Iranian Journal of Mathematical Sciences and Informatics
Vol. 19, No. 2 (2024), pp 51-60
DOI: 10.61186/ijmsi.19.2.51

G-Injective Envelope of Separable G-C*-algebras

Ali Mahmoodi, Mohammad R. Mardanbeigi*

Department of Mathematics, Faculty of Science, Science and Research Branch, Islamic Azad University(IAU), Tehran, Iran

E-mail: mahmoodi326@gmail.com
E-mail: mrmardanbeigi@srbiau.ac.ir

Abstract. Argerami and Farenick have found conditions for the injective envelope of a separable C^*-algebra to be a von Neumann algebra. In this paper, we introduce an equivalent version of this result by finding conditions for the G-injective envelope of a separable G-C^*-algebra A to be a von Neumann algebra, when G is a discrete group acting on A.

Keywords: G-W^*-algebra, G-AW^*-algebra, G-Injective envelope, G-Regular monotone completion, Type I C^*-algebra, G-invariant Essential ideal, G-Local multiplier algebra, Discrete group.

2020 Mathematics subject classification: 46L05, 20M12.

1. Introduction

1.1. **Notice.** In 1979, Hamana [7, theorem 4.1] used the Arveson extension theorem to prove that any C^*-algebra has an injective envelope which is unique up to *-isomorphism. Indeed, he showed that if A is a C^*-algebra, then the image of a unit-preserving idempotent contractive linear map φ of an Arveson injective extension B into itself, is the injective envelope of A. Later, in 1985, Hamana found an equivariant version of his result [9] by showing that there exists a unique G-injective envelope $(I_G(A), \kappa)$, for any G-operator system A, such that if $(B, \acute{\kappa})$ is any G-injective envelope of A, there exists a complete

*Corresponding Author

Received 9 November 2019; Accepted 6 February 2021

order isomorphism $\varphi : I_G(A) \longrightarrow B$, satisfying $\varphi \circ \kappa = \acute{\kappa}$, where G is a discrete group acting on A and B.

On the other hand, an injective operator system is unitally and completely order isomorphic to a unital, monotone complete AW^*-algebra [5, 12]. In the above cited result of Hamana, if $\varphi : B \longrightarrow B$ is a minimal A-projection, then the multiplication on $I_G(A) = \varphi(B)$ is given by the Choi-Effros product, that is, by

$$x \circ y = \varphi(xy), \quad x, y \in I_G(A)$$

and the involution and norm on $I_G(A)$ are inherited from B. Furthermore, if A is a unital G-C^*-algebra, then A embeds into its G-injective envelope as a G-invariant unital C^*-subalgebra. In the case when $G = \{1\}$, the above product yields a C^*-algebra injective structure on the injective envelope $I(A)$ of A.

In this paper, we extend a result of M. Argerami and D. R. Farenick [2, Theorem 1.2] to the setting of discrete C^*-dynamics. In the next section, we set up the terminology and notations for G-C^*-algebras and G-W^*-algebras. In the main result of the paper in section 3, we show that parts $(i), (ii)$ and (v) of Theorem 1.2 in [2] remain equivalent in separable G-C^*-algebras for discrete C^*-dynamics.

2. G-C^*-algebras

Let $B(H)$ and $K(H)$ be the set of bounded and compact operators on a complex Hilbert space H, respectively. A C^*-algebra A is a W^*-algebra if A, as a Banach space, is the dual space X^* of some (in fact, unique) Banach space X. It is a classical fact that a C^*-algebra A is a W^*-algebra iff A has a representation as a von Neumann algebra of operators acting on some complex Hilbert space. A C^*-algebra A is an AW^*-algebra if the left annihilator of each right ideal in A is of the form Ap, for some projection $p \in A$, or equivalently, if every maximal abelian C^*-subalgebra $D \subseteq A$ is monotone complete [3]. Any W^*-algebra is an AW^*-algebra, but the converse is not true, i.e., there exists AW^*-algebras that fail to have any faithful representation as a von Neumann algebra.

In the category of C^*-algebras and completely positive (c.p.) linear maps, the pair (B, κ) is an extension of a C^*-algebra A, if B is a C^*-algebra and $\kappa : A \longrightarrow B$ is a c.p. map. A C^*-algebra A is *injective* if we can extend any A-valued completely positive linear map of subspace S of a C^*-algebra C to an A-valued completely positive linear map of the C^*-algebra C. An extension (B, κ) of a C^*-algebra A is called the *injective envelope* of A if B is injective and the only completely positive linear map of B into itself that fixes each element of $\kappa(A)$, is the identity map id_B. In [7], Hamana proved that any C^*-algebra has a unique injective envelope. Following Choi and Effros [4], he considered a completely positive linear map ϕ of the C^*-algebra B into itself, and observed

[DOI: 10.61186/ijmsi.19.2.51]

that $Im(\phi)$ with multiplication "$\circ$", $x \circ y = \phi(xy)$ for all $x, y \in Im(\phi)$, and involution and norm induced by those of B, is a unital C^*-algebra. The C^*-algebra $Im(\phi)$ is denoted by $C^*(\phi)$. Hamana proved that $C^*(\phi)$ is injective if B is injective in this category. Finally, if A is a C^*-algebra, there exists an injective C^*-algebra C containing A as a C^*-subalgebra, by the Arveson extension theorem (which asserts that the algebra of bounded operators on a complex Hilbert space is injective). By [7, Theorem 3.4], there exists a minimal A-projection ϕ on C. If $B = C^*(\phi)$ and κ is the canonical inclusion of A into B, then (B, κ) is an injective envelope of A.

In this section we generalize some of the results obtained in the category of C^*-algebras and completely positive linear maps to the category of G-C^*-algebras and completely positive G-linear maps. We assume throughout this paper that G is a discrete group.

A G-C^*-algebra is a C^*-algebra which equipped with an action of G by automorphisms. In other words, a G-C^*-algebra A is a C^*-algebra and a left G-module. Given two G-C^*-algebras A and B, the unital completely positive linear map $\varphi : A \longrightarrow B$ is *G-equivariant*, if $\varphi(g \cdot a) = g \cdot \varphi(a)$, for any $g \in G$ and $a \in A$. A G-C^*-algebra B can be viewed as a C^*-algebra over the discrete group algebra $L^1(G)$ with the module operation defined by

$$f \cdot x = \int f(g)\theta_g(x)dg \quad , \quad f \in L^1(G), x \in B$$

One could define the category of G-W^*-algebras and G-injective objects in this category in an analogous manner. A G-C^*-algebra B is a *G-W^*-algebra* if B is a W^*-algebra with the $L^1(G)$-module structure such that the map $x \mapsto f \cdot x$ in B is positive and normal, for each $f \in L^1(G)^+$.

A G-C^*-algebra A is said to be *G-injective* if for any G-C^*-algebras B and C, any G-equivariant complete isometry $\kappa : B \longrightarrow C$ and any G-equivariant u.c.p map $\varphi : B \longrightarrow A$, there exists a G-equivariant u.c.p map $\tilde{\varphi} : C \longrightarrow A$ satisfying $\tilde{\varphi} \circ \kappa = \varphi$, i.e., the following diagram commutes,

$$\begin{tikzcd} B \arrow[r, "\kappa"] \arrow[dr, "\varphi"'] & C \arrow[d, "\tilde{\varphi}"] \\ & A \end{tikzcd}$$

This simply means that G-equivariant u.c.p maps into A have G-equivariant u.c.p extensions.

Suppose that A and B are G-C^*-algebras. We say that;
(i) (B, κ) is a *G-extension* of A, if $\kappa : A \longrightarrow B$ be a G-equivariant and u.c.p $*$-monomorphism.
(ii) The G-extension (B, κ) is *G-essential* if for any G-C^*-algebra C and any G-equivariant u.c.p map $\varphi : B \longrightarrow C$, φ is completely isometric whenever $\varphi \circ \kappa$ is.

[DOI: 10.61186/ijmsi.19.2.51]

(iii) The G-extension (B,κ) is *G-rigid* if the only G-equivariant u.c.p map $\varphi : B \longrightarrow B$ satisfying $\varphi \circ \kappa = \kappa$ is the identity map id_B.

The pair (B,κ) is a *G-injective envelope* of A, if (B,κ) is G-essential , G-rigid and B is G-injective.

Throughout this paper, we denote the G-injective envelope of a G-C^*-algebra A by $I_G(A)$. When G is trivial we are back to the notations of injectivity for C^*-algebras, as well as plain essentiality and rigidity of extensions.

Let A be a unital G-C^*-algebra and let $\theta : G \longrightarrow Aut(A)$ be a G-action. Writing $\theta_g = \theta(g)$, for all $g \in G$, by injectivity each $\theta_g : A \longrightarrow A$ $(a \longrightarrow g \cdot a)$ extends to a $*$-isomorphism $I_G(A) \longrightarrow I_G(A)$, still denoted by θ_g. Due to rigidity, one can show that $\theta_g \circ \theta_h = \theta_{gh}$ on $I_G(A)$, for all $g, h \in G$, so that $I_G(A)$ becomes a unital G-C^*-algebra containing A as a G-invariant C^*-subalgebra. Further, the inclusion $A \hookrightarrow I_G(A)$ is a G-essential extension of A.

In [9], Hamana proved that there exist a unique G-injective envelope $(I_G(A),\kappa)$, for any G-operator system A, such that if $(B,\acute{\kappa})$ is any other G-injective envelope of A, there exists a complete order isomorphism $\varphi : I_G(A) \longrightarrow B$ satisfying $\varphi \circ \kappa = \acute{\kappa}$.

Let H be a complex Hilbert space and A be an operator system in $B(H)$, then $\ell^\infty(G,A)$ becomes a G-operator subsystem of $B(H \otimes \ell^2(G))$ with the action of G given by the left translation, i.e.,

$$(gf)(h) = f(g^{-1}h), \quad g,h \in G, \quad f \in \ell^\infty(G,A)$$

and each $f \in \ell^\infty(G,A)$ is acting on $H \otimes \ell^2(G)$ by $f(\xi \otimes \delta_g) = f(g)\xi \otimes \delta_g$, for $\xi \in H$ and $g \in G$.

Hamana showed that if A is an injective operator system, then $\ell^\infty(G,A)$ is G-injective, and that any G-injective G-operator system is injective.

If $A \subseteq B$ and B is a G-injective G-operator system, then an A-projection on B is a G-equivariant u.c.p map $\varphi : B \longrightarrow B$ satisfying $\varphi|_A = id_A$. A partial ordering on the set of A-projections on B can be defined by $\varphi \prec \psi$, for A-projections $\varphi, \psi : B \longrightarrow B$ if $\varphi \circ \psi = \psi \circ \varphi = \varphi$.

By the Zorn's lemma, there exists a minimal A-projection $\varphi : B \longrightarrow B$ on the set of seminorms induced by A-projection on B. In this argument, letting $\kappa : A \longrightarrow B$ be the inclusion map, then $(\varphi(B),\kappa)$ is a G-rigid and G-C^*-injective extension of A. Therefore, $(\varphi(B),\kappa)$ is the G-injective envelope of A.

A canonical G-injective G-operator system is $\ell^\infty(G,B)$, where B is an injective C^*-algebra. Let A be a unital G-C^*-algebra and B be a unital injective C^*-algebra containing A Let $\kappa : A \longrightarrow \mathcal{M} = \ell^\infty(G,B)$ be the G-equivariant injective $*$-homomorphism given by

$$\kappa(x)(g) = g^{-1}x, \quad x \in A, \quad g \in G.$$

[DOI: 10.61186/ijmsi.19.2.51]

Then there is a $\kappa(A)$-projection $\varphi : \mathcal{M} \longrightarrow \mathcal{M}$ such that $(\varphi(\mathcal{M}), \kappa)$ is the G-injective envelope of A. Thus, for any injective extension B of a unital G-C^*-algebra A, the map $\kappa : A \longrightarrow \ell^\infty(G, B)$ is the canonical inclusion map.

Any injective operator system is unitally and completely order isomorphic to a unital, monotone complete AW^*-algebra [5, 12]. In our setting, if $A \subseteq B$ are as above and $\varphi : B \longrightarrow B$ is a minimal A-projection, then the multiplication on $I_G(A) = \varphi(B)$ is given by the Choi-Effros product, i.e., by

$$x \circ y = \varphi(xy), \quad x, y \in I_G(A)$$

and the involution and norm on $I_G(A)$ are inherited from B [7]. Further, if A is a unital G-C^*-algebra, then A embeds into its G-injective envelope as a G-invariant unital C^*-subalgebra. In the case when $G = \{1\}$, the above product yields a C^*-algebra injective structure on the injective envelope $I(A)$ of A.

A G-C^*-algebra A is a *G-monotone complete* if underlying C^*-algebra A is a monotone complete. A G-W^*-algebra is G-monotone complete if the underlying W^*-algebra is so as a C^*-algebra. A linear subspace A of a G-C^*-algebra B is called G-C^*-subalgebra of B, written $A \preceq B$, if A is a G-C^*-algebra in the restricted action of G.

Given two G-C^*-algebra $A \preceq B$, A is said to be *G-closed* in B if $y \in B$ and $g \cdot y \in A$, for all $g \in L^1(G)$, imply $y \in A$. For any G-C^*-algebras $A \preceq B$ the smallest G-closed G-C^*-subalgebra of B containing A is called the *G-closure* of A in B, written G-$cl_B A$, i.e., $G\text{-}cl_B A = \{y \in B : f \cdot y \in A \text{ for all } f \in L^1(G)\}$. A G-C^*-algebra A is *G-complete* if for any G-C^*-algebra B with $A \preceq B$, A is a G-closed in B.

A *G-regular completion* of a G-C^*-algebra A is a G-C^*-algebra, written $\overline{A}_G$, such that;
(1) $\overline{A}_G$ is G-complete,
(2) $A \preceq \overline{A}_G$,
(3) If $A \preceq B$ and B is G-C^*-complete, there are a G-C^*-algebra B' with $A \preceq B' \preceq B$ and a G-isomorphism $\psi : \overline{A}_G \longrightarrow B'$ with $\psi|_A = id_A$.

In fact, the $\overline{A}_G$ is the smallest G-complete containing A. Hence, $\overline{A}_G$ exists and is unique. Now the Hamana's construction [9] of $\overline{A}_G$ is via the G-injective envelope of A. Namely, $\overline{A}_G$ is the G-closure of A in $I_G(A)$.
For each G-C^*-algebra A, there is a representation in which

$$A \preceq \overline{A}_G \preceq I_G(A),$$

where each containment is as a G-C^*-subalgebra. An important feature of this sequence of containments is that $\overline{A}_G$ is G-monotone closed in $I_G(A)$

An ideal I of A is *essential* if $K \cap I \neq \{0\}$, for any non-zero ideal $K \subseteq A$. Equivalently, if $aI = 0$, for all $a \in A$, then $a = 0$. Any essential ideal is necessarily non-zero. The multiplier algebra $M(A)$ of a C^*-algebra A is a C^*-subalgebra of the enveloping von Neumann algebra A^{**} that consists of all $x \in A^{**}$ for which $xa \in A$ and $ax \in A$, for all $a \in A$.

[DOI: 10.61186/ijmsi.19.2.51]

An essential ideal I of a G-C^*-algebra A is *G-essential ideal* if I is G-invariant. For a G-invariant ideal I of A, the *G-multiplier algebra* $M_G(I)$ of I is the G-regular completion of the multiplier algebra $M(I)$, endowed with the canonical strictly continuous action of G,that is, $M_G(I) = \overline{M(I)}_G$.

If $J \subseteq A$ is a G-invariant ideal, then J^{**} is identified with the closure of J in A^{**} with respect to the strong operator topology. Thus, if J and K are G-invariant ideals of A, and if $J \subseteq K$, then $M_G(J) \succeq M_G(K) \succeq M_G(A)$.

Consider the G-multiplier algebra $M_G(J)$ of any G-essential ideal J of A. If $\varepsilon_G(A)$ is the set of G-essential ideals of A, partially ordered by reverse inclusion, then the set $\xi(A)$ of G-multiplier algebras $M_G(K)$ of $K \in \varepsilon_G(A)$ is a directed system of G-C^*-algebras. We define a *G-local multiplier algebra*, denoted by $M_G^{loc}(A)$, as follows

$$M_G^{loc}(A) = \varinjlim\{M_G(K); K \in \varepsilon_G(A)\}.$$

In fact, the $M_G^{loc}(A)$ is defined to be the C^*-direct limit over the downward directed system $K \in \varepsilon_G(A)$, and $M_G^{loc}(A)$ is realized by idealizers in $I_G(A)$ of G-essential ideals of A. By an argument similar to [6, Corollary 4.3]

$$M_G^{loc}(A) = cl\left(\bigcup_{K\in\varepsilon_G(A)}\{x \in I_G(A); xK + Kx \subseteq K\}\right)$$

where the closure is with respect to the norm topology of $I_G(A)$. Thus,

$$A \preceq M_G^{loc}(A) \preceq I_G(A)$$

is an inclusion of G-C^*-subalgebras.

Lemma 2.1. *If A is a G-C^*-algebra for which $I_G(A)$ is a G-W^*-algebra, then $\overline{A}_G$ is a G-W^*-algebra.*

Proof. Suppose that $I_G(A)$ is a G-W^*-algebra. Then $I_G(A)$ is represented as a von Neumann algebra acting on a Hilbert space. We assume that $\{h_\alpha\}_\alpha$ be any bounded increasing net in $(\overline{A}_G)_{sa}$. Because $I_G(A)$ is G-monotone complete, $\{h_\alpha\}_\alpha$ has a least upper h such that $h = \lim_\alpha h_\alpha = \sup_\alpha h_\alpha$ in the strong operator topology. Since, $\overline{A}_G$ is G-monotone closed in $I_G(A)$, $h \in \overline{A}_G$. Thus $\overline{A}_G$ is a G-C^*-algebra of operators in which the limit of every bounded increasing net of hermitian elements again belongs to $\overline{A}_G$. Therefore, $\overline{A}_G$ is a G-W^*-algebra by [10, lemma 1]. $\square$

Proposition 2.2. *For any G-C^*-algebra A the G-closure of A in its G-injective envelope $I_G(A)$ is the G-regular completion $\overline{A}_G$ of A.*

Proof. Let A_1 be the G-closure of A in $I_G(A)$ and $A \preceq B$, then $A \preceq B \preceq B_1$ for some G-injective B_1, and there are an idempotent G-morphism $\phi : B_1 \longrightarrow B_1$ and a G-isomorphism $\psi : I_G(A) \longrightarrow \phi(B_1)$ such that $\phi|_A = id_A = \psi|_A$. We have $G\text{-}cl_{B_1}A \preceq \phi(B_1)$. Indeed, if $b \in G\text{-}cl_{B_1}A$, then $f \cdot b \in A$ for all $f \in L^1(G)$ and $f \cdot b = \phi(f \cdot b) = f \cdot \phi(b)$ in B_1 for all $f \in L^1(G)$; hence $b = \phi(b) \in \phi(B_1)$. Thus

$$G\text{-}cl_{\phi(B_1)}A = (G\text{-}cl_{B_1}A) \cap \phi(B_1) = G\text{-}cl_{B_1}A.$$

Further, since ψ is a G-isomorphism and $\psi|_A = id_A$, we have $\psi(A_1) = G - cl_{\phi(B_1)}A$, and so $\psi(A_1) = G - cl_{B_1}A$. First we assume that $y \in \psi(A_1)$, then there is a $a_1 \in A_1$ such that $y = \psi(a_1) \in \phi(B_1)$. On the other hand, since A_1 is a G-closure of A, $f \cdot a_1 \in A$ for all $f \in L^1(G)$, and since $\psi|_A = id_A$, we have

$$f \cdot y = f \cdot \psi(a_1) = \psi(f \cdot a_1) = f \cdot a_1 \in A.$$

Hence, $y \in G\text{-}cl_{\phi(B_1)}A$.
Now, let $y \in G - cl_{\phi(B_1)}A$. By definition, we have $f \cdot y \in A$ and $y \in \phi(B_1)$. Suppose that $b_1 \in B_1$, with $y = \phi(b_1)$. Since ψ is a G-isomorphism, there exists $a_1 \in I_G(A)$ such that $y = \phi(b_1) = \psi(a_1)$. On the other hand, since A_1 is a G-closure of A in $I_G(A)$,

$$\psi(f \cdot a_1) = f \cdot \psi(a_1) = f \cdot y \in A \Rightarrow f \cdot a_1 \in A \Rightarrow a_1 \in A_1 \Rightarrow y = \psi(a_1) \in \psi(A_1).$$

If $A_1 = A$, namely, A is G-closed in $I_G(A)$. Then so is A in $\phi(B_1)$, and $A = G\text{-}cl_{B_1}A$. Hence, $A = G\text{-}cl_B A$, that is, A is G-closed in B. Since $A \preceq B \preceq B_1$, $G\text{-}cl_B A \preceq G\text{-}cl_{B_1}A$. As B is arbitrary, this means that A is G-complete.

Next, suppose that A is arbitrary, but B is G-complete. Since $I_G(A_1) = I_G(A)$ and A_1 is G-closed in $I_G(A)$, it follows from the foregoing argument that A_1 is G-complete. As B is G-complete, $G\text{-}cl_{B_1}A \preceq G\text{-}cl_{B_1}B = B$, and $\psi(A_1) = G\text{-}cl_{B_1}A \preceq B$ with $\psi(A_1) \cong A_1$. Therefore, A_1 is the G-regular completion of A.

Finally, let only that $A \preceq B$. By the above argument to $A \preceq B \preceq \overline{B}_G$, there is a G-isomorphism ψ of A_1 onto $G\text{-}cl_{\overline{B}_G}A$ with $\psi|_A = id_A$. Hence, since $A \preceq G\text{-}cl_B A \preceq G\text{-}cl_{\overline{B}_G}A$, $G\text{-}cl_B A$ is isomorphic to the G-C^*-subalgebra $\psi^{-1}(G\text{-}cl_B A)$ of A_1. □

3. Separable C^*-algebra of a discrete group

The main result of this paper is Theorem (3.4) on separable discrete C^*-dynamics. Before turning to the proof of Theorem (3.4), we prove some preliminary results. We need the notion of *covariant representation* and the relation between G-local multiplier algebra and G-regular completion of G-C^*-algebras.

Definition 3.1. A C^*-algebra A is called *elementary* if $A \cong K(H)$ for some Hilbert space H.

The separable elementary C^*-algebras are the finite-dimensional matrix algebras and the C^*-algebras of compact operators of separable infinite-dimensional Hilbert space. Every elementary C^*-algebra is simple and the converse is true when the C^*-algebra is of type I. If A is a C^*-subalgebra of $K(H)$ acting irreducibly on Hilbert space H, then A is elementary.

Definition 3.2. A *covariant representation* of a G-C^*-algebra A is a pair (π, σ) where (π, H) is a representation of A , (σ, H) is a unitary representations of G,

[DOI: 10.61186/ijmsi.19.2.51]

such that

$$\sigma(g)\pi(a)\sigma(g)^{-1} = \pi(\theta_g(a)) = \pi(g \cdot a)$$

for every $a \in A$, $g \in G$.
A covariant representation (π, σ) of a G-C^*-algebra A on a Hilbert space H is *normal* if (π, H) is normal.

Proposition 3.3. $\overline{M_G^{loc}(A)} = \overline{A}_G$ *for every G-C^*-algebra A.*

Proof. Since $M_G^{loc}(A)$ is G-equivariant *-isomorphically embedded into $I_G(A)$, extending the canonical G-equivariant *-monomorphism of A into $I_G(A)$, the G-C^*-algebra $I_G(A)$ serves as an injective G-extension of the G-C^*-algebra $M_G^{loc}(A)$. Therefore, the identity map on $M_G^{loc}(A)$ admits a unique G-extension to a G-equivariant completely positive map of $I_G(A)$ into itself with the same completely bounded norm one. Since $A_G \preceq M_G^{loc}(A) \preceq I_G(A)$ by construction and $I_G(A)$ is the G-injective envelope of A, $I_G(A)$ has to be the G-injective envelope of $M_G^{loc}(A)$. Since the G-regular completion of a G-C^*-algebra B is the G-monotone closure of B in the G-injective envelope $I_G(A)$,

$$A_G \preceq M_G^{loc}(A) \preceq \overline{A}_G \preceq I_G(A) = I(M_G^{loc}(A))$$

implies that $\overline{A}_G \preceq \overline{M_G^{loc}(A)} \preceq \overline{\overline{A}}_G$. Thus, $\overline{M_G^{loc}(A)} = \overline{A}_G$. □

Theorem 3.4. *The following statements are equivalent for a separable G-C^*-algebra A:*
(i) $\overline{A}_G$ is a G-W^*-algebra.
(ii) $I_G(A)$ is a G-W^*-algebra.
(iii) A contains a G-invariant minimal essential ideal that is G-isomorphic to a direct sum of elementary G-C^*-algebras.

Proof. By Lemma (2.1), the proof of (ii)⇒(i) is clear.
(ii)⇒(iii): We have divided the proof into two stages. In the first stage, let us first show that there exists a faithful representation $\pi : A_G \longrightarrow B(H)$ such that the von Neumann algebra $\pi(A_G)''$ is generated by its minimal projections, each of which is contained in $\pi(A_G)$. For this, let $I_G(A)$ be a G-W^*-algebra. By [11, lemma 7.4.9], there is a faithful G-equivariant representation $\widetilde{\pi} : I_G(A) \longrightarrow B(H)$ such that $\pi(A_G)$ is a G-C^*-subalgebra of $\widetilde{\pi}(I_G(A))$, with $\pi = \widetilde{\pi}|_{(A_G)}$. Without loss of generality, suppose that $I_G(A)$ is a von Neumann algebra acting on a Hilbert space. Since the G-regular completion $\overline{A}_G$ of A_G is G-monotone closed in $I_G(A)$ and because $I_G(A)$ is a von Neumann algebra, $\overline{A}_G$ is a von Neumann algebra by Lemma (2.1). Thus, $A_G'' \subseteq \overline{A}_G'' = \overline{A}_G$, A_G'' being the double commutant of A_G.

Now, let ω be a normal state on von Neumann algebra A_G'' that is faithful on A_G. Assume that $\omega(h) = 0$, where $h \in A_G''^{+}$. Because $h = \sup\{k \in A^+; k \leq h\}$, we have $0 \leq \omega(k) \leq \omega(h) = 0$, for each $k \in A_G^+$ with $k \leq h$. Thus $\omega(k) = 0$, which implies that $k = 0$ because ω is faithful on A. Hence, $h = 0$ and so

[DOI: 10.61186/ijmsi.19.2.51]

ω is faithful on A''_G. Namely, any normal state $\omega \in A''_G$ is faithful precisely when its restriction $\omega|_{A_G}$ to A_G is faithful. By [13, P. 139], because A_G is separable and order dense in A''_G, A''_G is generated by its minimal projections, each of which is contained in A_G. Furthermore, since A''_G is a direct product of type I factors by [3, lemma 2.2], A''_G is injective by [3, corollory 2.3]. Because $A_G \subseteq A''_G \subseteq I_G(A)$, we conclude that $A''_G = \overline{A}_G = I_G(A)$, by minimality of the injective envelope.

The second stage, without loss of generality, assumes that A_G is already represented as a subalgebra of $B(H)$ and that $M = A''_G$ is generated by its minimal projections, each of with lie in A_G. Let $K \subseteq A_G$ be the ideal of A_G generated by the minimal projections of M. We claim that K is an essential ideal, minimal among all essential ideals of A_G. Suppose that $J \subseteq A_G$ is a nonzero ideal. Choose any nonzero $h \in J^+$. There is a strictly positive λ in the spectrum $\sigma(h)$ of h. Let $e \in M$ be the spectral projection $e = e^h([\lambda, +\infty))$, where e^h denotes the spectral resolution of h. Thus, $0 \neq \lambda e \leq he$, and there is a minimal projection p of M such that $ep = pe = p$ and $0 \neq \lambda p = \lambda p^2 = p\lambda p \leq php \in J \cap K$. Then $J \cap K \neq \{0\}$.

By [3, lemma 2.2], since $M = A''_G$ is generated by its minimal projections, M is a discrete type I von Neumann algebra. Therefore, there is a faithful normal covariant *-representation γ of M on a Hilbert space H of the form $H = \bigoplus_n H_n$ by [11, lemma 7.4.9], such that

$$\gamma(K) \subseteq \gamma(A_G) \subseteq \gamma(M) = \prod_n B(H_n)$$

It fact, the minimal projections of any $B(H_n)$ are minimal projection of $\gamma(M)$. Hence, elements of $\gamma(K)$. Moreover, if e is a minimal projection of $\prod_n B(H_n)$, $e \in B(H_n)$, for some $n \in N$. Therefore, $\bigoplus_n K(H_n) \subseteq \gamma(K)$. Since $\gamma(K)$ is the smallest G-C^*-algebra that contains the minimal projections of $\gamma(M)$, it follows that $\gamma(K) = \bigoplus_n K(H_n)$. Since, $K \cong \bigoplus_n K(H_n)$, K is G-invariant minimal essential ideal of A_G.

(iii)⇒(ii): Suppose that A_G has a G-invariant minimal essential ideal K such that $K \cong \bigoplus_n K(H_n)$. Thus, by [1, Lemma 1.2.21],

$$M(K) = M(\bigoplus_n K(H_n)) = \prod_n M(K(H_n)) = \prod_n B(H_n),$$

and this shows that $M(K)$ is a type I W^*-algebra. Since K is a G-invariant minimal essential ideal of A_G, by [1, Remark 2.3.7] $M(K) = M_G^{loc}(A)$. Hence, $M_G^{loc}(A)$ is an injective G-W^*-algebra. We know that $A_G \subseteq M_G^{loc}(A) \subseteq I_G(A)$ as G-C^*-subalgebras, it must be that $M_G^{loc}(A) = I_G(A)$ by definition of injective envelope, and this is precisely the proof of the G-W^*-algebra of $I_G(A)$.

(i)⇒(ii): For the G-W^*-algebra $\overline{A}_G$, $\overline{A}_G = A''_G$ by the proof of (ii)⇒(iii). Since A''_G is a direct product of type I factors, so A''_G is injective. Therefore,

[DOI: 10.61186/ijmsi.19.2.51]

$\overline{A}_G$ is injective. Hence, $\overline{A}_G = I_G(A)$, which yields that $I_G(A)$ is a G-W^*-algebra. $\square$

EXAMPLE 3.5. by [8, lemma 2.2], $A = \ell^\infty(G, B(H))$ is G-injective, where G acts trivially on $B(H)$. Thus $I_G(A) = A$ which is a G-W^*-algebra. Now the minimal essential ideal of A is $c_0(G) \otimes K(H)$ which is essential ideal and dense and is direct sum of $|G|$-copies of elementary C^*-algebras $\mathbb{C} \otimes K(H)$ [This is an infinite direct sum if the cardinal $|G|$ is not finite]. Also A is already G-complete, so the G-closure of A is A itself, which is a G-W^*-algebra.

Acknowledgments

The authors are grateful to the anonymous referees for their careful reading of this paper and constructive corrections and valuable comments on this paper, which have considerably improved the presentation of this paper.The authors have special thanks to Professor M. Amini for helpful comments which improved the paper.

References

1. P. Ara, M. Mathieu, *Local Multipliers of C^*-algebras*, Springer Monographs in Mathematics, London, 2003.
2. M. Argerami, D. R. Farenick, Local Multipliers Algebras, Injective Envelopes, and Type I W^*-algebras, *J. Operator Theory*, **59**, (2008), 237-245.
3. M. Argerami, D. R. Farenick, Injective Envelopes of Separable C^*-algebras, *Philos. Trans. R. Soc. A*, **179**, (2005), 43-63.
4. M.-D. Choi, E. G. Effros, Separable Nuclear C^*-algebras and Injectivity, *Duke Math. J*, **43**, (1976), 309-322.
5. M. D. choi, E. G. Effros, Injectivity and Operator Spaces, *Journal of Functional Analysis*, **24**, (1977), 156-209.
6. M. Frank, V. I. Paulsen, Injective Envelopes of C^*-algebras as Operator Modules, *Pacific J. Math*, **212**, (2003), 57-69.
7. M. Hamana, Injective Envelopes of C^*-algebras, *J. Math. Soc. Japen*, **31**, (1979), 181-197.
8. M. Hamana, Injective Envelope of C^*-dynamical Systems, *Tôhoku Math. Journ*, **37**, (1985), 463-487.
9. M. Hamana, Injective Envelope of Dynamical Systems, *Toyama Math. J*, **34**, (2011), 23-86.
10. R. V. Kadison, Operator Algebras with a Faithful Weakly-closed Representation, *Ann. of Math*, **64**, (1956), 175-181.
11. G. K. Pedersen, *C^*-algebras and Their Automorphism Groups*, London Mathematical Society, Monographs 14, 1979.
12. K. Saitô, J. D. M. Wright, *Monotone Complete C^*-algebras and Generic Dynamics*, Springer-Varlag, 2015.
13. M. Takesaki, *Theory in Operator Algebras I*, Encylopaedia of Mathematical Sciences, Springer-Verlag, New York, **124**, 2001.

[DOI: 10.61186/ijmsi.19.2.51]