

## G-Injective Envelope of Separable G-C\*-algebras

Ali Mahmoodi, Mohammad R. Mardanbeigi\*

Department of Mathematics, Faculty of Science, Science and Research  
Branch, Islamic Azad University(IAU), Tehran, Iran

E-mail: mahmoodi326@gmail.com

E-mail: mrmardanbeigi@srbiau.ac.ir

ABSTRACT. Argerami and Farenick have found conditions for the injective envelope of a separable  $C^*$ -algebra to be a von Neumann algebra. In this paper, we introduce an equivalent version of this result by finding conditions for the  $G$ -injective envelope of a separable  $G$ - $C^*$ -algebra  $A$  to be a von Neumann algebra, when  $G$  is a discrete group acting on  $A$ .

**Keywords:**  $G$ - $W^*$ -algebra,  $G$ - $AW^*$ -algebra,  $G$ -Injective envelope,  $G$ -Regular monotone completion, Type I  $C^*$ -algebra,  $G$ -invariant Essential ideal,  $G$ -Local multiplier algebra, Discrete group.

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### 1. INTRODUCTION

1.1. **Notice.** In 1979, Hamana [7, theorem 4.1] used the Arveson extension theorem to prove that any  $C^*$ -algebra has an injective envelope which is unique up to  $*$ -isomorphism. Indeed, he showed that if  $A$  is a  $C^*$ -algebra, then the image of a unit-preserving idempotent contractive linear map  $\varphi$  of an Arveson injective extension  $B$  into itself, is the injective envelope of  $A$ . Later, in 1985, Hamana found an equivariant version of his result [9] by showing that there exists a unique  $G$ -injective envelope  $(I_G(A), \kappa)$ , for any  $G$ -operator system  $A$ , such that if  $(B, \hat{\kappa})$  is any  $G$ -injective envelope of  $A$ , there exists a complete

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\*Corresponding Author

order isomorphism  $\varphi : I_G(A) \longrightarrow B$ , satisfying  $\varphi \circ \kappa = \acute{\kappa}$ , where  $G$  is a discrete group acting on  $A$  and  $B$ .

On the other hand, an injective operator system is unital and completely order isomorphic to a unital, monotone complete  $AW^*$ -algebra [5, 12]. In the above cited result of Hamana, if  $\varphi : B \longrightarrow B$  is a minimal  $A$ -projection, then the multiplication on  $I_G(A) = \varphi(B)$  is given by the Choi-Effros product, that is, by

$$x \circ y = \varphi(xy), \quad x, y \in I_G(A)$$

and the involution and norm on  $I_G(A)$  are inherited from  $B$ . Furthermore, if  $A$  is a unital  $G$ - $C^*$ -algebra, then  $A$  embeds into its  $G$ -injective envelope as a  $G$ -invariant unital  $C^*$ -subalgebra. In the case when  $G = \{1\}$ , the above product yields a  $C^*$ -algebra injective structure on the injective envelope  $I(A)$  of  $A$ .

In this paper, we extend a result of M. Argerami and D. R. Farenick [2, Theorem 1.2] to the setting of discrete  $C^*$ -dynamics. In the next section, we set up the terminology and notations for  $G$ - $C^*$ -algebras and  $G$ - $W^*$ -algebras. In the main result of the paper in section 3, we show that parts (i), (ii) and (v) of Theorem 1.2 in [2] remain equivalent in separable  $G$ - $C^*$ -algebras for discrete  $C^*$ -dynamics.

## 2. $G$ - $C^*$ -ALGEBRAS

Let  $B(H)$  and  $K(H)$  be the set of bounded and compact operators on a complex Hilbert space  $H$ , respectively. A  $C^*$ -algebra  $A$  is a  $W^*$ -algebra if  $A$ , as a Banach space, is the dual space  $X^*$  of some (in fact, unique) Banach space  $X$ . It is a classical fact that a  $C^*$ -algebra  $A$  is a  $W^*$ -algebra iff  $A$  has a representation as a von Neumann algebra of operators acting on some complex Hilbert space. A  $C^*$ -algebra  $A$  is an  $AW^*$ -algebra if the left annihilator of each right ideal in  $A$  is of the form  $Ap$ , for some projection  $p \in A$ , or equivalently, if every maximal abelian  $C^*$ -subalgebra  $D \subseteq A$  is monotone complete [3]. Any  $W^*$ -algebra is an  $AW^*$ -algebra, but the converse is not true, i.e., there exists  $AW^*$ -algebras that fail to have any faithful representation as a von Neumann algebra.

In the category of  $C^*$ -algebras and completely positive (c.p.) linear maps, the pair  $(B, \kappa)$  is an extension of a  $C^*$ -algebra  $A$ , if  $B$  is a  $C^*$ -algebra and  $\kappa : A \longrightarrow B$  is a c.p. map. A  $C^*$ -algebra  $A$  is *injective* if we can extend any  $A$ -valued completely positive linear map of subspace  $S$  of a  $C^*$ -algebra  $C$  to an  $A$ -valued completely positive linear map of the  $C^*$ -algebra  $C$ . An extension  $(B, \kappa)$  of a  $C^*$ -algebra  $A$  is called the *injective envelope* of  $A$  if  $B$  is injective and the only completely positive linear map of  $B$  into itself that fixes each element of  $\kappa(A)$ , is the identity map  $id_B$ . In [7], Hamana proved that any  $C^*$ -algebra has a unique injective envelope. Following Choi and Effros [4], he considered a completely positive linear map  $\phi$  of the  $C^*$ -algebra  $B$  into itself, and observed

that  $Im(\phi)$  with multiplication "  $\circ$  ",  $x \circ y = \phi(xy)$  for all  $x, y \in Im(\phi)$ , and involution and norm induced by those of  $B$ , is a unital  $C^*$ -algebra. The  $C^*$ -algebra  $Im(\phi)$  is denoted by  $C^*(\phi)$ . Hamana proved that  $C^*(\phi)$  is injective if  $B$  is injective in this category. Finally, if  $A$  is a  $C^*$ -algebra, there exists an injective  $C^*$ -algebra  $C$  containing  $A$  as a  $C^*$ -subalgebra, by the Arveson extension theorem (which asserts that the algebra of bounded operators on a complex Hilbert space is injective). By [7, Theorem 3.4], there exists a minimal  $A$ -projection  $\phi$  on  $C$ . If  $B = C^*(\phi)$  and  $\kappa$  is the canonical inclusion of  $A$  into  $B$ , then  $(B, \kappa)$  is an injective envelope of  $A$ .

In this section we generalize some of the results obtained in the category of  $C^*$ -algebras and completely positive linear maps to the category of  $G$ - $C^*$ -algebras and completely positive  $G$ -linear maps. We assume throughout this paper that  $G$  is a discrete group.

A  $G$ - $C^*$ -algebra is a  $C^*$ -algebra which equipped with an action of  $G$  by automorphisms. In other words, a  $G$ - $C^*$ -algebra  $A$  is a  $C^*$ -algebra and a left  $G$ -module. Given two  $G$ - $C^*$ -algebras  $A$  and  $B$ , the unital completely positive linear map  $\varphi : A \rightarrow B$  is  $G$ -equivariant, if  $\varphi(g \cdot a) = g \cdot \varphi(a)$ , for any  $g \in G$  and  $a \in A$ . A  $G$ - $C^*$ -algebra  $B$  can be viewed as a  $C^*$ -algebra over the discrete group algebra  $L^1(G)$  with the module operation defined by

$$f \cdot x = \int f(g)\theta_g(x)dg \quad , \quad f \in L^1(G), x \in B$$

One could define the category of  $G$ - $W^*$ -algebras and  $G$ -injective objects in this category in an analogous manner. A  $G$ - $C^*$ -algebra  $B$  is a  $G$ - $W^*$ -algebra if  $B$  is a  $W^*$ -algebra with the  $L^1(G)$ -module structure such that the map  $x \mapsto f \cdot x$  in  $B$  is positive and normal, for each  $f \in L^1(G)^+$ .

A  $G$ - $C^*$ -algebra  $A$  is said to be  $G$ -injective if for any  $G$ - $C^*$ -algebras  $B$  and  $C$ , any  $G$ -equivariant complete isometry  $\kappa : B \rightarrow C$  and any  $G$ -equivariant u.c.p map  $\varphi : B \rightarrow A$ , there exists a  $G$ -equivariant u.c.p map  $\tilde{\varphi} : C \rightarrow A$  satisfying  $\tilde{\varphi} \circ \kappa = \varphi$ , i.e., the following diagram commutes,

$$\begin{array}{ccc} B & \xrightarrow{\kappa} & C \\ & \searrow \varphi & \downarrow \tilde{\varphi} \\ & & A \end{array}$$

This simply means that  $G$ -equivariant u.c.p maps into  $A$  have  $G$ -equivariant u.c.p extensions.

Suppose that  $A$  and  $B$  are  $G$ - $C^*$ -algebras. We say that;

- (i)  $(B, \kappa)$  is a  $G$ -extension of  $A$ , if  $\kappa : A \rightarrow B$  be a  $G$ -equivariant and u.c.p \*-monomorphism.
- (ii) The  $G$ -extension  $(B, \kappa)$  is  $G$ -essential if for any  $G$ - $C^*$ -algebra  $C$  and any  $G$ -equivariant u.c.p map  $\varphi : B \rightarrow C$ ,  $\varphi$  is completely isometric whenever  $\varphi \circ \kappa$  is.

(iii) The  $G$ -extension  $(B, \kappa)$  is  $G$ -rigid if the only  $G$ -equivariant u.c.p map  $\varphi : B \rightarrow B$  satisfying  $\varphi \circ \kappa = \kappa$  is the identity map  $id_B$ .

The pair  $(B, \kappa)$  is a  $G$ -injective envelope of  $A$ , if  $(B, \kappa)$  is  $G$ -essential,  $G$ -rigid and  $B$  is  $G$ -injective.

Throughout this paper, we denote the  $G$ -injective envelope of a  $G$ - $C^*$ -algebra  $A$  by  $I_G(A)$ . When  $G$  is trivial we are back to the notations of injectivity for  $C^*$ -algebras, as well as plain essentiality and rigidity of extensions.

Let  $A$  be a unital  $G$ - $C^*$ -algebra and let  $\theta : G \rightarrow Aut(A)$  be a  $G$ -action. Writing  $\theta_g = \theta(g)$ , for all  $g \in G$ , by injectivity each  $\theta_g : A \rightarrow A$  ( $a \rightarrow g \cdot a$ ) extends to a  $*$ -isomorphism  $I_G(A) \rightarrow I_G(A)$ , still denoted by  $\theta_g$ . Due to rigidity, one can show that  $\theta_g \circ \theta_h = \theta_{gh}$  on  $I_G(A)$ , for all  $g, h \in G$ , so that  $I_G(A)$  becomes a unital  $G$ - $C^*$ -algebra containing  $A$  as a  $G$ -invariant  $C^*$ -subalgebra. Further, the inclusion  $A \hookrightarrow I_G(A)$  is a  $G$ -essential extension of  $A$ .

In [9], Hamana proved that there exist a unique  $G$ -injective envelope  $(I_G(A), \kappa)$ , for any  $G$ -operator system  $A$ , such that if  $(B, \kappa)$  is any other  $G$ -injective envelope of  $A$ , there exists a complete order isomorphism  $\varphi : I_G(A) \rightarrow B$  satisfying  $\varphi \circ \kappa = \kappa$ .

Let  $H$  be a complex Hilbert space and  $A$  be an operator system in  $B(H)$ , then  $\ell^\infty(G, A)$  becomes a  $G$ -operator subsystem of  $B(H \otimes \ell^2(G))$  with the action of  $G$  given by the left translation, i.e.,

$$(gf)(h) = f(g^{-1}h), \quad g, h \in G, \quad f \in \ell^\infty(G, A)$$

and each  $f \in \ell^\infty(G, A)$  is acting on  $H \otimes \ell^2(G)$  by  $f(\xi \otimes \delta_g) = f(g)\xi \otimes \delta_g$ , for  $\xi \in H$  and  $g \in G$ .

Hamana showed that if  $A$  is an injective operator system, then  $\ell^\infty(G, A)$  is  $G$ -injective, and that any  $G$ -injective  $G$ -operator system is injective.

If  $A \subseteq B$  and  $B$  is a  $G$ -injective  $G$ -operator system, then an  $A$ -projection on  $B$  is a  $G$ -equivariant u.c.p map  $\varphi : B \rightarrow B$  satisfying  $\varphi|_A = id_A$ . A partial ordering on the set of  $A$ -projections on  $B$  can be defined by  $\varphi \prec \psi$ , for  $A$ -projections  $\varphi, \psi : B \rightarrow B$  if  $\varphi \circ \psi = \psi \circ \varphi = \varphi$ .

By the Zorn's lemma, there exists a minimal  $A$ -projection  $\varphi : B \rightarrow B$  on the set of seminorms induced by  $A$ -projection on  $B$ . In this argument, letting  $\kappa : A \rightarrow B$  be the inclusion map, then  $(\varphi(B), \kappa)$  is a  $G$ -rigid and  $G$ - $C^*$ -injective extension of  $A$ . Therefore,  $(\varphi(B), \kappa)$  is the  $G$ -injective envelope of  $A$ .

A canonical  $G$ -injective  $G$ -operator system is  $\ell^\infty(G, B)$ , where  $B$  is an injective  $C^*$ -algebra. Let  $A$  be a unital  $G$ - $C^*$ -algebra and  $B$  be a unital injective  $C^*$ -algebra containing  $A$ . Let  $\kappa : A \rightarrow \mathcal{M} = \ell^\infty(G, B)$  be the  $G$ -equivariant injective  $*$ -homomorphism given by

$$\kappa(x)(g) = g^{-1}x, \quad x \in A, \quad g \in G.$$

Then there is a  $\kappa(A)$ -projection  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  such that  $(\varphi(\mathcal{M}), \kappa)$  is the  $G$ -injective envelope of  $A$ . Thus, for any injective extension  $B$  of a unital  $G$ - $C^*$ -algebra  $A$ , the map  $\kappa : A \rightarrow \ell^\infty(G, B)$  is the canonical inclusion map.

Any injective operator system is unital and completely order isomorphic to a unital, monotone complete  $AW^*$ -algebra [5, 12]. In our setting, if  $A \subseteq B$  are as above and  $\varphi : B \rightarrow B$  is a minimal  $A$ -projection, then the multiplication on  $I_G(A) = \varphi(B)$  is given by the Choi-Effros product, i.e., by

$$x \circ y = \varphi(xy), \quad x, y \in I_G(A)$$

and the involution and norm on  $I_G(A)$  are inherited from  $B$  [7]. Further, if  $A$  is a unital  $G$ - $C^*$ -algebra, then  $A$  embeds into its  $G$ -injective envelope as a  $G$ -invariant unital  $C^*$ -subalgebra. In the case when  $G = \{1\}$ , the above product yields a  $C^*$ -algebra injective structure on the injective envelope  $I(A)$  of  $A$ .

A  $G$ - $C^*$ -algebra  $A$  is a  $G$ -monotone complete if underlying  $C^*$ -algebra  $A$  is a monotone complete. A  $G$ - $W^*$ -algebra is  $G$ -monotone complete if the underlying  $W^*$ -algebra is so as a  $C^*$ -algebra. A linear subspace  $A$  of a  $G$ - $C^*$ -algebra  $B$  is called  $G$ - $C^*$ -subalgebra of  $B$ , written  $A \preceq B$ , if  $A$  is a  $G$ - $C^*$ -algebra in the restricted action of  $G$ .

Given two  $G$ - $C^*$ -algebra  $A \preceq B$ ,  $A$  is said to be  $G$ -closed in  $B$  if  $y \in B$  and  $g \cdot y \in A$ , for all  $g \in L^1(G)$ , imply  $y \in A$ . For any  $G$ - $C^*$ -algebras  $A \preceq B$  the smallest  $G$ -closed  $G$ - $C^*$ -subalgebra of  $B$  containing  $A$  is called the  $G$ -closure of  $A$  in  $B$ , written  $G\text{-cl}_B A$ , i.e.,  $G\text{-cl}_B A = \{y \in B : f \cdot y \in A \text{ for all } f \in L^1(G)\}$ . A  $G$ - $C^*$ -algebra  $A$  is  $G$ -complete if for any  $G$ - $C^*$ -algebra  $B$  with  $A \preceq B$ ,  $A$  is a  $G$ -closed in  $B$ .

A  $G$ -regular completion of a  $G$ - $C^*$ -algebra  $A$  is a  $G$ - $C^*$ -algebra, written  $\overline{A}_G$ , such that;

- (1)  $\overline{A}_G$  is  $G$ -complete,
- (2)  $A \preceq \overline{A}_G$ ,
- (3) If  $A \preceq B$  and  $B$  is  $G$ - $C^*$ -complete, there are a  $G$ - $C^*$ -algebra  $B'$  with  $A \preceq B' \preceq B$  and a  $G$ -isomorphism  $\psi : \overline{A}_G \rightarrow B'$  with  $\psi|_A = id_A$ .

In fact, the  $\overline{A}_G$  is the smallest  $G$ -complete containing  $A$ . Hence,  $\overline{A}_G$  exists and is unique. Now the Hamana's construction [9] of  $\overline{A}_G$  is via the  $G$ -injective envelope of  $A$ . Namely,  $\overline{A}_G$  is the  $G$ -closure of  $A$  in  $I_G(A)$ .

For each  $G$ - $C^*$ -algebra  $A$ , there is a representation in which

$$A \preceq \overline{A}_G \preceq I_G(A),$$

where each containment is as a  $G$ - $C^*$ -subalgebra. An important feature of this sequence of containments is that  $\overline{A}_G$  is  $G$ -monotone closed in  $I_G(A)$

An ideal  $I$  of  $A$  is *essential* if  $K \cap I \neq \{0\}$ , for any non-zero ideal  $K \subseteq A$ . Equivalently, if  $aI = 0$ , for all  $a \in A$ , then  $a = 0$ . Any essential ideal is necessarily non-zero. The multiplier algebra  $M(A)$  of a  $C^*$ -algebra  $A$  is a  $C^*$ -subalgebra of the enveloping von Neumann algebra  $A^{**}$  that consists of all  $x \in A^{**}$  for which  $xa \in A$  and  $ax \in A$ , for all  $a \in A$ .

An essential ideal  $I$  of a  $G$ - $C^*$ -algebra  $A$  is  $G$ -essential ideal if  $I$  is  $G$ -invariant. For a  $G$ -invariant ideal  $I$  of  $A$ , the  $G$ -multiplier algebra  $M_G(I)$  of  $I$  is the  $G$ -regular completion of the multiplier algebra  $M(I)$ , endowed with the canonical strictly continuous action of  $G$ , that is,  $M_G(I) = \overline{M(I)}_G$ .

If  $J \subseteq A$  is a  $G$ -invariant ideal, then  $J^{**}$  is identified with the closure of  $J$  in  $A^{**}$  with respect to the strong operator topology. Thus, if  $J$  and  $K$  are  $G$ -invariant ideals of  $A$ , and if  $J \subseteq K$ , then  $M_G(J) \succeq M_G(K) \succeq M_G(A)$ .

Consider the  $G$ -multiplier algebra  $M_G(J)$  of any  $G$ -essential ideal  $J$  of  $A$ . If  $\varepsilon_G(A)$  is the set of  $G$ -essential ideals of  $A$ , partially ordered by reverse inclusion, then the set  $\xi(A)$  of  $G$ -multiplier algebras  $M_G(K)$  of  $K \in \varepsilon_G(A)$  is a directed system of  $G$ - $C^*$ -algebras. We define a  $G$ -local multiplier algebra, denoted by  $M_G^{loc}(A)$ , as follows

$$M_G^{loc}(A) = \varprojlim \{M_G(K); K \in \varepsilon_G(A)\}.$$

In fact, the  $M_G^{loc}(A)$  is defined to be the  $C^*$ -direct limit over the downward directed system  $K \in \varepsilon_G(A)$ , and  $M_G^{loc}(A)$  is realized by idealizers in  $I_G(A)$  of  $G$ -essential ideals of  $A$ . By an argument similar to [6, Corollary 4.3]

$$M_G^{loc}(A) = cl \left( \bigcup_{K \in \varepsilon_G(A)} \{x \in I_G(A); xK + Kx \subseteq K\} \right)$$

where the closure is with respect to the norm topology of  $I_G(A)$ . Thus,

$$A \preceq M_G^{loc}(A) \preceq I_G(A)$$

is an inclusion of  $G$ - $C^*$ -subalgebras.

**Lemma 2.1.** *If  $A$  is a  $G$ - $C^*$ -algebra for which  $I_G(A)$  is a  $G$ - $W^*$ -algebra, then  $\overline{A}_G$  is a  $G$ - $W^*$ -algebra.*

*Proof.* Suppose that  $I_G(A)$  is a  $G$ - $W^*$ -algebra. Then  $I_G(A)$  is represented as a von Neumann algebra acting on a Hilbert space. We assume that  $\{h_\alpha\}_\alpha$  be any bounded increasing net in  $(\overline{A}_G)_{sa}$ . Because  $I_G(A)$  is  $G$ -monotone complete,  $\{h_\alpha\}_\alpha$  has a least upper  $h$  such that  $h = \lim_\alpha h_\alpha = \sup_\alpha h_\alpha$  in the strong operator topology. Since,  $\overline{A}_G$  is  $G$ -monotone closed in  $I_G(A)$ ,  $h \in \overline{A}_G$ . Thus  $\overline{A}_G$  is a  $G$ - $C^*$ -algebra of operators in which the limit of every bounded increasing net of hermitian elements again belongs to  $\overline{A}_G$ . Therefore,  $\overline{A}_G$  is a  $G$ - $W^*$ -algebra by [10, lemma 1].  $\square$

**Proposition 2.2.** *For any  $G$ - $C^*$ -algebra  $A$  the  $G$ -closure of  $A$  in its  $G$ -injective envelope  $I_G(A)$  is the  $G$ -regular completion  $\overline{A}_G$  of  $A$ .*

*Proof.* Let  $A_1$  be the  $G$ -closure of  $A$  in  $I_G(A)$  and  $A \preceq B$ , then  $A \preceq B \preceq B_1$  for some  $G$ -injective  $B_1$ , and there are an idempotent  $G$ -morphism  $\phi : B_1 \rightarrow B_1$  and a  $G$ -isomorphism  $\psi : I_G(A) \rightarrow \phi(B_1)$  such that  $\phi|_A = id_A = \psi|_A$ . We have  $G-cl_{B_1} A \preceq \phi(B_1)$ . Indeed, if  $b \in G-cl_{B_1} A$ , then  $f \cdot b \in A$  for all  $f \in L^1(G)$  and  $f \cdot b = \phi(f \cdot b) = f \cdot \phi(b)$  in  $B_1$  for all  $f \in L^1(G)$ ; hence  $b = \phi(b) \in \phi(B_1)$ . Thus

$$G-cl_{\phi(B_1)} A = (G-cl_{B_1} A) \cap \phi(B_1) = G-cl_{B_1} A.$$

Further, since  $\psi$  is a  $G$ -isomorphism and  $\psi|_A = id_A$ , we have  $\psi(A_1) = G - cl_{\phi(B_1)}A$ , and so  $\psi(A_1) = G - cl_{B_1}A$ . First we assume that  $y \in \psi(A_1)$ , then there is a  $a_1 \in A_1$  such that  $y = \psi(a_1) \in \phi(B_1)$ . On the other hand, since  $A_1$  is a  $G$ -closure of  $A$ ,  $f \cdot a_1 \in A$  for all  $f \in L^1(G)$ , and since  $\psi|_A = id_A$ , we have

$$f \cdot y = f \cdot \psi(a_1) = \psi(f \cdot a_1) = f \cdot a_1 \in A.$$

Hence,  $y \in G-cl_{\phi(B_1)}A$ .

Now, let  $y \in G - cl_{\phi(B_1)}A$ . By definition, we have  $f \cdot y \in A$  and  $y \in \phi(B_1)$ . Suppose that  $b_1 \in B_1$ , with  $y = \phi(b_1)$ . Since  $\psi$  is a  $G$ -isomorphism, there exists  $a_1 \in I_G(A)$  such that  $y = \phi(b_1) = \psi(a_1)$ . On the other hand, since  $A_1$  is a  $G$ -closure of  $A$  in  $I_G(A)$ ,

$$\psi(f \cdot a_1) = f \cdot \psi(a_1) = f \cdot y \in A \Rightarrow f \cdot a_1 \in A \Rightarrow a_1 \in A_1 \Rightarrow y = \psi(a_1) \in \psi(A_1).$$

If  $A_1 = A$ , namely,  $A$  is  $G$ -closed in  $I_G(A)$ . Then so is  $A$  in  $\phi(B_1)$ , and  $A = G-cl_{B_1}A$ . Hence,  $A = G-cl_B A$ , that is,  $A$  is  $G$ -closed in  $B$ . Since  $A \preceq B \preceq B_1$ ,  $G-cl_B A \preceq G-cl_{B_1}A$ . As  $B$  is arbitrary, this means that  $A$  is  $G$ -complete.

Next, suppose that  $A$  is arbitrary, but  $B$  is  $G$ -complete. Since  $I_G(A_1) = I_G(A)$  and  $A_1$  is  $G$ -closed in  $I_G(A)$ , it follows from the foregoing argument that  $A_1$  is  $G$ -complete. As  $B$  is  $G$ -complete,  $G-cl_{B_1}A \preceq G-cl_{B_1}B = B$ , and  $\psi(A_1) = G-cl_{B_1}A \preceq B$  with  $\psi(A_1) \cong A_1$ . Therefore,  $A_1$  is the  $G$ -regular completion of  $A$ .

Finally, let only that  $A \preceq B$ . By the above argument to  $A \preceq B \preceq \overline{B}_G$ , there is a  $G$ -isomorphism  $\psi$  of  $A_1$  onto  $G-cl_{\overline{B}_G}A$  with  $\psi|_A = id_A$ . Hence, since  $A \preceq G-cl_B A \preceq G-cl_{\overline{B}_G}A$ ,  $G-cl_B A$  is isomorphic to the  $G$ - $C^*$ -subalgebra  $\psi^{-1}(G-cl_B A)$  of  $A_1$ . □

### 3. SEPARABLE $C^*$ -ALGEBRA OF A DISCRETE GROUP

The main result of this paper is Theorem (3.4) on separable discrete  $C^*$ -dynamics. Before turning to the proof of Theorem (3.4), we prove some preliminary results. We need the notion of *covariant representation* and the relation between  $G$ -local multiplier algebra and  $G$ -regular completion of  $G$ - $C^*$ -algebras.

**Definition 3.1.** A  $C^*$ -algebra  $A$  is called *elementary* if  $A \cong K(H)$  for some Hilbert space  $H$ .

The separable elementary  $C^*$ -algebras are the finite-dimensional matrix algebras and the  $C^*$ -algebras of compact operators of separable infinite-dimensional Hilbert space. Every elementary  $C^*$ -algebra is simple and the converse is true when the  $C^*$ -algebra is of type I. If  $A$  is a  $C^*$ -subalgebra of  $K(H)$  acting irreducibly on Hilbert space  $H$ , then  $A$  is elementary.

**Definition 3.2.** A *covariant representation* of a  $G$ - $C^*$ -algebra  $A$  is a pair  $(\pi, \sigma)$  where  $(\pi, H)$  is a representation of  $A$ ,  $(\sigma, H)$  is a unitary representations of  $G$ ,

such that

$$\sigma(g)\pi(a)\sigma(g)^{-1} = \pi(\theta_g(a)) = \pi(g \cdot a)$$

for every  $a \in A$ ,  $g \in G$ .

A covariant representation  $(\pi, \sigma)$  of a  $G$ - $C^*$ -algebra  $A$  on a Hilbert space  $H$  is *normal* if  $(\pi, H)$  is normal.

**Proposition 3.3.**  $\overline{M_G^{loc}(A)} = \overline{A_G}$  for every  $G$ - $C^*$ -algebra  $A$ .

*Proof.* Since  $M_G^{loc}(A)$  is  $G$ -equivariant  $*$ -isomorphically embedded into  $I_G(A)$ , extending the canonical  $G$ -equivariant  $*$ -monomorphism of  $A$  into  $I_G(A)$ , the  $G$ - $C^*$ -algebra  $I_G(A)$  serves as an injective  $G$ -extension of the  $G$ - $C^*$ -algebra  $M_G^{loc}(A)$ . Therefore, the identity map on  $M_G^{loc}(A)$  admits a unique  $G$ -extension to a  $G$ -equivariant completely positive map of  $I_G(A)$  into itself with the same completely bounded norm one. Since  $A_G \preceq M_G^{loc}(A) \preceq I_G(A)$  by construction and  $I_G(A)$  is the  $G$ -injective envelope of  $A$ ,  $I_G(A)$  has to be the  $G$ -injective envelope of  $M_G^{loc}(A)$ . Since the  $G$ -regular completion of a  $G$ - $C^*$ -algebra  $B$  is the  $G$ -monotone closure of  $B$  in the  $G$ -injective envelope  $I_G(A)$ ,

$$A_G \preceq M_G^{loc}(A) \preceq \overline{A_G} \preceq I_G(A) = I(M_G^{loc}(A))$$

implies that  $\overline{A_G} \preceq \overline{M_G^{loc}(A)} \preceq \overline{A_G}$ . Thus,  $\overline{M_G^{loc}(A)} = \overline{A_G}$ .  $\square$

**Theorem 3.4.** *The following statements are equivalent for a separable  $G$ - $C^*$ -algebra  $A$ :*

- (i)  $\overline{A_G}$  is a  $G$ - $W^*$ -algebra.
- (ii)  $I_G(A)$  is a  $G$ - $W^*$ -algebra.
- (iii)  $A$  contains a  $G$ -invariant minimal essential ideal that is  $G$ -isomorphic to a direct sum of elementary  $G$ - $C^*$ -algebras.

*Proof.* By Lemma (2.1), the proof of (ii) $\Rightarrow$ (i) is clear.

(ii) $\Rightarrow$ (iii): We have divided the proof into two stages. In the first stage, let us first show that there exists a faithful representation  $\pi : A_G \rightarrow B(H)$  such that the von Neumann algebra  $\pi(A_G)''$  is generated by its minimal projections, each of which is contained in  $\pi(A_G)$ . For this, let  $I_G(A)$  be a  $G$ - $W^*$ -algebra. By [11, lemma 7.4.9], there is a faithful  $G$ -equivariant representation  $\tilde{\pi} : I_G(A) \rightarrow B(H)$  such that  $\pi(A_G)$  is a  $G$ - $C^*$ -subalgebra of  $\tilde{\pi}(I_G(A))$ , with  $\pi = \tilde{\pi}|_{A_G}$ . Without loss of generality, suppose that  $I_G(A)$  is a von Neumann algebra acting on a Hilbert space. Since the  $G$ -regular completion  $\overline{A_G}$  of  $A_G$  is  $G$ -monotone closed in  $I_G(A)$  and because  $I_G(A)$  is a von Neumann algebra,  $\overline{A_G}$  is a von Neumann algebra by Lemma (2.1). Thus,  $A_G'' \subseteq \overline{A_G}'' = \overline{A_G}$ ,  $A_G''$  being the double commutant of  $A_G$ .

Now, let  $\omega$  be a normal state on von Neumann algebra  $A_G''$  that is faithful on  $A_G$ . Assume that  $\omega(h) = 0$ , where  $h \in A_G''^+$ . Because  $h = \sup\{k \in A^+; k \leq h\}$ , we have  $0 \leq \omega(k) \leq \omega(h) = 0$ , for each  $k \in A_G^+$  with  $k \leq h$ . Thus  $\omega(k) = 0$ , which implies that  $k = 0$  because  $\omega$  is faithful on  $A$ . Hence,  $h = 0$  and so

$\omega$  is faithful on  $A_G''$ . Namely, any normal state  $\omega \in A_G''$  is faithful precisely when its restriction  $\omega|_{A_G}$  to  $A_G$  is faithful. By [13, P. 139], because  $A_G$  is separable and order dense in  $A_G''$ ,  $A_G''$  is generated by its minimal projections, each of which is contained in  $A_G$ . Furthermore, since  $A_G''$  is a direct product of type I factors by [3, lemma 2.2],  $A_G''$  is injective by [3, corollary 2.3]. Because  $A_G \subseteq A_G'' \subseteq I_G(A)$ , we conclude that  $A_G'' = \overline{A_G} = I_G(A)$ , by minimality of the injective envelope.

The second stage, without loss of generality, assumes that  $A_G$  is already represented as a subalgebra of  $B(H)$  and that  $M = A_G''$  is generated by its minimal projections, each of which lie in  $A_G$ . Let  $K \subseteq A_G$  be the ideal of  $A_G$  generated by the minimal projections of  $M$ . We claim that  $K$  is an essential ideal, minimal among all essential ideals of  $A_G$ . Suppose that  $J \subseteq A_G$  is a nonzero ideal. Choose any nonzero  $h \in J^+$ . There is a strictly positive  $\lambda$  in the spectrum  $\sigma(h)$  of  $h$ . Let  $e \in M$  be the spectral projection  $e = e^h([\lambda, +\infty))$ , where  $e^h$  denotes the spectral resolution of  $h$ . Thus,  $0 \neq \lambda e \leq he$ , and there is a minimal projection  $p$  of  $M$  such that  $ep = pe = p$  and  $0 \neq \lambda p = \lambda p^2 = p\lambda p \leq php \in J \cap K$ . Then  $J \cap K \neq \{0\}$ .

By [3, lemma 2.2], since  $M = A_G''$  is generated by its minimal projections,  $M$  is a discrete type I von Neumann algebra. Therefore, there is a faithful normal covariant  $*$ -representation  $\gamma$  of  $M$  on a Hilbert space  $H$  of the form  $H = \bigoplus_n H_n$  by [11, lemma 7.4.9], such that

$$\gamma(K) \subseteq \gamma(A_G) \subseteq \gamma(M) = \prod_n B(H_n)$$

It fact, the minimal projections of any  $B(H_n)$  are minimal projection of  $\gamma(M)$ . Hence, elements of  $\gamma(K)$ . Moreover, if  $e$  is a minimal projection of  $\prod_n B(H_n)$ ,  $e \in B(H_n)$ , for some  $n \in N$ . Therefore,  $\bigoplus_n K(H_n) \subseteq \gamma(K)$ . Since  $\gamma(K)$  is the smallest  $G$ - $C^*$ -algebra that contains the minimal projections of  $\gamma(M)$ , it follows that  $\gamma(K) = \bigoplus_n K(H_n)$ . Since,  $K \cong \bigoplus_n K(H_n)$ ,  $K$  is  $G$ -invariant minimal essential ideal of  $A_G$ .

(iii) $\Rightarrow$ (ii): Suppose that  $A_G$  has a  $G$ -invariant minimal essential ideal  $K$  such that  $K \cong \bigoplus_n K(H_n)$ . Thus, by [1, Lemma 1.2.21],

$$M(K) = M\left(\bigoplus_n K(H_n)\right) = \prod_n M(K(H_n)) = \prod_n B(H_n),$$

and this shows that  $M(K)$  is a type I  $W^*$ -algebra. Since  $K$  is a  $G$ -invariant minimal essential ideal of  $A_G$ , by [1, Remark 2.3.7]  $M(K) = M_G^{loc}(A)$ . Hence,  $M_G^{loc}(A)$  is an injective  $G$ - $W^*$ -algebra. We know that  $A_G \subseteq M_G^{loc}(A) \subseteq I_G(A)$  as  $G$ - $C^*$ -subalgebras, it must be that  $M_G^{loc}(A) = I_G(A)$  by definition of injective envelope, and this is precisely the proof of the  $G$ - $W^*$ -algebra of  $I_G(A)$ .

(i) $\Rightarrow$ (ii): For the  $G$ - $W^*$ -algebra  $\overline{A_G}$ ,  $\overline{A_G} = A_G''$  by the proof of (ii) $\Rightarrow$ (iii). Since  $A_G''$  is a direct product of type I factors, so  $A_G''$  is injective. Therefore,

$\overline{A}_G$  is injective. Hence,  $\overline{A}_G = I_G(A)$ , which yields that  $I_G(A)$  is a  $G$ - $W^*$ -algebra.  $\square$

EXAMPLE 3.5. by [8, lemma 2.2],  $A = \ell^\infty(G, B(H))$  is  $G$ -injective, where  $G$  acts trivially on  $B(H)$ . Thus  $I_G(A) = A$  which is a  $G$ - $W^*$ -algebra. Now the minimal essential ideal of  $A$  is  $c_0(G) \otimes K(H)$  which is essential ideal and dense and is direct sum of  $|G|$ -copies of elementary  $C^*$ -algebras  $\mathbb{C} \otimes K(H)$  [This is an infinite direct sum if the cardinal  $|G|$  is not finite]. Also  $A$  is already  $G$ -complete, so the  $G$ -closure of  $A$  is  $A$  itself, which is a  $G$ - $W^*$ -algebra.

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#### REFERENCES

1. P. Ara, M. Mathieu, *Local Multipliers of  $C^*$ -algebras*, Springer Monographs in Mathematics, London, 2003.
2. M. Argerami, D. R. Farenick, Local Multipliers Algebras, Injective Envelopes, and Type I  $W^*$ -algebras, *J. Operator Theory*, **59**, (2008), 237-245.
3. M. Argerami, D. R. Farenick, Injective Envelopes of Separable  $C^*$ -algebras, *Philos. Trans. R. Soc. A*, **179**, (2005), 43-63.
4. M.-D. Choi, E. G. Effros, Separable Nuclear  $C^*$ -algebras and Injectivity, *Duke Math. J.*, **43**, (1976), 309-322.
5. M. D. choi, E. G. Effros, Injectivity and Operator Spaces, *Journal of Functional Analysis*, **24**, (1977), 156-209.
6. M. Frank, V. I. Paulsen, Injective Envelopes of  $C^*$ -algebras as Operator Modules, *Pacific J. Math*, **212**, (2003), 57-69.
7. M. Hamana, Injective Envelopes of  $C^*$ -algebras, *J. Math. Soc. Japen*, **31**, (1979), 181-197.
8. M. Hamana, Injective Envelope of  $C^*$ -dynamical Systems, *Tōhoku Math. Journ*, **37**, (1985), 463-487.
9. M. Hamana, Injective Envelope of Dynamical Systems, *Toyama Math. J.*, **34**, (2011), 23-86.
10. R. V. Kadison, Operator Algebras with a Faithful Weakly-closed Representation, *Ann. of Math*, **64**, (1956), 175-181.
11. G. K. Pedersen,  *$C^*$ -algebras and Their Automorphism Groups*, London Mathematical Society, Monographs 14, 1979.
12. K. Saitō, J. D. M. Wright, *Monotone Complete  $C^*$ -algebras and Generic Dynamics*, Springer-Verlag, 2015.
13. M. Takesaki, *Theory in Operator Algebras I*, Encylopaedia of Mathematical Sciences, Springer-Verlag, New York, **124**, 2001.