi.19.2.77]

(a) $f(0_R) = 0_S$;
(b) $f(1_R) = 1_S$; and
(c) $f(r + r') = f(r) + f(r')$ and $f(rr') = f(r)f(r')$ for all r and r' in R.

We now begin considering some properties of morphisms of semirings.

Proposition 2.4. (cf. [5, Proposition 8.37]) Let $f : R \to S$ be a morphism of semirings.

(a) If H is an ideal of S, then $f^{-1}(H)$ is an ideal of R. Moreover, if H is subtractive then so is $f^{-1}(H)$.
(b) If f is a surjective morphism and I is an ideal of R, then $f(I)$ is an ideal of S.
(c) If f is a surjective morphism, then the kernel of f is a subtractive ideal of R.
(d) If f is a surjective morphism, then u is a unit in R if and only if $f(u)$ is a unit in S.

Proof. Parts (a) and (b) follows from [5, Proposition 8.37] and (c) follows from (a) since $\ker(f) = f^{-1}(\{0\})$. The necessary part of (d) is clear since $1_S = f(1_R) = f(uu^{-1}) = f(u)f(u^{-1})$. Conversely, let $u \in R$, $I = (u)$ an ideal of R, and $f(u)$ be a unit in S. Clearly $f(I) = S$ since $f(u)$ is a unit in S and so $I = R$. Otherwise, $I \neq R$ implies $1_R \in R \setminus I$, which implies $1_S = f(1_R) \notin f(I) = S$, yielding a contradiction. Thus u is a unit in R. □

Remark 2.5. (As defined on page 68 of [5]), an ideal I of a semiring R defines an *equivalence relation* $=_I$ on R called the *Bourne relation*, given by $r =_I r'$ if and only if there exist elements a and a' of I satisfying $r + a = r' + a'$. Note that if $r =_I r'$ and $s =_I s'$ in R, then $r + s =_I r' + s'$ and $rs =_I r's'$. We denote the set of all *equivalence classes* of elements of R under this relation by R/I and will denote the equivalence class of an element r of R by r/I. Clearly this relation is a *congruence* (i.e., an equivalence relation which is compatible with two binary operations of R) and, consequently, R/I is well-defined for any ideal I of R. Also, $a \in I$ implies $a \in 0/I$ since $a =_I 0$ by the fact that $a + 0 = 0 + a$. Thus $I \subseteq 0/I$. Moreover, if I is a subtractive ideal of R, then $0/I = I$ since $a + i = 0 + j \in I$ implies $a \in I$. Thus, for any subtractive ideal I of R, the factor semiring R/I and the surjective morphism $f : R \to R/I$, given by $r \mapsto r/I$, is well defined and its kernel is I. See also Example 9.1 and Proposition 9.10 in [5].

We end this section by recalling some more definitions from [5] and write them here for the sake of completeness as follows.

• A semiring with no nonzero zero divisors is called an *entire (= semidomain)*. A *semifield* is a semiring in which every nonzero element has a *multiplicative inverse*. A semiring R is *zerosumfree* if and only if $r + r' = 0$ implies

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that $r = r' = 0$. A semiring R is said to be *simple* if $1 + r = 1$ for each $r \in R$. Let R be a semiring and $G(R) = \{r \in R \mid 1+r \in U(R)\}$. A semiring R is called a *Gelfand semiring* when $G(R) = R$. Clearly, every simple semiring is Gelfand. Of course, bounded distributive lattices are among Gelfand semirings. But the class of the Gelfand semirings is quite wider as Example 1.4 in [12] shows(cf.[5, Example 3.38]).

3. B- and B_J-semirings with some preliminary results on the stable range of commutative semirings

In this section, we first recall some definitions and results from section 2 of [12], respectively, Definitions 3.1, 3.3, and Remark 3.6 and then discuss some more properties of B-type semirings. That is, we extend the notion of the stable range of a commutative ring ([4]) to the stable range of a commutative semiring (or an *n-stable semiring*, for short) (Definition 3.1) and merely focus on some simple results and properties of n-stable semirings, B- and B_J-semirings, where B-semirings can be regarded as a generalization of a subclass of *2-stable rings* and B_J-semirings are exactly a natural extension of B-rings to semirings (see Definition 3.3).

Definition 3.1. Let R be a commutative semiring and $s \geq 1$ an integer. An integer $n \geq 1$ is said to be in the *stable range* of R (or simply, R is n-stable) if every unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq n$, of elements of R is stable. The semiring R is said to be *n_J-stable* if every unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq n$, of elements of R with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J_k(R)$ is stable.

Remark 3.2. It is clear that if R is n-stable, then it is m-stable for any integer $m \geq n$. Note that the term "R is n-stable" is used in [11] (for convenience) and is exactly the same as the statement "n is in the stable range of R", which is used by D. Estes and J. Ohm [4, page 345].

Definition 3.3. A commutative semiring R is said to be a *B-semiring* [resp *B_J-semiring*] whenever for any unimodular sequence $(a_1, a_2, \ldots, a_s, a_{s+1})$, $s \geq 2$, of elements in R [resp. with $(a_1, a_2, \ldots, a_{s-1}) \nsubseteq J_k(R)$], there exists an element b in R such that $(a_1, a_2, \ldots, a_s + ba_{s+1}) = R$.

Remark 3.4. In the above definition, it is clear that the definition of a B_J-semiring is exactly the same as the definition of a B-ring whenever R is assumed to be a ring as defined in [10] and obviously, any B-semiring is a B_J-semiring.

The following example provides a trivial instance of a class of B-semirings.

Example 3.5. A semifield is a B-semiring (consequently, a B_J-semiring). That is, $1 \in (a_1, a_2, \ldots, a_n, a_{n+1}) = (a_1, a_2, \ldots, a_n + ba_{n+1})$, where $b = 0$ when $a_n \neq 0$; or $b = 1$ when $a_n = 0$. Moreover, besides some trivial examples of

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semifields such as semifields of nonnegative reals and nonnegative rationals, see [5, Proposition 7.8] that states: If I is a subtractive maximal ideal of a commutative semiring R, then R/I is a semifield.

We now recall some of the results from [12] for the sake of comparison and completeness.

Remark 3.6. The following facts are true in a commutative semiring.

(a) If all unimodular sequences of size $n+1$ ($n \geq 1$ a fixed integer) of a commutative semiring R are stable, then any unimodular sequence of size larger than n is stable (see [12, Theorem 2.6]).

(b) A commutative semiring R is a B-semiring [resp. B_J-semiring] if and only if for any unimodular sequence (a_1, a_2, a_3) of R [resp. with $a_1 \notin J_k(R)$], there exists an element $b \in R$ such that $(a_1, a_2 + ba_3) = R$ (see [12, Theorem 2.7]).

(c) Let $n \geq 1$ be a fixed integer and R a commutative semiring in which every maximal ideal is subtractive. Then R is n-stable if and only if R is n_J-stable (see [12, Theorem 2.8]).

(d) Let R be a B_J-semiring in which every maximal ideal is subtractive. Then R is 2-stable (see [12, Corollary 2.9]).

(e) Let R be a commutative semiring in which every maximal ideal is subtractive [in particular, R is a subtractive semiring (i.e., a semiring in which every ideal is subtractive)] and $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 1$, a unimodular sequence of R. Then $(a_1, a_2, \ldots, a_i + a_{n+1}, \ldots, a_n) = R$ provided that $a_i \in J_k(R)$ for some $1 \leq i \leq n$. Further, $(a_1, a_2, \ldots, a_i + a_{n+1}, \ldots, a_n) = R$ for each $1 \leq i \leq n$ provided that $a_{n+1} \in J_k(R)$ (see [12, Proposition 2.18]).

(f) Let R be a Gelfand semiring and let $(a_1, a_2, \ldots, a_n, a_{n+1})$, $n \geq 1$, be a unimodular sequence of R. Then $1 \in (a_1, a_2, \ldots, a_n + ba_{n+1})$ for some $b \in R$ (i.e., one is in the stable range of R). In other words, we may simply say R is a B-semiring when $n \geq 2$ (see [12, Theorem 2.10]).

(g) A simple semiring is a B-semiring (consequently, a B_J-semiring), see [12, Corollary 2.11].

(h) A semiring R is a B-semiring (consequently, a B_J-semiring) provided that R is a semiring in which every maximal ideal is strong (see [12, Corollary 2.11 and Proposition 2.15]).

• In view of the above remark, Part (b), we need only consider the unimodular triples instead of arbitrary unimodular $(n+1)$-tuples, $n > 2$, in our study of B- and B_J-semirings.

We now, similar to Remark 3.6(e), show the stability of a unimodular sequence of a semiring under some special conditions.

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Proposition 3.7. *Let $(a_1, a_2, \dots, a_n, a_{n+1})$, $n \geq 1$, be a unimodular sequence in a semiring R such that a_n or a_{n+1} is in $J_k(R)$. Then:*

(a) $(a_1, a_2, \dots, a_n + a_{n+1}) = R$ *provided that every maximal ideal of R is subtractive.*

(b) $(a_1, a_2, \dots, a_n + a_{n+1}) = R$ *provided that $(a_1, a_2, \dots, a_n + a_{n+1})$ is a subtractive ideal of R.*

(c) $(a_1, a_2, \dots, a_n + a_{n+1}) = R$ *provided that R is a subtractive semiring (i.e., a semiring in which every ideal is subtractive).*

Proof. Parts (a) and (c) follows from the fact that every ideal in a semiring is contained in a maximal ideal [5, Proposition 5.47]. Part (b) follows from [15, Corollary 2.2] that states any subtractive ideal in a semiring is contained in a maximal subtractive ideal. □

Proposition 3.8. *The direct product of any family of semirings is a B-semiring if and only if each factor of the product is a B-semiring.*

Proof. Let $\{R_i \mid i \in I\}$ be a family of semirings and $R = \Pi_{i\in I} R_i$. The necessary part is an immediate consequence of Theorem 3.13(a) below, which states that the homomorphic image of a B-semiring is a B-semiring. To prove the sufficiency, let R_i be a B-semiring for each $i \in I$. Suppose $R = (\{a_i\}, \{b_i\}, \{c_i\})$. Thus $1_R = \{1_i\} \in (\{a_i\}, \{b_i\}, \{c_i\})$, where 1_i is the identity element of R_i for each $i \in I$. Consequently $1_i \in (a_i, b_i, c_i)$ for each $i \in I$, which implies $1_i \in (a_i, b_i + d_i c_i)$, where $d_i \in R_i$ for each $i \in I$. Therefore, $1_R = \{1_i\} \in (\{a_i\}, \{b_i\} + \{d_i\}\{c_i\})$ and so by virtue of Remark 3.6(b), R is a B-semiring. □

We now introduce a class of B-type semirings that are defined with respect to a nonempty subset S of a semiring R.

Definition 3.9. Let S be a nonempty subset of a semiring R. R is said to be a B-semiring [resp. B_J-semiring] with respect to S or R is an S-relative B-semiring [resp. an S-relative B_J-semiring] if for any ideal $(a_1, a_2, \dots, a_n, a_{n+1})$, $n \geq 2$, of R and $a \in S$ [resp. with $(a_1, a_2, \dots, a_{n-1}) \not\subseteq J_k(R)$] such that $1 + a \in (a_1, a_2, \dots, a_n, a_{n+1})$, then there exists $b \in R$ such that $1 + a \in (a_1, a_2, \dots, a_{n-1}, a_n + b a_{n+1})$.

Remark 3.10. From the above definition, a B-semiring [resp. B_J-semiring] is a $\{0\}$-relative (or simply, 0-relative) B-semiring [resp. 0-relative) B_J-semiring]. Clearly, every B-semiring [resp. B_J-semiring] with respect to a nonempty subset S of R is a B-semiring [resp. B_J-semiring] provided $0 \in S$. Moreover, every SB-semiring [resp. SB_J-semiring] (Definition 4.1) is an S-relative B-semiring [resp. S-relative B_J-semiring] for each nonempty subset S of R. Also, let $S \subseteq T$ be two nonempty subsets of a semiring R. Then R is an S-relative B-semiring [resp. S-relative B_J-semiring] if R is a T-relative B-semiring [resp. T-relative B_J-semiring]. Clearly, R is a B-semiring [resp. B_J-semiring] if

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and only if R is a $G(R)$-relative B-semiring [resp. $G(R)$-relative B_J-semiring], where $G(R) = \{a \in R \mid 1 + a \in U(R)\}$.

• From the above remark, it is clear that the class of SB- and SB_J-semirings (Definition 4.1) are contained in the class of S-relative B- and S-relative B_J-semirings, respectively. Further, the class of S-relative B- and S-relative B_J-semirings are contained in the class of B- and B_J-semirings, respectively, provided that $0 \in S$.

Example 3.11. In Remark 3.6(f and g), it is shown that a Gelfand semiring R [in particular, a simple semiring] is a B-semiring. Thus from the above remark, R is an R-relative B-semiring or equivalently an S-relative B-semiring for any nonempty subset S of R.

We now provide a criterion for the study of S-relative B- and S-relative B_J-semirings (see Remark 3.6(b)).

Theorem 3.12. (cf. [13, Theorem 2]) Let S be a nonempty subset of a semiring R. A semiring R is an S-relative B-semiring [resp. an S-relative B_J-semiring] if and only if for every $a \in S$ and $c_1, c_2, c_3 \in R$ with $1 + a \in (c_1, c_2, c_3)$ [resp. $c_1 \notin J_k(R)$], it follows that $1 + a \in (c_1, c_2 + bc_3)$ for some $b \in R$.

Proof. The proof is similar to the proof of Remark 3.6(b)) by replacing 1 with $1 + a$ (see also the proof of Theorem 4.3 below). □

• In view of the above theorem, we need only consider the sequences of size three that satisfy B-stability condition with respect to a nonempty subset S of R in our study of S-relative B- and S-relative B_J-semirings.

We now consider the homomorphic image of B- and B_J-semirings. Also, Abdolyousefi and Chen in [1, Lemma 2.9] show the similar result for J-stable rings and they refer to the work of the second author [13, Theorem 3] that shows the homomorphic image of a B-ring is a B-ring. Further, they show how the classes of J-stable rings and B-rings coincide with each other (see the paragraph preceding Theorem 2.5 and Remark 2.6 in [1]).

Theorem 3.13. (cf. [13, Theorem 3]) Let $f : R \to S$ be a surjective morphism of semirings.

(a) If R is a B-semiring, then so is S.
(b) If R is a B_J-semiring, then so is S provided that $f(J_k(R)) \subseteq J_k(S)$.

Proof. We write a proof for Part (b) and leave the other part to the reader. By virtue of Remark 3.6(b), it suffices to argue only for unimodular sequences of size three. Suppose R is a B_J-semiring and let $1_S \in (x_1, x_2, x_3)$ with $x_1 \notin J_k(S)$, where $x_1, x_2, x_3 \in S$. Thus $f(1_R) = 1_S = \sum s_i x_i$ for some $s_i \in S$, where $i = 1, 2, 3$. Therefore, $f(1_R) = \sum f(r_i) f(a_i) = \sum f(r_i a_i) = f(\sum r_i a_i)$

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for some r_i and a_i in R, where $f(r_i) = s_i$ and $f(a_i) = x_i$ and $i = 1, 2, 3$. Clearly $1_R \in (a_1, a_2, a_3)$ and $a_1 \notin J_k(R)$ by hypothesis (see also Proposition 2.4). Thus $1_R \in (a_1, a_2 + ba_3)$ for some $b \in R$ since R is a B_J-semiring. Consequently $f(1_R) = 1_S \in (f(a_1), f(a_2) + f(b)f(a_3)) = (x_1, x_2 + sx_3)$, where $s = f(b)$. □

Corollary 3.14. *Let I be a proper ideal of a semiring R. Then R/I is a B-semiring when R is a B-semiring.*

Proof. Clearly $r \mapsto r/I$ defines a surjective morphism from R to R/I, where $r \in R$. Now the proof follows directly from Part (a) of the above theorem. See also Example 9.1 and Proposition 9.10 in [5]. □

We conclude this section with extending the above corollary to an S-relative B-semiring.

Theorem 3.15. *Let I be a proper ideal of a semiring R and S a nonempty subset of R. Then R/I is an S/I-relative B-semiring when R is an $S+I$-relative B-semiring. Further, if $S \cap I \neq \emptyset$ [in particular, if $0 \in S$], then R/I is also a B-semiring.*

Proof. By virtue of Theorem 3.12, it suffices to argue only for sequences of size three. Suppose $1/I + s/I \in (a_1/I, a_2/I, a_3/I)$, where $s \in S$. Thus $(1+s)/I = \sum(r_i/I)(a_i/I) = \sum(r_i a_i)/I$ for some $r_i \in R$, where $i = 1, 2, 3$. Therefore, $1 + s + a = r_1a_1 + r_2a_2 + r_3a_3 + a'$ for some $a, a' \in I$ by definition. Thus $1 + s + a \in (a_1, a_2, r_3a_3 + a')$. Now by hypothesis, there exists $b \in R$ such that $1+s+a \in (a_1, a_2+b(r_3a_3+a'))$. Consequently, $(1/I)+(s/I) \in (a_1/I, (a_2/I)+(br_3)/I(a_3/I))$ and the proof of the first part is complete. The "further" part is clear since $a/I = 0/I$ when $a \in I$ (see also Remarks 2.5 and 3.10). □

4. SB- and SB_J-semirings

We now turn our attention to the study of SB-semirings [resp. SB_J-semirings] (Definition 4.1). Our main objective here is to compare the theory of this particular subclass of B-semirings [resp. B_J-semirings] with that of B-semirings [resp. B_J-semirings] given in the previous section.

Definition 4.1. R is defined to be a *strongly B-semiring* (or SB-semiring for short) [resp. strongly B_J-semiring (or SB_J-semiring for short)] if $d \in (a_1, \ldots, a_n, a_{n+1})$, $n \geq 2$, [resp. with $(a_1, a_2, \ldots, a_{n-1}) \not\subseteq J_k(R)$] implies that there exists b in R such that $d \in (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$.

Remark 4.2. In the above definition, it is clear that the definition of an SB_J-semiring is exactly the same as the definition of an SB-ring whenever R is assumed to be a ring as defined in [10] and obviously, any B-semiring [resp. SB-semiring] is a B_J-semiring [resp. SB_J-semiring]. Also, it is clear that any SB-semiring [resp. SB_J-semiring] is a B-semiring [resp. B_J-semiring]. See the following diagram.

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$$SB\text{-semiring} \to B\text{-semiring} \to B_J\text{-semiring}$$
$$SB\text{-semiring} \to SB_J\text{-semiring} \to B_J\text{-semiring}$$

We now provide a criterion for the study of SB- and SB_J-semirings (see Remark 3.6(b)).

Theorem 4.3. *A semiring R is an SB-semiring [resp. SB_J-semiring] if and only if for every $s, c_1, c_2, c_3 \in R$ with $s \in (c_1, c_2, c_3)$ [resp. $c_1 \notin J_k(R)$], it follows that $s \in (c_1, c_2 + bc_3)$ for some $b \in R$.*

Proof. The proof is essentially similar to the proof of Lemma 3.1 of [10]. The necessity clearly follows from the definition of an SB-semiring [resp. SB_J-semiring]. We just give a proof for the SB_J-semiring case and leave the other part to the reader. To prove the sufficient part, assume that $a_1, a_2, \ldots, a_n, a_{n+1}$, $n \geq 2$, is a sequence in R with $(a_1, a_2, \ldots, a_{n-1}) \nsubseteq J_k(R)$ and let $r \in (a_1, a_2, \ldots, a_n, a_{n+1})$. Without loss of generality, we may assume that $a_{n-1} \notin J_k(R)$. Suppose $r = \sum_{i=1}^{n+1} a_i x_i$ and let $s = a_{n-1}x_{n-1} + a_n x_n + a_{n+1}x_{n+1}$ for some $x_i \in R$. Then $r \in (a_1, a_2, \ldots, a_{n-2}, s)$ and $s \in (a_{n-1}, a_n, a_{n+1})$. Since $a_{n-1} \notin J_k(R)$, $s \in (a_{n-1}, a_n + ba_{n+1})$ for some $b \in R$. Therefore $r \in (a_1, a_2, \ldots, a_{n-2}, s) \subseteq (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$, and the proof is complete. □

Remark 4.4. We can also prove the above theorem by using the same argument as in the proof of [12, Theorem 2.7] (see Remark 3.6(b)).

• In view of the above theorem, we need only consider triples instead of arbitrary $(n+1)$-tuples, $n > 2$, in our study of SB-semirings [resp. SB_J-semirings].

We now provide a sharper result than Theorem 4.3 for the study of SB- and SB_J-semirings when the underlying semiring is subtractive.

Theorem 4.5. *Let R be a subtractive semiring. Then R is an SB-semiring [resp. SB_J-semiring] if and only if for every sequence (c_1, c_2, c_3) of R [resp. with $c_1 \notin J_k(R)$], there exists $b \in R$ such that $c_3 \in (c_1, c_2 + bc_3)$. Further, if $A = (a_1, a_2, \ldots, a_n, a_{n+1})$, then $A = (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$ for some $b \in R$.*

Proof. The necessity clearly follows from the definition of an SB-semiring [resp. SB_J-semiring]. We just give a proof for the SB_J-semiring case and leave the other part to the reader. To prove the sufficient part, assume that $a_1, a_2, \ldots, a_n, a_{n+1}$, $n \geq 2$, is a sequence in R with $(a_1, a_2, \ldots, a_{n-1}) \nsubseteq J_k(R)$ and let $r \in (a_1, a_2, \ldots, a_n, a_{n+1})$. Without loss of generality, we may assume that $a_{n-1} \notin J_k(R)$. Since $a_{n+1} \in (a_{n-1}, a_n, a_{n+1})$, there exist $b \in R$ such that $a_{n+1} \in (a_{n-1}, a_n + ba_{n+1})$ by hypothesis. Hence $r \in (a_1, a_2, \ldots, a_{n-1}, a_n, a_{n+1}) = (a_1, a_2, \ldots, a_{n-1}, a_n + ba_{n+1})$, where the equality holds by the subtractive assumption and the proof is complete. □

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• In view of the above theorem, we see that a subtractive semiring R is an SB-semiring [resp. SB_J-semiring] if for any sequence (a_1, a_2, a_3) of R [resp. with $a_1 \notin J_k(R)$], there exists $b \in R$ such that $a_3 \in (a_1, a_2 + ba_3)$. Clearly, the above theorem is a good criterion to check whether a ring is an SB-ring since every ideal in a ring is subtractive.
We now consider the homomorphic image of SB- and SB_J-semirings.

Theorem 4.6. *Let $\phi : R \to S$ be a surjective morphism of semirings.*

(a) *If R is an SB-semiring, then so is S.*
(b) *If R is an SB_J-semiring, then so is S provided that $\phi(J_k(R)) \subseteq J_k(S)$.*

Proof. We write a proof for the SB_J-semiring case and leave the other part to the reader. Let $\overline{R}$ be the image of R under the homomorphism ϕ, and let $\overline{d} \in (\overline{a_1}, \overline{a_2}, \overline{a_3}$ with $\overline{a_1} \notin J_k(\overline{R})$, where $\overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{d} \in \overline{R}$. suppose that $\overline{d} = \sum_{i=1}^{3} \overline{a_i x_i}$ for some $\overline{x_i} \in \overline{R}$ and let $\phi(a_i) = \overline{a_i}$, $\phi x_i = \overline{x_i}$ for $i = 1, 2, 3$. Let $d = \sum_{i=1}^{3} a_i x_i$. Since by hypothesis $\phi(J_k(R)) \subseteq J_k(\overline{R})$, we have $a_1 \notin J_k(R)$ and so $d \in (a_1, a_2 + ba_3)$ for some $b \in R$. Since $\phi(d) = \overline{d}$, we have $\overline{d} \in (\overline{a_1}, \overline{a_2} + \overline{b}\overline{a_3})$, where $\phi(b) = \overline{b}$. Hence by Theorem 4.3, $S = \overline{R}$ is an SB_J-semiring. □

Corollary 4.7. *Let I be an ideal of a semiring R. Then R/I is an SB-semiring when R is an SB-semiring.*

Proof. Clearly $r \mapsto r/I$ defines a surjective morphism from R to R/I, where $r \in R$. Now the proof follows directly from Part (a) of the above theorem. See also Example 9.1 and Proposition 9.10 in [5]. □

Proposition 4.8. *The direct product of any family of semirings is an SB-semiring if and only if each factor of the product is an SB-semiring.*

Proof. The necessary part follows directly from Theorem 4.6(a) and the proof of the sufficient part is similar to the proof of Proposition 3.8. □

In the following theorem, we partially characterize the B-stability condition of $R[x]$ (the semiring of polynomials over a semiring R), which is somewhat a counterpart to [10, Theorem 2.7] and [resp. [10, Theorem 3.4]] that states: $R[x]$ is a B-ring [resp. an SB-ring] if and only if R is completely primary (i.e., a ring consisting of units and nilpotents) [resp. a field].

Theorem 4.9. *Let $R[x]$ be the semiring of polynomials over a semiring R.*

(a) *Let I be a proper ideal of R. If $R[x]$ is a B-semiring with respect to the ideal $I[x]$ of $R[x]$, then I is not a strong ideal of R. In other words, if I is a strong proper ideal of R, then $R[x]$ can not be a B-semiring with respect to the ideal $I[x]$ in $R[x]$.*
(b) *If $R[x]$ is a B-semiring, then R is not a zerosumfree semiring.*
(c) *If $R[x]$ is an SB-semiring, then R is not a zerosumfree semiring.*

Remark 4.10. Clearly, $\mathbb{Z}$ (ring of integers) is not zerosumfree as a semiring since rings can not be zerosumfree by the fact that $-1+1=0$. Thus, the converse of Parts (b) and (c) of the above theorem are not true in general since by Theorem 2.7 of [10], $\mathbb{Z}[x]$ is not a B-ring [consequently, not an SB-ring]. We can also directly conclude from [10, Theorem 3.4] that $\mathbb{Z}[x]$ is not an SB-ring since $\mathbb{Z}$ is not a field.

Proof. (a): Suppose to the contrary that I is a strong proper ideal of R. Let $a \in I$ and $1+ax \in (x^2, x, 1+ax)$. If $R[x]$ is a B-semiring with respect to $I[x]$, then $1+ax \in (x^2, x+b(x)(1+ax))$ for some $b(x) \in R[x]$ by definition. Let $1+ax = x^2 f(x) + (x+b(x)(1+ax))g(x)$, where $f(x), g(x) \in R[x]$. Let f_i, g_i, and b_i represent the coefficient of x^i in the polynomials $f(x)$, $g(x)$, and $b(x)$, respectively. Now by equating the corresponding coefficients in the above equation, we get $(x+b(x)+axb(x))g(x) = (x+b_0+b_1x+\cdots+ab_0x+\cdots)g(x)$, which implies $g_0x+b_0g_0+b_1g_0x+ab_0g_0x+b_0g_1x$ and so $1=g_0b_0$ and $a = g_0+b_1g_0+ab_0g_0+b_0g_1$. Thus if I is strong in R, then $g_0 \in I$ since $a \in I$ by the assumption, which implies $1 \in I$ and leads to a contradiction.

(b): The proof follows directly from Part (a) by setting $I = \{0\}$ and using the fact that $\{0\}$ is a strong ideal of R if and only if R is a zerosumfree semiring. (c): The proof is very much similar to Part (a) by replacing $1+ax$ with r and we write it here for the sake of comparison and completeness. Notice that (c) is an immediate consequence of (b) since an SB-semiring is a B-semiring. Suppose to the contrary that R is a zerosumfree semiring. Let $r \in R$ with $r \neq 0$. Then $r \in (x^2, x, r)$. If $R[x]$ is an SB-semiring, then $r \in (x^2, x+rb(x))$ for some $b(x) \in R[x]$. Let $r = x^2 f(x) + (x+rb(x))g(x)$, where $f(x), g(x) \in R[x]$. Let f_i, g_i, and b_i represent the coefficient of x^i in the polynomials $f(x)$, $g(x)$, and $b(x)$, respectively. Equating coefficients in the above equation gives $r = rb_0g_0$ and $0 = g_0 + r(b_0g_1 + g_0b_1)$. Now if R is zerosumfree, then $g_0 = 0$, which implies $r = 0$ and leads to a contradiction. □

Example 4.11. Let R be the semiring of nonnegative reals, or nonnegative rationals, or nonnegative integers, respectively, with usual addition and multiplication. Clearly, R is a commutative, zerosumfree semiring which is not additively idempotent and by Theorem 4.9(b), $R[x]$ is not a B-semiring [consequently, not an SB-semiring]. For more examples of zerosumfree semirings, see [5].

Corollary 4.12. *If R is an additively idempotent semiring [in particular, a simple semiring], then $R[x]$ is not a B-semiring [consequently, not an SB-semiring].*

Proof. The proof is immediate from Part (b) of the above theorem since every additively idempotent semiring [in particular, a (simple semiring], which is not a ring, is zerosumfree. Note that $a+b=0$ implies $a = a+a+b = a+b+b = b$

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in any additively idempotent semiring and also a simple semiring is additively idempotent since $1 + 1 = 1$. □

We close this section with two examples related to the stability of polynomial semirings.

Example 4.13. Let R be a semiring (ring) and $S = \text{ideal}(R)$ be the semiring of ideals of R under the addition and multiplication of the ideals of R. Then by the above Corollary, $S[x]$ is not a B-semiring [consequently, not an SB-semiring] since S is a simple semiring (i.e., $R + A = R$ for any ideal A of R).

Example 4.14. (cf. [5, Example 5.1]) If A is an infinite set, then the family $\text{fsub}(A)$ of all finite subsets of A is a strong ideal of the semiring $(\text{sub(A)}, \cup, \cap)$. Thus, by Theorem 4.9(a), $R[x]$ is not a B-semiring with respect to $I[x]$ when $I = \text{fsub}(A)$ and $R = (\text{sub(A)}, \cup, \cap)$.

5. Some special cases

In this section we study the B-stability condition of some special classes of semirings such as regular semirings (Theorem 5.2) and construct a semiring R by combining a semidomain D and a semifield F, $D \subseteq F$, and show that D is a B-semidomain when R is a B-semiring (Proposition 5.3). We also discuss an example regarding the stability condition of a polynomial semiring (Example 5.4). Finally, we write a brief note on matrix completion of commutative rings and close the paper by posing a question related to the matrix completion of B- and 2-stable semirings.

We will use the following proposition to prove Theorem 5.2 as a comparison to [10, Corollary 3.2] that states: "a *regular ring* is an SB-ring".

• An element a of R is *complemented* if and only if there exists an element c of R satisfying $ac = ca = 0$ and $a + c = 1$. This element c of R is the *complement* of $a \in R$. If a has a complement, it is unique. Denote the set of all *complemented elements* of R by $\text{comp}(R)$. This set is nonempty since $0 \in \text{comp(R)}$. See Chapter 4 of [5] for a detailed study of complemented elements in semirings.

Proposition 5.1. (cf. [10, Lemma 3.4]) *Let R be a B-semiring and $a \in \text{comp}(R)$. Suppose $a \in (a_1, a_2, \ldots, a_{n-1}, a_n)$, $n \geq 3$. Then $a \in (a_1, a_2, \ldots, a_{n-2}, a_{n-1} + ba_n)$ for some $b \in R$.*

Proof. Since $a \in \text{comp}(R)$, there exists $c \in R$ such that $a + c = 1$ and $ac = 0$. Let $a = \sum a_i x_i = \sum (a_i a)(x_i a)$. Thus, $1 = c + a = (a_1 a)(x_1 a) + c + \sum (a_i a)(x_i a)$ implies $1 = (a_1 a)(x_1 a) + c + \sum (a_i a)(x_i a) = (a_1 a)(x_1 a) + c + \sum (a_i a)(x_i a)$. Hence, $1 = (a_1 a + c)(x_1 a + c) + \sum (a_i a)(x_i a)$. Thus, $1 \in (a_1 a + c, a_2 a, \ldots, a_n a)$. Thus, since R is a B-semiring, we have $1 \in (a_1 a + c, a_2 a, \ldots, a_{n-2} a, a_{n-1} a +$

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$ba_na)$ for some $b \in R$. Therefore, $a \in (a_1a, a_2a, \ldots, a_{n-2}a, a_{n-1}a + ba_na) \subseteq (a_1, a_2, \ldots, a_{n-2}, a_{n-1} + ba_n)$. □

Theorem 5.2. (cf. [10, Corollary 3.2]) *Every plain simple yoked multiplicatively regular semiring is an SB-semiring.*

Proof. By [5, Example 4.5] if R is a plain simple yoked semiring, then $\text{comp}(R) = I^{\times}(R)$ (the set of all multiplicatively idempotent elements of R). Let $r \in (a_1, a_2, a_3)$ where $a_1, a_2, a_3 \in R$. Since R is a regular semiring, then $r = rxr$ for some $x \in R$. Consequently, rx is a multiplicatively idempotent element of R. Now, the result follows directly from the above proposition and the fact that every simple semiring is a B-semiring by [12, Corollary 2.11] (see Remark 3.6(g)). □

Proposition 5.3. (cf. [10, Theorem 2.6] and [13, Theorem 5]) *Let D be a semidomain and F a semifield containing D. Let $R = \{(a_1, \ldots, a_k, a, a, \ldots) \mid a_i \in F, a \in D\}$, where k is a nonnegative integer (k may be different for distinct elements of R). The operations in R are componentwise addition and multiplication. Let $\phi : R \to D$ be a map given by $(a_1, a_2, \ldots, a_k, a, a, \ldots) \mapsto a$. Then:*

(a) *If R is a B-semiring, then D is a B-semidomain.*
(b) *If R is a B_J-semiring, then D is a B_J-semidomain provided $\phi(J_k(R)) \subseteq J_k(D)$.*
(c) *If R is an SB-semiring, then D is an SB-semidomain.*
(d) *If R is an SB_J-semiring, then D is an SB_J-semidomain provided $\phi(J_k(R)) \subseteq J_k(D)$.*

Proof. Clearly $1_F = dd^{-1}$ for each $0 \neq d \in D$ and hence $1_D = 1_F 1_D = (dd^{-1})1_D = d^{-1}(d1_D) = d^{-1}d = 1_F$. So $1_R = (1, 1, 1, \ldots)$. It can easily be seen that D is a homomorphic image of R under the map given by $(a_1, a_2, \ldots, a_k, a, a, \ldots) \mapsto a$. Now the proof of Parts (a), (b); (c), and (d) follows directly from Theorem 3.13; and Theorem 4.6, respectively. □

We now discuss and provide an example regarding the stability condition of a polynomial semiring.

Example 5.4. Clearly, from Theorem 3.13 [resp. 4.6], if $R[x]$ (the semiring of polynomials over a semiring R) is a B-semiring [resp. an SB-semiring], then so is R under the morphism $\phi : R[x] \to R$ given by $a_0 + a_1x + \cdots + a_nx^n \mapsto a_0$, where $a_i \in R$ for each $0 \leq i \leq n$. See also Theorem 4.9 that shows $R[x]$ is not a B-semiring [resp. an SB-semiring] when R is zerosumfree. Further, since a simple semiring R is a B-semiring ([12, Corollary 2.11]; see Remark 3.6(g)), the converse of this example need not be true in general since by Corollary 4.12, $R[x]$ is not a B-semiring when R is a simple semiring. We also, by a trivial example of a B-semiring, show that the converse of this example need

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not be true in general. Let R be the semiring of nonnegative reals, which is a B-semiring by Example 3.5. Now, by Theorem 4.9, $R[x]$ is not a B-semiring since R is zerosumfree.

We conclude this paper with a brief note on the notion of *(strongly) Completable rings* which follows with a somewhat similar question on semirings being Completable.

In [8], there is a discussion of matrix completions over different types of rings with many references related to this context. Completable rings have been extensively studied, largely in connection with Serre's Problem (now the Quillen-Suslin Theorem), which can be phrased as: polynomial rings in finitely many variables over fields are completable [9]. In 1981 Gustafson, Moore, and Reiner [6] extended Hermite's classic result along a different course, showing that $\mathbb{Z}$ (or more generally any Dedekind domain) is *very strongly completable*, i.e., given an $m \times n$ matrix A $(m < n)$ and an element d of the ideal generated by its $m \times m$ minors, we can extend A to an $n \times n$ matrix with determinant d. Nearly thirty years later, Gustafson, Robinson, Richter, and Wardlaw [7] returned to the topic, using a similar technique to show that principal ideal rings are very strongly completable.

The literature on outer product rings and very strongly completable rings (as described in [8]) has focused almost exclusively on the Noetherian case. These results are often deep, with proofs that do not typically generalize to non-Noetherian rings at all, so it is likely to be extremely difficult to achieve the same level of understanding of the general case. However, Juett and Williams in [8] achieve a significant expansion of the theory of outer product rings and very strongly completable rings by providing non-Noetherian generalizations of some of the examples given in the introduction of their paper [8].

Also, in [1], there is a discussion of matrix completions over J-stable rings and in [1, Theorem 4.11], it is shown that every J-stable ring is strongly completable. The authors in the paragraph preceding [1, Corollary 4.2] refer to [11, Corollary 2.1], which is a typo and should be "[11, Corollary 2.11]" that states every 2-stable ring is completable.

Finally, we end this paper with a question related to the matrix completion of a B- or 2-stable commutative semiring.

Question: Under what condition(s) a unimodular sequence of a B-semiring or 2-stable semiring R can be completed to a square matrix whose *permanent* is a unit of R. For the definition and some properties of the permanent of a (square) matrix see [3] and Chapter 17 of [5].

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