

## On bi-bases of $\Gamma$ -semihypergroups

Apipray Sarakam, Pongpich Kotheop, Samkhan Hobanthad\*

Department of Mathematics, Faculty of Science, Buriram Rajabhat  
University, Buriram, Thailand, 31000

E-mail: [apiplay0805@gmail.com](mailto:apiplay0805@gmail.com)

E-mail: [Pongpich.iot@bru.ac.th](mailto:Pongpich.iot@bru.ac.th)

E-mail: [samkan.hb@bru.ac.th](mailto:samkan.hb@bru.ac.th)

**ABSTRACT.** This paper focuses on the  $\Gamma$ -semihypergroups. Our goal seeks to find the conditions of sub- $\Gamma$ -semihypergroup using bi-bases properties. We provide definitions and explain some properties of bi-bases in  $\Gamma$ -semihypergroups. The findings extend the results from bi-bases of  $\Gamma$ -semigroups. The findings demonstrate that if  $B$  is a bi-bases of a  $\Gamma$ -semihypergroup  $H$ ; then,  $B$  is a sub- $\Gamma$ -semihypergroup of  $H$  if and only if for any  $b, c \in B$  and  $\gamma \in \Gamma, b \in b\gamma c$  or  $c \in b\gamma c$ .

**Keywords:**  $\Gamma$ -semihypergroup, bi-bases, sub- $\Gamma$ -semihypergroup.

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### 1. INTRODUCTION AND PRELIMINARIES

F. Marty [1] created hyperstructure theory. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Fabrici [2, 3] introduced the concepts of one-sided and two-sided bases of a semigroup which were extended to ordered semigroups by T. Changpas and P. Summaprab [9]. Later,

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\*Corresponding Author

P. Kummoon and T. Changpas [6] defined and identified some properties of bi-bases in semigroups, and they [7] extended the results to  $\Gamma$ -semigroups. This paper attempts to find the condition of sub- $\Gamma$ -semihypergroup using bi-bases properties in  $\Gamma$ -semihypergroups and begins by introducing the concept of bi-bases of  $\Gamma$ -semihypergroup and extend the results of bi-bases in  $\Gamma$ -semigroups to  $\Gamma$ -semihypergroups. In this section, the authors begin by recalling terminologies of  $\Gamma$ -semihypergroups as follows:

Let  $H$  be a nonempty set. Then, the map  $\circ : H \times H \rightarrow P^*(H)$  where  $P^*(H)$  is the family of nonempty subset of  $H$ . The system  $(H, \circ)$  is called a semihypergroup if for every  $x, y, z \in H$ ,  $x \circ (y \circ z) = (x \circ y) \circ z$ . If  $A$  and  $B$  are two nonempty subsets of  $H$ , then, we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b; \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\} \text{ for all } x \in H.$$

A nonempty subset  $A$  of semihypergroup  $H$  is called a subsemihypergroup of  $H$  if  $A \circ A \subseteq A$ .

**Definition 1.1.** [8] Let  $H$  and  $\Gamma$  be two nonempty sets. Then,  $H$  is called a  $\Gamma$ -semihypergroup if  $\Gamma$  is a set of hyperoperation on  $H$  and for every  $\alpha, \beta \in \Gamma$ ,  $x\alpha(y\beta z) = (x\alpha y)\beta z$  for all  $x, y, z \in H$ .

If  $A$  and  $B$  are two nonempty subsets of  $H$ , we denote

$$A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B = \cup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

Let  $(H, \circ)$  be a semihypergroup and  $\Gamma = \{\circ\}$ . Then,  $H$  is a  $\Gamma$ -semihypergroup. Clearly, every semihypergroup is a  $\Gamma$ -semihypergroup.

**Definition 1.2.** [5] Let  $H$  be a  $\Gamma$ -semihypergroup. A nonempty subset  $A$  of  $H$  is called a sub- $\Gamma$ -semihypergroup of  $H$  if  $A\Gamma A \subseteq A$ . A sub- $\Gamma$ -semihypergroup  $A$  of  $H$  is called a bi- $\Gamma$ -hyperideal of  $H$  if  $A\Gamma H\Gamma A \subseteq A$ .

**Proposition 1.3.** [8] Let  $H$  be a  $\Gamma$ -semihypergroup and  $B_i$  be a bi- $\Gamma$ -hyperideal of  $H$  for any  $i \in I$ . If  $\bigcap_{i \in I} B_i \neq \emptyset$ ; then,  $\bigcap_{i \in I} B_i$  is a bi- $\Gamma$ -hyperideal of  $H$ .

Let  $A$  be a nonempty subset of a  $\Gamma$ -semihypergroup  $H$  and define the set of all bi- $\Gamma$ -hyperideal of  $H$  containing  $A$  as follows:

$$K = \{B \mid B \text{ is a bi-}\Gamma\text{-hyperideal of } H \text{ containing } A\}.$$

Clearly,  $K \neq \emptyset$ , because  $H \in K$ . Suppose  $(A)_b = \bigcap_{B \in K} B$ . This indicates seen that  $A \subseteq (A)_b$ . By proposition 1.3,  $(A)_b$  is a bi- $\Gamma$ -hyperideal of  $H$ . Moreover,  $(A)_b$  is the smallest bi- $\Gamma$ -hyperideal of  $H$  containing  $A$ .

**Proposition 1.4.** Let  $A$  be a nonempty subset of a  $\Gamma$ -semihypergroup  $H$ . Then,

$$(A)_b = A \cup A\Gamma A \cup A\Gamma H\Gamma A$$

*Proof.* Suppose  $B = A \cup A\Gamma A \cup A\Gamma H\Gamma A$ . Clearly,  $A \subseteq B$ . Consider  $B\Gamma B = (A \cup A\Gamma A \cup A\Gamma H\Gamma A)\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A) \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq B$ . Hence,  $B$  is a sub- $\Gamma$ -semihypergroup of  $H$  containing  $A$ . Consider  $B\Gamma H\Gamma B = (A \cup A\Gamma A \cup A\Gamma H\Gamma A)\Gamma H\Gamma(A \cup A\Gamma A \cup A\Gamma H\Gamma A) \subseteq A\Gamma H\Gamma A \subseteq B$ . Therefore,  $B$  is a bi- $\Gamma$ -hyperideal of  $H$  containing  $A$ . Let  $C$  be any bi- $\Gamma$ -hyperideal of  $H$  containing  $A$ . Thus,  $A \subseteq C$ . Since  $C$  is a sub- $\Gamma$ -semihypergroup of  $H$ ; so,  $A\Gamma A \subseteq C\Gamma C \subseteq C$ .  $A\Gamma H\Gamma A \subseteq C\Gamma H\Gamma C \subseteq C$  because  $C$  is bi- $\Gamma$ -hyperideal of  $H$ . Hence,  $B = A \cup A\Gamma A \cup A\Gamma H\Gamma A \subseteq C$ . Thus,  $B$  is the smallest bi- $\Gamma$ -hyperideal of  $H$  containing  $A$ . Therefore,  $(A)_b = A \cup A\Gamma A \cup A\Gamma H\Gamma A$ .  $\square$

$(A)_b$  is called the bi- $\Gamma$ -hyperideal of  $H$  generated by  $A$ .

**Definition 1.5.** Let  $H$  be a  $\Gamma$ -semihypergroup. A nonempty subset  $B$  of  $H$  is called a bi-bases of  $H$  if it satisfies the following two conditions.

1.  $H = (B)_b$  (i.e.,  $H = B \cup B\Gamma B \cup B\Gamma H\Gamma B$ ).
2. If  $A$  is a nonempty subset of  $B$  such that  $H = (A)_b$ ; then,  $A = B$ .

EXAMPLE 1.6. Let  $H = \{x, y, z, w\}$  and  $\Gamma = \{\beta, \alpha\}$  be the sets of hyperoperations defined below

$\beta$	$x$	$y$	$z$	$w$	$\alpha$	$x$	$y$	$z$	$w$
$x$	$\{x\}$	$\{x, y\}$	$\{z, w\}$	$\{w\}$	$x$	$\{x, y\}$	$\{x, y\}$	$\{z, w\}$	$\{w\}$
$y$	$\{x, y\}$	$\{x, y\}$	$\{z, w\}$	$\{w\}$	$y$	$\{x, y\}$	$\{y\}$	$\{z, w\}$	$\{w\}$
$z$	$\{z, w\}$	$\{z, w\}$	$\{z\}$	$\{w\}$	$z$	$\{z, w\}$	$\{z, w\}$	$\{z\}$	$\{w\}$
$w$	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$	$w$	$\{w\}$	$\{w\}$	$\{w\}$	$\{w\}$

N. Yaqoob [4] showed that  $H$  is a  $\Gamma$ -semihypergroup. Consider  $(A)_1 = \{x\}$  and  $(A)_2 = \{y\}$ ; so,  $(A)_1$  and  $(A)_2$  are bi-bases of  $H$ .

## 2. MAIN RESULTS

In this section, we characterize bi-bases of  $\Gamma$ -semihypergroups and show conditions of sub- $\Gamma$ -semihypergroup using bi-bases properties.

**Lemma 2.1.** *Let  $B$  be a bi-bases of a  $\Gamma$ -semihypergroup  $H$  and  $a, b \in B$ . If  $a \in b\Gamma b \cup b\Gamma H\Gamma b$ ; then,  $a = b$ .*

*Proof.* Let  $a, b \in B$ . Suppose  $a \in b\Gamma b \cup b\Gamma H\Gamma b$  and  $a \neq b$ . Setting  $A = B \setminus \{a\}$ ; then,  $A \subseteq B$ . From  $a \neq b$ , so  $b \in A$ . Hence,  $(A)_b \subseteq (B)_b = H$ . Let  $x \in H = (B)_b$ . Then,  $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$ . There are three cases to consider.

Case 1:  $x \in B$ .

Subcase 1.1:  $x \neq a$ . Then,  $x \in B \setminus \{a\} = A \subseteq (A)_b$ .

Subcase 1.2:  $x = a$ . So,  $x = a \in b\Gamma b \cup b\Gamma H\Gamma b \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b$ .

Case 2 :  $x \in B\Gamma B$ . Hence,  $x \in b_1\gamma b_2$  for some  $b_1, b_2 \in B$  and  $\gamma \in \Gamma$ .

Subcase 2.1:  $b_1 = a$  and  $b_2 = a$ . By assumption,

$$\begin{aligned} x \in b_1\gamma b_2 &= a\gamma a \\ &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\ &= b\Gamma b\Gamma b\Gamma b \cup b\Gamma b\Gamma b\Gamma H\Gamma b \cup b\Gamma H\Gamma b\Gamma b\Gamma b \cup b\Gamma H\Gamma b\Gamma b\Gamma H\Gamma b \\ &\subseteq A\Gamma A\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma A\Gamma H\Gamma A \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b \end{aligned}$$

Subcase 2.2:  $b_1 \neq a$  and  $b_2 = a$ . By assumption and  $A = B \setminus \{a\}$ ,

$$\begin{aligned} x \in b_1\gamma b_2 &= b_1\gamma a \subseteq (B \setminus \{a\})\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\ &= (B \setminus \{a\})\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma b\Gamma H\Gamma b \\ &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b \end{aligned}$$

Subcase 2.3:  $b_1 = a$  and  $b_2 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; then,

$$\begin{aligned} x \in b_1\gamma b_2 &= a\gamma b_2 \subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma(B \setminus \{a\}) \\ &= b\Gamma b\Gamma(B \setminus \{a\}) \cup b\Gamma H\Gamma b\Gamma(B \setminus \{a\}) \\ &\subseteq A\Gamma A\Gamma A \cup A\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.4:  $b_1 \neq a$  and  $b_2 \neq a$ . From  $A = B \setminus \{a\}$ , so  $x \in b_1\gamma b_2 \subseteq (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b$ .

Case 3:  $x \in B\Gamma H\Gamma B$ . Hence,  $x \in b_3\gamma_1 h\gamma_2 b_4$  for some  $b_3, b_4 \in B$ ,  $\gamma_1, \gamma_2 \in \Gamma$  and  $h \in H$ .

Subcase 3.1:  $b_3 = a$  and  $b_4 = a$ , consider

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 &= a\gamma_1 h\gamma_2 a \\ &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\ &= b\Gamma b\Gamma H\Gamma b\Gamma b \cup b\Gamma b\Gamma H\Gamma b\Gamma H\Gamma b \cup b\Gamma H\Gamma b\Gamma H\Gamma b\Gamma b \cup b\Gamma H\Gamma b\Gamma H\Gamma b\Gamma b \\ &\subseteq A\Gamma A\Gamma H\Gamma A\Gamma A \cup A\Gamma A\Gamma A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A\Gamma A \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.2:  $b_3 \neq a$  and  $b_4 = a$ . By assumption and  $A = B \setminus \{a\}$ ; so,

$$\begin{aligned} x \in b_3\gamma_1 h\gamma_2 b_4 &= b_3\gamma_1 h\gamma_2 a \\ &\subseteq (B \setminus \{a\})\Gamma H\Gamma(b\Gamma b \cup b\Gamma H\Gamma b) \\ &= (B \setminus \{a\})\Gamma H\Gamma b\Gamma b \cup (B \setminus \{a\})\Gamma H\Gamma b\Gamma H\Gamma b \\ &\subseteq A\Gamma H\Gamma A\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.3 :  $b_3 = a$  and  $b_4 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; hence,

$$\begin{aligned} x \in b_3\gamma_1h\gamma_2b_4 &= a\gamma_1h\gamma_2b_4 \\ &\subseteq (b\Gamma b \cup b\Gamma H\Gamma b)\Gamma H\Gamma(B \setminus \{a\}) \\ &= b\Gamma b\Gamma H\Gamma(B \setminus \{a\}) \cup b\Gamma H\Gamma b\Gamma H\Gamma(B \setminus \{a\}) \\ &\subseteq A\Gamma A\Gamma H\Gamma A \cup A\Gamma H\Gamma A\Gamma H\Gamma A \subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4:  $b_3 \neq a$  and  $b_4 \neq a$ . From  $A = B \setminus \{a\}$ ; then,  $x \in b_3\gamma_1h\gamma_2b_4 \subseteq (B \setminus \{a\})\Gamma H\Gamma(B \setminus \{a\}) = A\Gamma H\Gamma A \subseteq (A)_b$ . Thus,  $(A)_b = H$ . This is a contradiction. Therefore,  $a = b$ .  $\square$

**Lemma 2.2.** *Let  $B$  be a bi-bases of a  $\Gamma$ -semihypergroup  $H$  and  $a, b, c \in B$ . If  $a \in c\Gamma b \cup c\Gamma H\Gamma b$ , then  $a = b$  or  $a = c$ .*

*Proof.* Let  $a, b, c \in B$  and  $h \in H$ . Assume  $a \in c\Gamma b \cup c\Gamma H\Gamma b$  such that  $a \neq b$  and  $a \neq c$ . Setting  $A = (B \setminus \{a\})$ . Hence,  $A \subseteq B$ , then  $(A)_b \subseteq (B)_b = H$ . From  $a \neq b$  and  $a \neq c$ ; so,  $b, c \in A$ . Let  $x \in H$ . Since  $(B)_b = H$ ; thus,  $x \in B \cup B\Gamma b \cup B\Gamma H\Gamma b$ . There are three cases to consider.

Case 1:  $x \in B$ .

Subcase 1.1:  $x \neq a$ . Then,  $x \in B \setminus \{a\} = A \subseteq (A)_b$ .

Subcase 1.2:  $x = a$ . Thus,  $x = a \in c\Gamma b \cup c\Gamma H\Gamma b \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b$ .

Case 2:  $x \in B\Gamma b$ . Then,  $x \in b_1\gamma b_2$  for some  $b_1, b_2 \in B$  and  $\gamma \in \Gamma$ .

Subcase 2.1:  $b_1 = a$  and  $b_2 = a$ . By assumption,

$$x \in b_1\gamma b_2 = a\gamma a \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 2.2:  $b_1 \neq a$  and  $b_2 = a$ . By assumption and  $A = B \setminus \{a\}$ ; then,  $x \in b_1\gamma b_2 = b_1\gamma a \subseteq (B \setminus \{a\})\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b$ .

Subcase 2.3:  $b_1 = a$  and  $b_2 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; so,

$$x \in b_1\gamma b_2 = a\gamma b_2 \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma(B \setminus \{a\}) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 2.4:  $b_1 \neq a$  and  $b_2 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; thus,

$$x \in b_1\gamma b_2 \subseteq (B \setminus \{a\})\Gamma(B \setminus \{a\}) = A\Gamma A \subseteq (A)_b.$$

Case 3:  $x \in B\Gamma H\Gamma b$ . Hence,  $x \in b_3\gamma_1h\gamma_2b_4$  for some  $b_3, b_4 \in B$ ,  $\gamma_1, \gamma_2 \in \Gamma$  and  $h \in H$ .

Subcase 3.1:  $b_3 = a$  and  $b_4 = a$ . Then,

$$x \in b_3\gamma_1h\gamma_2b_4 = a\gamma_1h\gamma_2a \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 3.2:  $b_3 \neq a$  and  $b_4 = a$ . By assumption and  $A = B \setminus \{a\}$ ; so,

$$x \in b_3\gamma_1h\gamma_2b_4 = b_3\gamma_1h\gamma_2a \subseteq (B \setminus \{a\})\Gamma H\Gamma(c\Gamma b \cup c\Gamma H\Gamma b) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 3.3:  $b_3 = a$  and  $b_4 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; thus,

$$x \in b_3\gamma_1h\gamma_2b_4 = a\gamma_1h\gamma_2b_4 \subseteq (c\Gamma b \cup c\Gamma H\Gamma b)\Gamma H\Gamma(B \setminus \{a\}) \subseteq A\Gamma H\Gamma A \subseteq (A)_b.$$

Subcase 3.4:  $b_3 \neq a$  and  $b_4 \neq a$ . From  $A = B \setminus \{a\}$ ; then,

$$x \in b_3\gamma_1h\gamma_2b_4 \in (B \setminus \{a\})\Gamma H\Gamma(B \setminus \{a\}) = A\Gamma H\Gamma A \subseteq (A)_b.$$

From all cases,  $(A)_b = H$ . This is a contradiction. Therefore,  $a = b$  or  $a = c$ .  $\square$

**Definition 2.3.** Let  $H$  be a  $\Gamma$ -semihypergroup. Define a quasi-order on  $H$  by, for any  $a, b \in H$ ,  $a \leq_b b \Leftrightarrow (a)_b \subseteq (b)_b$ .

In example 1.6,  $A_1 = \{x\}$  and  $A_2 = \{y\}$  are bi-bases of  $H$ . Since  $(x)_b \subseteq (y)_b$ , so  $x \leq_b y$  and since  $(y)_b \subseteq (x)_b$ , then  $y \leq_b x$ . From  $x \leq_b y$  and  $y \leq_b x$ , but  $x \neq y$ . Therefore,  $\leq_b$  is not a partial order on  $H$ . This shows that the order  $\leq_b$  defined above is not, in general, a partial order.

**Lemma 2.4.** Let  $B$  be a bi-bases of a  $\Gamma$ -semihypergroup  $H$  and  $a, b \in B$ . If  $a \neq b$ , then neither  $a \leq_b b$  or  $b \leq_b a$ .

*Proof.* Let  $a, b \in B$ . Suppose  $a \neq b$ . If  $a \leq_b b$ ; hence,  $(a)_b \subseteq (b)_b$ . Thus,  $a \in (a)_b \subseteq (b)_b = \{b\} \cup b\Gamma b \cup b\Gamma H\Gamma b$ . By Lemma 2.1,  $a = b$ . This is a contradiction. If  $b \leq_b a$ , can be proved similarly.  $\square$

**Lemma 2.5.** Let  $B$  be a bi-bases of a  $\Gamma$ -semihypergroup  $H$ . For all  $a, b, c \in B$ ,  $\gamma_1, \gamma_2 \in \Gamma$  and  $h \in H$ .

1. If  $a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c$ ; then,  $a = b$  or  $a = c$ .
2. If  $a \in b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma b\gamma_1 h\gamma_2 c \cup b\gamma_1 h\gamma_2 c\Gamma H\Gamma b\gamma_1 h\gamma_2 c$ ; then,  $a = b$  or  $a = c$ .

*Proof.* Let  $a, b, c \in B$ ,  $\gamma \in \Gamma$  and  $h \in H$ .

(1.) Assume  $a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c$  such that  $a \neq b$  and  $a \neq c$ . Consider  $A = B \setminus \{a\}$ . Clearly,  $A \subseteq B$ , thus  $(A)_b \subseteq (B)_b = H$ . From  $a \neq b$  and  $a \neq c$ ; so,  $b, c \in A$ . Let  $x \in H$ . Since  $(B)_b = H$ , so  $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$ . There are three cases to consider.

Case 1:  $x \in B$ .

Subcase 1.1:  $x \neq a$ . Then,  $x \in B \setminus \{a\} = A \subseteq (A)_b$ .

Subcase 1.2:  $x = a$ . By assumption, so

$$x = a \in b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c \subseteq A\Gamma A \cup A\Gamma H\Gamma A \subseteq (A)_b.$$

Case 2:  $x \in B\Gamma B$ . Thus,  $x \in b_1\gamma b_2$  for some  $b_1, b_2 \in B$  and  $\gamma \in \Gamma$ .

Subcase 2.1 :  $b_1 = a$  and  $b_2 = a$ . By assumption, then

$$\begin{aligned} x \in b_1\gamma b_2 &= a\gamma a \\ &\subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b \end{aligned}$$

Subcase 2.2:  $b_1 \neq a$  and  $b_2 = a$ . By assumption and  $A = B \setminus \{a\}$ ; then,

$$\begin{aligned} x \in b_1\gamma b_2 &= b_1\gamma a \subseteq (B \setminus \{a\})\Gamma (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.3:  $b_1 = a$  and  $b_2 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; so,

$$\begin{aligned} x \in b_1\gamma b_2 &= a\gamma b_2 \subseteq (b\gamma_1 c \cup b\gamma_1 c\Gamma b\gamma_1 c \cup b\gamma_1 c\Gamma H\Gamma b\gamma_1 c)\Gamma (B \setminus \{a\}) \\ &\subseteq A\Gamma H\Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.4:  $b_1 \neq a$  and  $b_2 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; thus,  
 $x \subseteq b_1 \gamma b_2 \subseteq (B \setminus \{a\}) \Gamma (B \setminus \{a\}) = A \Gamma A \subseteq (A)_b$ .

Case 3:  $x \in B \Gamma H \Gamma B$ . Hence,  $x \in b_3 \gamma_1 h \gamma_2 b_4$  for some  $b_3, b_4 \in B$  and for some  $\gamma_1, \gamma_2 \in \Gamma$  and for some  $h \in H$ .

Subcase 3.1:  $b_3 = a$  and  $b_4 = a$ . By assumption, then

$$\begin{aligned} x &\in b_3 \gamma_1 h \gamma_2 b_4 \\ &= a \gamma_1 h \gamma_2 a \\ &\subseteq (b \gamma_1 c \cup b \gamma_1 c \Gamma b \gamma_1 c \cup b \gamma_1 c \Gamma H \Gamma b \gamma_1 c) \Gamma H \Gamma (b \gamma_1 c \cup b \gamma_1 c \Gamma b \gamma_1 c \cup b \gamma_1 c \Gamma H \Gamma b \gamma_1 c) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.2:  $b_3 \neq a$  and  $b_4 = a$ . By assumption and  $A = B \setminus \{a\}$ ; so,

$$\begin{aligned} x \in b_3 \gamma_1 h \gamma_2 b_4 &= b_3 \gamma_1 h \gamma_2 a \subseteq (B \setminus \{a\}) \Gamma H \Gamma (b \gamma_1 c \cup b \gamma_1 c \Gamma b \gamma_1 c \cup b \gamma_1 c \Gamma H \Gamma b \gamma_1 c) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.3:  $b_3 = a$  and  $b_4 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; thus,

$$\begin{aligned} x \in b_3 \gamma_1 h \gamma_2 b_4 &= b_3 \gamma_1 h \gamma_2 a \subseteq (b \gamma_1 c \cup b \gamma_1 c \Gamma b \gamma_1 c \cup b \gamma_1 c \Gamma H \Gamma b \gamma_1 c) \Gamma H \Gamma (B \setminus \{a\}) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4:  $b_3 \neq a$  and  $b_4 \neq a$ . By assumption and  $A = B \setminus \{a\}$ ; hence,  
 $x \in b_3 \gamma b_4 \subseteq (B \setminus \{a\}) \Gamma (B \setminus \{a\}) = A \Gamma A \subseteq (A)_b$ .

From all cases,  $(A)_b = H$ . This is a contradiction. Therefore,  $a = b$  and  $a = c$ .  
 (2.) Assume  $a \in b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma H \Gamma b \gamma_1 h \gamma_2 c$  such that  $a \neq b$  and  $a \neq c$ . Setting  $A = B \setminus \{a\}$ . Then,  $A \subseteq B$ . Thus,  $(A)_b \subseteq (B)_b = H$ . From  $a \neq b$  and  $a \neq c$ ; so,  $b, c \in A$ . Let  $x \in H$ . Since  $(B)_b = H$ ; then,  $x \in B \cup B \Gamma B \cup B \Gamma H \Gamma B$ . There are three cases to consider.

Case 1:  $x \in B$ .

Subcase 1.1:  $x \neq a$ . Then,  $x \in B \setminus \{a\} = A \subseteq (A)_b$ .

Subcase 1.2:  $x = a$ . By assumption, then

$$x = a \in b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma H \Gamma b \gamma_1 h \gamma_2 c \subseteq A \Gamma H \Gamma A \subseteq (A)_b.$$

Case 2:  $x \in B \Gamma B$ . Thus,  $x \in b_1 \gamma b_2$  for some  $b_1, b_2 \in B$  and  $\gamma \in \Gamma$ .

Subcase 2.1:  $b_1 = a$  and  $b_2 = a$ . By assumption, thus

$$\begin{aligned} x \in b_1 \gamma b_2 &= a \gamma a \\ &\subseteq (b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma H \Gamma b \gamma_1 h \gamma_2 c) \Gamma \\ &\quad (b \gamma_1 h \gamma_2 c \cup a \gamma b \gamma_1 h \gamma_2 c \Gamma a \gamma b \gamma_1 h \gamma_2 c \cup a \gamma b \gamma_1 h \gamma_2 c \Gamma H \Gamma a \gamma b \gamma_1 h \gamma_2 c) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.2:  $b_1 \neq a$  and  $b_2 = a$ . By assumption and  $A = B \setminus \{a\}$ ; so,

$$\begin{aligned} x \in b_1 \gamma b_2 &= b_1 \gamma a \\ &\subseteq (B \setminus \{a\}) \Gamma (b \gamma_1 h \gamma_2 c \cup a \gamma b \gamma_1 h \gamma_2 c \Gamma a \gamma b \gamma_1 h \gamma_2 c \cup a \gamma b \gamma_1 h \gamma_2 c \Gamma H \Gamma a \gamma b \gamma_1 h \gamma_2 c) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.3:  $b_1 = a$  and  $b_2 \neq a$ . By assumption  $A = B \setminus \{a\}$ ; hence,

$$\begin{aligned} x \in b_1 \gamma b_2 &= a \gamma b_2 \\ &\subseteq (b \gamma_1 h \gamma_2 c \cup a \gamma b \gamma_1 h \gamma_2 c \Gamma a \gamma b \gamma_1 h \gamma_2 c \cup a \gamma b \gamma_1 h \gamma_2 c \Gamma H \Gamma a \gamma b \gamma_1 h \gamma_2 c) \Gamma (B \setminus \{a\}) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 2.4:  $b_1 \neq a$  and  $b_2 \neq a$ . From  $A = B \setminus \{a\}$ ; then,

$$x \in b_1 \gamma b_2 \in (B \setminus \{a\}) \Gamma (B \setminus \{a\}) = A \Gamma A \subseteq (A)_b.$$

Case 3:  $x \in B \Gamma H \Gamma B$ . Hence,  $x \in b_3 \gamma h \gamma b_4$  for some  $b_3, b_4 \in B$ ,  $\gamma_1, \gamma_2 \in \Gamma$  and  $h \in H$ .

Subcase 3.1:  $b_1 = a$  and  $b_2 = a$ . By assumption, so

$$\begin{aligned} x \in b_3 \gamma_1 h \gamma_2 b_4 &= a \gamma_1 h \gamma_2 a \\ &\subseteq (b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma H \Gamma b \gamma_1 h \gamma_2 c) \Gamma H \Gamma \\ &\quad (b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma H \Gamma b \gamma_1 h \gamma_2 c) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.2:  $b_3 \neq a$  and  $b_4 = a$ . By assumption and  $A = B \setminus \{a\}$ ; thus,

$$\begin{aligned} x \in b_3 \gamma_1 h \gamma_2 b_4 &= b_3 \gamma h \gamma_2 a \\ &\subseteq (B \setminus \{a\}) \Gamma H \Gamma (b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma H \Gamma b \gamma_1 h \gamma_2 c) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.3:  $b_3 = a$  and  $b_4 \neq a$ . By assumption  $A = B \setminus \{a\}$ ; hence,

$$\begin{aligned} x \in b_3 \gamma h \gamma_2 b_4 &= a \gamma_1 h \gamma_2 b_4 \\ &\subseteq (b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma b \gamma_1 h \gamma_2 c \cup b \gamma_1 h \gamma_2 c \Gamma H \Gamma b \gamma_1 h \gamma_2 c) \Gamma H \Gamma (B \setminus \{a\}) \\ &\subseteq A \Gamma H \Gamma A \subseteq (A)_b. \end{aligned}$$

Subcase 3.4:  $b_3 \neq a$  and  $b_4 \neq a$ . From  $A = B \setminus \{a\}$ ; then,

$$x \in b_3 \gamma_1 h \gamma_2 b_4 \in (B \setminus \{a\}) \Gamma H \Gamma (B \setminus \{a\}) = A \Gamma H \Gamma A \subseteq (A)_b.$$

From all cases,  $(A)_b = H$ . This is a contradiction. Therefore,  $a = b$  or  $a = c$ .  $\square$

**Lemma 2.6.** *Let  $B$  be a bi-bases of a  $\Gamma$ -semihypergroup  $H$ .*

1. For any  $a, b, c \in B, \gamma_1 \in \Gamma$ , if  $a \neq b$  and  $a \neq c$ ; then,  $a \not\leq_b b \gamma_1 c$ .
2. For any  $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$  and  $h \in H$ , if  $a \neq b$  and  $a \neq c$ ; then,  $a \not\leq_b b \gamma_2 h \gamma_3 c$ .

*Proof.* (1.) Assume  $a \neq b$  and  $a \neq c$ . Suppose  $a \leq_b b \gamma_1 c$ , thus  $(a)_b \subseteq (b \gamma_1 c)_b$ . Hence,  $a \in (a)_b \subseteq (b \gamma_1 c)_b = b \gamma_1 c \cup b \gamma_1 c \Gamma b \gamma_1 c \cup b \gamma_1 c \Gamma H \Gamma b \gamma_1 c$ . By Lemma 2.5(1.), it follows  $a = b$  and  $a = c$ . This contradicts the assumption.

(2.) Assume  $a \neq b$  and  $a \neq c$ . Suppose  $a \leq_b b \gamma_2 h \gamma_3 c$ , then  $(a)_b \subseteq (b \gamma_2 h \gamma_3 c)_b$ . Thus,  $a \in (a)_b \subseteq (b \gamma_2 h \gamma_3 c)_b = b \gamma_2 h \gamma_3 c \cup b \gamma_2 h \gamma_3 c \Gamma b \gamma_2 h \gamma_3 c \cup b \gamma_2 h \gamma_3 c \Gamma H \Gamma b \gamma_2 h \gamma_3 c$ . By Lemma 2.5(2.), it follows  $a = b$  or  $a = c$ . This contradicts the assumption.  $\square$

**Theorem 2.7.** *A nonempty subset  $B$  of a  $\Gamma$ -semihypergroup  $H$  is a bi-bases of  $H$  if and only if  $B$  satisfies the following conditions.*

1. For any  $x \in H$ ,
  - 1.1. there exists  $b \in B$  such that  $x \leq_b b$  or
  - 1.2. there exists  $b_1, b_2 \in B$  and  $\gamma \in \Gamma$  such that  $x \leq_b b_1\gamma b_2$  or
  - 1.3. there exists  $b_3, b_4 \in B$  and  $\gamma_1, \gamma_2 \in \Gamma$  such that  $x \leq_b b_3\gamma_1 h\gamma_2 b_4$ .
2. For any  $a, b, c \in B$  and  $\gamma_1 \in \Gamma$ , if  $a \neq b$  and  $a \neq c$ ; then,  $a \not\leq_b b\gamma_1 c$ .
3. For any  $a, b, c \in B, \gamma_2, \gamma_3 \in \Gamma$  and  $h \in H$ , if  $a \neq b$  and  $a \neq c$ ; then,  $a \not\leq_b b\gamma_2 h\gamma_3 c$ .

*Proof.* Assume  $B$  is a bi-bases of  $H$ . Then,  $H = (B)_b$ . To show that (1.) holds, let  $x \in H$ . Thus,  $x \in B \cup B\Gamma B \cup B\Gamma H\Gamma B$ . We consider three cases.

case 1:  $x \in B$ . Then,  $x = b$  for some  $b \in B$ . This implies  $(x)_b \subseteq (b)_b$ . Therefore,  $x \leq_b b$ .

case 2:  $x \in B\Gamma B$ . Then,  $x \in b_1\gamma b_2$  for some  $b_1, b_2 \in B$  and  $\gamma \in \Gamma$ . This implies  $(x)_b \subseteq (b_1\gamma b_2)_b$ . Hence,  $x \leq_b b_1\gamma b_2$ .

case 3:  $x \in B\Gamma H\Gamma B$ . Then,  $x \in b_3\gamma_1 h\gamma_2 b_4$  for some  $b_3, b_4 \in B, h \in H$  and  $\gamma_1, \gamma_2 \in \Gamma$ . This implies  $(x)_b \subseteq (b_3\gamma_1 h\gamma_2 b_4)_b$ . Hence,  $x \leq_b b_3\gamma_1 h\gamma_2 b_4$ .

The validity of (2.) and (3.) follow from Lemma 2.6(1.) and Lemma 2.6(2.), respectively. Conversely, we will show that  $B$  is a bi-bases of  $H$ . Clearly,  $(B)_b \subseteq H$ . Let  $x \in H$ . By assumption, there exists  $b \in B$  such that  $x \leq_b b$ , thus  $(x)_b \subseteq (b)_b$ . Thus,  $x \in (x)_b \subseteq (b)_b = b \cup b\Gamma b \cup b\Gamma H\Gamma b \subseteq B \cup B\Gamma B \cup B\Gamma H\Gamma B = (B)_b$ . Then,  $H \subseteq (B)_b$ . Hence,  $H = (B)_b$ . Suppose  $H = (A)_b$ . Since  $A \subset B$ , there exists  $b \in B \setminus A$ . Since  $b \in B \subseteq H = (A)_b$ , so  $b \in (A)_b$ . Thus,  $b \in A \cup A\Gamma A \cup A\Gamma H\Gamma A$ . Since  $b \notin A$ , we have  $b \in A\Gamma A \cup A\Gamma H\Gamma A$ . There are two cases to consider.

case 1:  $b \in A\Gamma A$ . Thus,  $b \in a_1\gamma_1 a_2$  for some  $a_1, a_2 \in A$  and  $\gamma_1 \in \Gamma$ . From  $A \subseteq B$ , so  $a_1, a_2 \in B$ . Since  $b \notin A$ ; hence,  $b \neq a_1$  and  $b \neq a_2$ . From  $b \in a_1\gamma_1 a_2$ ; then,  $(b)_b \subseteq (a_1\gamma_1 a_2)_b$ . Hence,  $b \leq_b a_1\gamma_1 a_2$ . This contradicts to (2).

case 2:  $b \in A\Gamma H\Gamma A$ . Hence,  $b \in a_3\gamma_2 h\gamma_3 a_4$  for some  $a_3, a_4 \in A, h \in H$  and  $\gamma_2, \gamma_3 \in \Gamma$ . From  $A \subset B$ , so  $a_3, a_4 \in B$ . Since  $b \notin A$ , then  $b \neq a_3$  and  $b \neq a_4$ . From  $b \in a_3\gamma_2 h\gamma_3 a_4$ , so  $(b)_b \subseteq (a_3\gamma_2 h\gamma_3 a_4)_b$ . Therefore,  $b \leq_b a_3\gamma_2 h\gamma_3 a_4$ . This contradicts to (3.).  $\square$

**Theorem 2.8.** *Let  $B$  be a bi-bases of a  $\Gamma$ -semihypergroup  $H$ . Then  $B$  is a sub- $\Gamma$ -semihypergroup of  $H$  if and only if for any  $b, c \in B$  and  $\gamma \in \Gamma, b \in b\gamma c$  or  $c \in b\gamma c$*

*Proof.* Suppose  $B$  is a sub- $\Gamma$ -semihypergroup of  $H$  and  $b \notin b\gamma c$  and  $c \notin b\gamma c$  for any  $b, c \in B$  and  $\gamma \in \Gamma$ . Assume  $a \in b\gamma c$ , then  $a \neq b$  and  $a \neq c$ . Hence,  $a \in b\gamma c \cup b\gamma c\Gamma b\gamma c \cup b\gamma c\Gamma H\Gamma b\gamma c$ . By Lemma 2.5(1.),  $a = b$  or  $a = c$ . This is a contradiction. Conversely, assume  $b \in b\gamma c$  or  $c \in b\gamma c$  for any  $b, c \in B$ . We will show that  $B$  is a sub- $\Gamma$ -semihypergroup of  $H$ . Let  $a \in B\Gamma B$ . Hence,  $a \in b\gamma c$  for some  $b, c \in B$  and  $\gamma \in \Gamma$ . This implies  $a \in b\gamma c \cup b\gamma c\Gamma b\gamma c \cup b\gamma c\Gamma H\Gamma b\gamma c$ .

By Lemma 2.5(1.),  $a = b$  or  $a = c$ . Hence,  $a \in \{b, c\} \subseteq B$ . Therefore  $B$  is a sub- $\Gamma$ -semihypergroup of  $H$ .  $\square$

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