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K-G-Frames and G-Atomic Systems in Hilbert Pro-C*-Modules

Akbar Nazari, Zahra Ahmadi Moosavi*

Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran

E-mail: nazari@uk.ac.ir
E-mail: zmoosavi@math.uk.ac.ir

Abstract. In this paper, the concept of K-g-frames in Hilbert pro-C^*-modules is introduced, and their basic properties are examined. The conditions under which family of operators can form a K-g-frame on Hilbert pro-C^*-modules are derived. Moreover g-atomic systems in Hilbert pro-C^*-modules are introduced and their properties are examined.

Keywords: Hilbert modules over pro-C^*-algebras, K-G-frames, Frame operators, Bounded module maps, G-atomic systems.

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1. Introduction

The concept of frames is an extension of the concept of orthonormal bases in a Hilbert space H. Actually, a frame is a sequence of (countable) elements in H such that every element in H has a representation as a linear combination of the frame elements and its elements are not necessarily independent. Furthermore, the concept of generalized frames or g-frames, which includes more other cases of generalizations of frames, was introduced by Sun [26]. More details about (discrete) frames and g-frames can be found in [7, 26] .

*Corresponding Author

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Redundancy is applied in areas such as: filter bank theory [5] by Bolcskei, Hlawatsch and Feichtinger, sigma-delta quantization [4] by Benedetto, Powell and Yilmaz, signal and image processing [6] by Candes and Donoho and wireless communications [12] by Heath and Paulraj. Redundancy for finite g-frames and infinite g-frames and its relationship with redundancy of other kinds of frames is introduced by Rahmani [22]. Redundancy of g-frames(g-frames) is also used in signal and image processing.

It is not usually possible to obtain a solution in closed form for integral and partial differential equations with fractional order, and hence numerical methods are employed to obtain approximate solutions. One of the most widely used methods for solving such equations is the use of wavelets. Wavelets are a family of functions, which are the transitions and extensions of the mother wavelet. One way to produce smooth wavelets is to use mutiresolution analysis, and frames theory.

Atomic systems for subspaces were first introduced by Feichtinger and Werther. In 2011, Gavruta [9] introduced the generalization of frames in Hilbert spaces to study the decomposition of atomic systems and discussed their properties.

K-frames have been discussed in [10] and [27]. In [29], the authors put forward the concept of K-g-frames, which are more general than ordinary g-frames. In this paper, we discuss K-g-frames, limited to the range of a bounded linear operator in Hilbert pro-C^*-modules. We also introduce g-atomic-systems in Hilbert pro-C^*-modules and examine their properties.

2. Hilbert pro-C^*-modules

A pro-C^*-algebra is a complete Hausdorff complex topological $*$-algebra A, whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_\lambda\}$ converges to zero if and only if $\rho(a_\lambda) \to 0$ for any continuous C^*-seminorm ρ on A and we have:

(1) $\rho(ab) \leq \rho(a)\rho(b)$;
(2) $\rho(a^*a) = \rho(a)^2$;

for all continuous C^*-seminorms ρ on A and for all $a, b \in A$.

If the topology of pro-C^*-algebra is determined by only countably many C^*-seminorms, then it is called a σ-C^*-algebra.

Let A be a unital pro-C^*-algebra with unit 1_A and let $a \in A$. Then, the spectrum $sp(a)$ of $a \in A$ is the set $\{\lambda \in \mathbb{C} : \lambda 1_A - a \text{ is not invertible}\}$. If A is not unital, then the spectrum is taken with respect to its unitization $\tilde{A}$.

If A^+ denotes the set of all positive elements of A, then A^+ is a closed convex cone such that $A^+ \cap (-A^+) = 0$. We denote, the set of all continuous C^*-seminorms on A by $S(A)$. For $\rho \in S(A)$, $\mathrm{k}er(\rho) = \{a \in A : \rho(a) = 0\}$ is a closed ideal in A. For each $\rho \in S(A)$, $A_\rho = A/\ker(\rho)$ is a C^*-algebra in the

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norm induced by ρ , defined as:

$$\|a+\ker(\rho)\|_{A_\rho} = \rho(a), \qquad \rho \in S(A).$$

We have $A = \varprojlim_{\rho} A_\rho$ (see [25]).

The canonical map from A to A_ρ for $\rho \in S(A)$, will be denoted by π_ρ and the image of $a \in A$ under π_ρ will be denoted by a_ρ. Hence $\ell^2(A_\rho)$ is a Hilbert A_ρ-module(see [14]), with the norm:

$$\|(\pi_\rho(a_i))_{i\in N}\| = \left[\rho(\sum_{i\in N} a_i a_i^*)\right]^{\frac{1}{2}}, \qquad \rho \in S(A), (\pi_\rho(a_i)_{i\in N} \in \ell^2(A_\rho)).$$

Example 2.1. Every C^*-algebra is a pro-C^*-algebra.

Example 2.2. [25] A product of C^*-algebras with the product topology is a pro-C^*-algebra.

Example 2.3. A closed $*$-algebra of a pro-C^*-algebra is a pro-C^*-algebra.

Notation 2.4. . *We write $a \geq 0$ if $a \in A^+$ and $a \leq b$ if $a - b \geq 0$.*

Proposition 2.5. .[11] *Let A be a unital pro-C^*-algebra with the identity 1_A.Then for any $\rho \in S(A)$, we have:*

1. $\rho(a) = \rho(a^*)$; *for all $a \in A$;*
2. $\rho(1_A) = 1$;
3. *If $a, b \in A^+$ and $a \leq b$, then $\rho(a) \leq \rho(b)$;*
4. *$a \leq b$ iff $a_\rho \leq b_\rho$;*
5. *If $1_A \leq b$, then b is invertible and $b^{-1} \leq 1_A$;*
6. *If $a, b \in A^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$;*
7. *If $a, b \in A$ and $a \leq b$ then $c^*ac \leq c^*bc$;*
8. *If $a, b \in A^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$.*

Definition 2.6. . A pre-Hilbert module over pro-C^*-algebra A, is a complex vector space E, which is also a left A-module compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ which is $\mathbb{C}$-linear and A-linear in its first variable and satisfies the following conditions:

1. $\langle x, y\rangle^* = \langle y, x\rangle$;
2. $\langle x, x\rangle \geq 0$;
3. $\langle x, x\rangle = 0$ iff $x = 0$;

for every $x, y \in E$. We say that E is a Hilbert A-module (or Hilbert pro-C^*-module over A) if E is complete with respect to the topology determined by the family of seminorms:

$$\overline{\rho}_E(x) = \sqrt{\rho(\langle x, x\rangle)}, \qquad x \in E, \rho \in S(A).$$

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Let E be a pre-Hilbert A-module. By [28, Lemma 2.1], for $\rho \in S(A)$ and for all $x, y \in E$, the following (Cauchy-Bunyakovskii) inequality holds:

$$(\rho\langle x, y\rangle)^2 \leq \rho(\langle x, x\rangle)\rho(\langle y, y\rangle).$$

Consequently, for each $\rho \in S(A)$, we have:

$$\overline{\rho}_E(ax) \leq \rho(a)\overline{\rho}(x), \qquad a \in A, x \in E.$$

Example 2.7. If A is a pro-C^*-algebra, then it is a Hilbert A-module with respect to the inner product defined by :

$$\langle a, b\rangle = ab^*, \qquad a, b \in A.$$

Example 2.8. Let E_i for $i \in N$, be a Hilbert A-module with the topology induced by the family of continuous $\{\rho_i\}_{\rho \in S(A)}$, defined as:

$$\overline{\rho}_i(x) = \sqrt{\rho(\langle x, x\rangle)}, \qquad x \in E_i.$$

The direct sum of $\{E_i\}_{i \in N}$ is defined as follows:

$$\bigoplus_{i \in N} E_i = \{(x_i) : x_i \in E_i \ , \ \sum_{i \in N}\langle x_i, x_i\rangle \text{ is convergent in } A\}.$$

It has been shown that the direct sum $\bigoplus_{i \in N} E_i$ is a Hilbert A-module with an A-valued inner product $\langle x, y\rangle = \sum_{i \in N}\langle x_i, y_i\rangle$, where $x = (x_i)_{i \in N}$ and $y = (y_i)_{i \in N}$ are in $\bigoplus_{i \in N} E_i$, pointwise operations and a topology determined by the family of seminorms (see [18, Example 3.2.3]).

$$\overline{\rho}(x) = \sqrt{\rho(\langle x, x\rangle)}, \qquad x \in \bigoplus_{i \in N} E_i \ , \ \rho \in S(A).$$

The direct sum of countable copies of a Hilbert module E is denoted by $\ell^2(E)$.

We recall that an element $a \in A$ (x *in* E) is bounded, if:

$$\|a\|_\infty = \sup\{\rho(a) \ ; \ \rho \in S(A)\} < \infty,$$
$$(\|x\|_\infty = \sup\{\overline{\rho}_E(x) \ ; \ \rho \in S(A)\} < \infty).$$

The set of all bounded elements in A (in E) will be denoted by $b(A)(b(E))$. We know that $b(A)$ is a C^*-algebra in the C^*-norm $\|.\|_\infty$ and $b(E)$ have the Hilbert $b(A)$-module [25, Proposition 1.11], and [28, Theorem 2.1].

Let $M \subset E$ be a closed sub-module of a Hilbert A-module E and let

$$M^\perp = \{y \in E : \ \langle x, y\rangle = 0 \ \text{ for all } x \in M\}.$$

Note that the inner product in Hilbert modules is separately continuous; hence, $M^\perp$ is a closed sub-module of the Hilbert A-module E. Also, a closed sub-module M in a Hilbert A-module E is considered orthogonally complementable if $E = M \bigoplus M^\perp$. A closed sub-module M in a Hilbert A-module E is called topologically complementable if there exists a closed sub-module in E such that $M \bigoplus N = E$, $N \bigcap M = \{0\}$.

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Let A be a pro-C^*-algebra, and E and F be two Hilbert A-modules. An A-module map $T : E \to F$ is said to bounded if for each $\rho \in S(A)$, there is $C_\rho > 0$ such that:

$$\overline{\rho}_F(Tx) \leq C\rho\ \overline{\rho}_E(x) \qquad (x \in E),$$

where $\overline{\rho}_E$ and $\overline{\rho}_F$ are continuous seminorms on E and F, respectively. A bounded A-module map from E to F is called an operator from E to F. We denote the set of all operators from E to F by $Hom_A(E,F)$, and set $Hom_A(E,F) = End_A(E,F)$.

Let $T \in Hom_A(E,F)$. We say T is adjointable if there exists an operator $T^* \in Hom_A(F,E)$ such that:

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,$$

holds for all $x \in E$, $y \in F$.

We denote by $Hom_A^*(E,F)$, the set of all adjointable operators from E to F and $End_A^*(E) = Hom_A^*(E,E)$

Proposition 2.9. [11] *Let $T : E \to F$ and $T^* : F \to E$ be two maps such that the equality*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,$$

holds for all $x \in E$, $y \in F$. Then $T \in Hom_A^*(E,E)$.

It is easy to see that for any $\rho \in S(A)$, the map defined by:

$$\hat{\rho}_{E,F}(T) = \sup\{\overline{\rho}_F(T(x) :\ x \in E,\ \overline{\rho}_E(x) \leq 1\}, \qquad T \in Hom_A(E,F),$$

is a seminorm on $Hom_A(E,F)$.

Definition 2.10. . Let E and F be two Hilbert modules over the pro-C^*-algebra A. Then the operator $T : E \to F$ is called uniformly bounded (below), if there exists $C > 0$ such that:

$$\overline{\rho}_F(Tx) \leq C\ \overline{\rho}_E(x), \tag{2.1}$$

$$(C\ \overline{\rho}_E(x) \leq \overline{\rho}_F(Tx)). \tag{2.2}$$

The number C in (2.1) is called an upper bound for T and we set:

$$\|T\|_\infty = \inf\{C : C \text{ is an upper bound for } T\}.$$

Clearly, in this case we have:

$$\hat{\rho}(T) \leq \|T\|_\infty, \qquad \forall \rho \in S(A).$$

Let T be an invertible element in $End_A^*(E)$ such that both are uniformly bounded. Then by [2, Proposition 3.2] for each $x \in E$ we have the following inequality:

$$\|T^{-1}\|_\infty^{-2}\langle x, x\rangle \leq \langle Tx, Tx\rangle \leq \|T\|_\infty^2\langle x, x\rangle. \tag{2.3}$$

The following proposition will be used in next section.

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Proposition 2.11. .[11] *Let T be a uniformly bounded below operator in $Hom_A(E,F)$. Then T is closed and injective.*

3. K-g-Frames in Hilbert pro-C^*-modules

Throughout this section, A is a pro-C^*-algebra, while U and V are two Hilbert A-modules. Also, $\{V_j\}_{j\in J}$ is a countable sequence of closed submodules of V, where J is a subset of Z.

Definition 3.1. . Let $K \in End_A^*(U)$ and $\Lambda = \{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$. A sequence $\{\Lambda_j\}_{j\in J}$ is called a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ if there exist two positive constants C and D such that for every $f \in U$,

$$C\langle K^*f, K^*f\rangle \le \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \le D\langle f, f\rangle.$$

The constants C and D are called g-frame bounds for Λ. The K-g-frame is called tight if $C = D$ and called a Parseval if $C = D = 1$. If only upper bound holds, then Λ is called a K-g-Bessel sequence.

Remark 3.2. . Every K-g-frame is a g-Bessel sequence for U with respect to $\{V_j\}_{j\in J}$.

Remark 3.3. . When K=I, a K-g-frame is a g-frame.

By Remark 3.2, if $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$, then $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence for U with respect to $\{V_j\}_{j\in J}$. So we can define the bounded linear operator $T : U \to \bigoplus_{j\in J} V_j$ as follows:

$$Tf = \{\Lambda_j f \ : j \in J\}, \text{ for all } f \in U.$$

The adjoint operator $T^* : \bigoplus_{j\in J} V_j \to U$ is given by:

$$T^*f = T^*(f_j) = \textstyle\sum_{j\in J} \Lambda_j^* f_j \text{ for all } f = \{f_j\}_{j\in J} \in \bigoplus_{j\in J} V_j.$$

Now we define the bounded linear operator $S : U \to U$ by:

$$Sf = T^*Tf = T^*\{\Lambda_j f\}_{j\in J} = \textstyle\sum_{j\in J} \Lambda_j^*\Lambda_j f, \text{ for all } f \in U.$$

We call T, T^* and S the analysis operator, the pre-frame operator and the frame operator, respectively. These operators play an important role in studying K-g-frame theory.

Example 3.4. Suppose that $\ell^2(A)$ is the set of all sequences $(a_n)_{n\in\mathbb{N}}$ of elements of a pro-C^*-algebra A such that the series $\sum_{i\in\mathbb{N}} a_i a_i^*$ is convergent in A. Then by [2, Example 3.2], $\ell^2(A)$ is a Hilbert module over A with respect to the pointwise operations and inner product defined by:

$$\langle (a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}\rangle = \sum_{i\in\mathbb{N}} a_i b_i^*.$$

Let $a = (a_i)_{i\in\mathbb{N}}$ and $b = (b_i)_{i\in\mathbb{N}}$ in $\ell^2(A)$. We define $ab = \{a_i b_i\}_{i\in\mathbb{N}}$ and $\overline{\rho}(a) = \sqrt{\rho(\langle a, a\rangle)}$ and $a^* := \{\overline{a_i}\}_{i\in\mathbb{N}}$ and $\langle a, b\rangle = ab^* = \sum_{i\in\mathbb{N}} a_i b_i^*$.

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Now, let $K \in End_A^*(\ell^2(A))$ and $j \in J := \mathbb{N}$. Also let $\{e_j\}_{j\in J}$ be the standard orthogonal basis of $\ell^2(A)$. For each $j \in J$, and set $\Lambda_j : \ell^2(A) \to \ell^2(A)$ such that $\Lambda_j a = (\delta_{i,j} a_j)_{i\in\mathbb{N}}$. Then we have:

$$\sum_{j\in J}\langle \Lambda_j a, \Lambda_j a\rangle = \sum_{j\in J}\langle (a_j),(a_j)\rangle = \langle a, a\rangle.$$

For fixed $N \in \mathbb{N}$, define $K : \ell^2(A) \to \ell^2(A)$ by:

$$Ke_j = \begin{cases} je_j & \text{if } j \le N \\ 0 & \text{if } j > N. \end{cases}$$

It is easy to check that K is adjointable and satisfies $K^* = K$. For any $a \in \ell^2(A)$, let $a = \sum_{j=1}^{\infty} c_j e_j$, then:

$$\langle K^*a, K^*a\rangle = \langle \sum_{j=1}^{N} jc_je_j, \sum_{j\in=1}^{N} jc_je_j\rangle = \sum_{j=1}^{N} j^2\langle c_j, c_j\rangle.$$

Therefore:

$$\frac{1}{N^2}\langle K^*a, K^*a\rangle = \sum_{j=1}^{N}(\frac{j}{N})^2\langle c_j, c_j\rangle \le \sum_{j=1}^{\infty}\langle c_j, c_j\rangle = \langle a, a\rangle = \sum_{j\in J}\langle \Lambda_j a, \Lambda_j a\rangle.$$

This shows that $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\ell^2(A)$ with bounds $(\frac{1}{N^2}, 1)$.

Remark 3.5. Let $k \in End_A^*(U)$ and $\Lambda = \{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ be a K-g-frame with the frame bounds C and D, respectively. Then:

$$C\langle K^*f, K^*f\rangle \le \langle Sf, f\rangle \le D\langle f, f\rangle.$$

For any $f \in U$.

Theorem 3.6. *Let $K \in End_A^*(U)$ and $\{\Lambda_j \in End_A^*(U, V_j)\}$ be a K-g-frame for U with the frame bounds C and D, respectively. Then the K-g-frame operator S is invertible, positive, self-adjoint and bounded.*

Proof. Suppose that $Sf = 0$ for any $f \in U$. By Remark 3.5 we observe that $\langle Sf, f\rangle = 0$, which implies S is invertible. Since $S = T^*T$ and $\langle Sf, f\rangle = \langle Tf, Tf\rangle, S$ is positive and a self-adjoint operator.

Now we show that T^* is bounded. Since $\{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ with bounds C and D, then for any finite subset $J_1 \subseteq J$, we have:

$$\begin{aligned}
\overline{\rho}_U(\sum_{j\in J_1}\Lambda_j^* g_j) &= \sup\{\rho\langle \sum_{j\in J_1}\Lambda_j^* g_j, f\rangle \ : \overline{\rho}_U(f) \le 1\} \\
&= \sup\{\rho\langle \sum_{j\in J_1}\langle \Lambda_j^* g_j, f\rangle \ : \ \overline{\rho}_U(f) \le 1\} \\
&= \sup\{\rho\langle \sum_{j\in J_1}\langle g_j, \Lambda_j f\rangle \ : \ \overline{\rho}_U(f) \le 1\} \\
&\le \sqrt{\rho \sum_{j\in J_1}\langle g_j, g_j\rangle} \ \sup\{\sqrt{\rho \sum_{j\in J_1}\langle \Lambda_j f, \Lambda_j f\rangle} : \overline{\rho}_U(f) \le 1\} \\
&\le \overline{\rho}(g)\ \sqrt{D}.
\end{aligned}$$

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Now, since the series $\sum_{j\in J_1}\langle g_j, g_j\rangle$ is convergent in A, the above inequality shows that $\sum_{j\in J_1}\Lambda_j^* g_j$ is also convergent. Hence, T^* is well-defined. On the other hand, since for any $g=\{g_j\}_{j\in J}\in\bigoplus_{j\in J}V_j$ and $f\in U$,

$$\begin{aligned}\langle Tf,(g_j)_j\rangle &= \langle(\Lambda_j f)_j,(g_j)_j\rangle\\ &= \sum_{j\in J}\langle\Lambda_j f, g_j\rangle\\ &= \sum_{j\in J}\langle f,\Lambda_j^* g_j\rangle\\ &= \langle f,\sum_{j\in J}\Lambda_j^* g_j\rangle\\ &= \langle f, T^*(g_j)_{j\in J}\rangle,\end{aligned}$$

therefor by Proposition 2.9 it follows that the analysis operator is adjoint of the transform operator . Also, for any $\rho\in S(A)$,we have:

$$\overline{\rho}(T^*(g))\leq\sqrt{D}\,\overline{\rho}(g).$$

Hence T^*, is uniformly bounded. Since $S=T^*T$ is a combination of two bounded operators, it is also bounded. □

Theorem 3.7. . *Let $K\in End_A^*(U)$ be surjective. Then the following statements are equivalent:*

(1) *The sequence $\Lambda=\{\Lambda_j\in End_A^*(U,V_j)\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$,*
(2) *for all $f\in U$ there exist $0<C<D<\infty$ such that:*

$$C\ \rho(\langle K^*f,K^*f\rangle)\leq\rho(\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle)\leq D\ \rho(\langle f,f\rangle), \tag{3.1}$$

(3) *The sequence $\Lambda=\{\Lambda_j\in End_A^*(U,V_j)\}_{j\in J}$ is a g-frame for U with respect to $\{V_j\}_{j\in J}$.*

Proof. 1⇒2). This is obvious.

2⇒3). Since $K\in End_A^*(U)$ is surjective, by [1, Proposition 2.2], $K^*\in End_A^*(U)$ is invertible. Then by [2, Proposition 2.3] we have:

$$\|(K^*)^{-1}\|_\infty^{-2}\rho\langle f,f\rangle\leq\rho\langle K^*f,K^*f\rangle.$$

$$C\|(K^*)^{-1}\|_\infty^{-2}\rho\langle f,f\rangle\leq C\ \rho\langle K^*f,K^*f\rangle.$$

Hence by (3.1):

$$C\|(K^*)^{-1}\|_\infty^{-2}\rho\langle f,f\rangle\leq\rho\sum_{j\in J}\langle\Lambda_j f,\Lambda_j f\rangle\leq D\|K^*\|_\infty^{2}\rho\langle f,f\rangle.$$

So $\Lambda=\{\Lambda_j\in End_A^*(U,V_j)\}_{j\in J}$ is a g-frame.

3 $\Rightarrow$1). $\Lambda = \{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ is a g-frame for U with respect to $\{V_j\}_{j\in J}$. Then there exist $C, D > 0$ such that for any $f \in U$,

$$C\langle f, f\rangle \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq D\langle f, f\rangle. \tag{3.2}$$

On the other hand, since $K \in End^*(U)$ is surjective, $K^* \in End^*(U)$ is invertible. Then we have: So:

$$\|K^*\|_\infty^{-2} C\langle K^* f, K^* f\rangle \leq C\langle f, f\rangle, \tag{3.3}$$

by (3.2) and (3.3) the proof is complete. □

Proposition 3.8. . *Let $K, L \in End_A^*(U)$ and $\Lambda = \{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ be a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ with frame bounds C and D. Then:*

(1) *If $T : U \to U$ is an isometry such that $K^*T = TK^*$. Then $\{\Lambda_j T^*\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ with the same bounds.*
(2) *$\{\Lambda_j T^*\}_{j\in J}$ is a L-K-g-frame with frame bounds $C, D\|L^*\|_\infty^2$.*

Proof. . Since $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ so by Proposition (2.5) we have:

$$C\ \rho\langle K^* f, K^* f\rangle \leq \rho\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq D\ \rho\langle f, f\rangle, \tag{3.4}$$

Hence for any $f \in U$,

$$\rho\sum_{j\in J}\langle \Lambda_j T f, \Lambda_j T f\rangle \leq D\ \rho\langle Tf, Tf\rangle. \tag{3.5}$$

On the other hand, for any $f \in U$ we have:

$$\begin{aligned}
\rho\sum_{j\in J}\langle \Lambda_j T f, \Lambda_j T f\rangle &\geq C\ \rho\langle K^*Tf, K^*Tf\rangle \\
&= C\ \rho\langle TK^*f, TK^*f\rangle \\
&= C\ \rho\langle K^* f, K^* f\rangle \\
&= C(\overline{\rho}(K^* f))^2,
\end{aligned}$$

which proves (1).

For proving (2), one may see that for any $f \in U$,

$$\begin{aligned}
C\ (\overline{\rho}(LK)^*(f))^2 &= C(\overline{\rho}(K^*L^*(f)))^2 \\
&\leq \rho\sum_{j\in J}\langle \Lambda_j L^* f, \Lambda_j L^* f\rangle \\
&\leq D\langle L^* f, L^* f\rangle \\
&\leq D\|L^*\|_\infty^2\ \rho\langle f, f\rangle.
\end{aligned}$$

□

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Theorem 3.9. . *Let $K \in End_A^*(U)$ and $\{\Lambda_j\}_{j\in J}$ be a g-frame for U with respect to $\{V_j\}_{j\in J}$ with frame bounds C and D. Then $\{\Lambda_j K^*\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ with frame bounds C and $D\|K\|_\infty^2$. The frame operator of $\{\Lambda_j K^*\}_{j\in J}$ is $S' = KSK^*$ where S is the frame operator of $\{\Lambda_j\}_{j\in J}$.*

Proof. . Since $\{\Lambda_j\}_{j\in J}$ is a g-frame for U with respect to $\{V_j\}_{j\in J}$ for any $f \in U$ we have:

$$\begin{aligned} C\ \rho\langle K^*f, K^*f\rangle &\leq \rho\sum_{j\in J}\langle \Lambda_j K^*f, \Lambda_j K^*f\rangle \\ &\leq D\ \rho\langle K^*f, K^*f\rangle \\ &\leq D\|K\|_\infty^2 \rho\langle f, f\rangle. \end{aligned}$$

But by definition of S:

$$SK^*f = \textstyle\sum_{j\in J} \Lambda_j^*\Lambda_j K^*f.$$

Thus

$$KSK^*f = \textstyle\sum_{j\in J} K\Lambda_j^*\Lambda_j K^*f = \sum_{j\in J}(\Lambda_j K^*)^*(\Lambda_j K^*)f.$$

Hence $S' = KSK^*$. □

Corollary 3.10. . *Let $K \in End_A^*(U)$ and $\{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ be a g-frame. Then $\{KS^{-1}\Lambda_j\}_{j\in J}$ is a K-g-frame, where S is the frame operator of $\{\Lambda_j\}_{j\in J}$.*

Theorem 3.11. . *Let K be surjective element in $End_A^*(U)$ and $\{\Lambda_j \in End_A^*(U, V_j)\}$ for all $j \in J$. A sequence $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ if and only if:*

$$Q : \{g_j\}_{j\in J} \to \sum_{j\in J} \Lambda_j^* g_j, \tag{3.6}$$

is a well-defined bounded linear operator from $\bigoplus_{j\in J} V_j$ onto U.

Proof. . If $\{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ with bounds C and D, then for any finite subset $J_1 \subseteq J$, we have:

$$\begin{aligned} \overline{\rho}_U(\sum_{j\in J_1} \Lambda_j^* g_j) &= \sup\{\rho\langle\sum_{j\in J_1} \Lambda_j^* g_j, f\rangle\ : \overline{\rho}_U(f) \leq 1\} \\ &= \sup\{\rho\sum_{j\in J_1}\langle g_j, \Lambda_j f\rangle\ :\ \overline{\rho}_U(f) \leq 1\} \\ &\leq \sqrt{\rho\sum_{j\in J_1}\langle g_j, g_j\rangle}\ \sup\{\sqrt{\rho\sum_{j\in J_1}\langle \Lambda_j f, \Lambda_j f\rangle} : \overline{\rho}_U(f) \leq 1\} \\ &\leq \overline{\rho}(g)\ \sqrt{D}. \end{aligned}$$

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Now, since the series $\sum_{j\in J_1}\langle g_j, g_j\rangle$ converges in A, the above inequality shows that $\sum_{j\in J_1}\Lambda_j^* g_j$ is also convergent in A. Hence, Q is well-defined. For $f\in U$, since $\{\Lambda_j\}_{j\in J}$ is a K-g-frame, then $f = Sg = \sum_{j\in J}\Lambda_j^*\Lambda_j g$ and $Q(\{\Lambda_j g\}_{j\in J}) = \sum_{j\in J}\Lambda_j^*\Lambda_j g = f$. This implies that the operator Q is onto. Conversely, if Q is a well-defined bounded linear operator from $\bigoplus V_j$ onto U, then for any $f\in U$ and any finite subset $J_1\subseteq J$, we have

$$\begin{aligned}
(\overline{\rho}_U\{\Lambda_j f\}_{j\in J_1})^2 &= \rho_U(\sum_{j\in J_1}\langle \Lambda_j f, \Lambda_j f\rangle \\
&= \rho_U(\sum_{j\in J_1}\langle f, \Lambda_j^*\Lambda_j f\rangle \\
&\leq \sqrt{\rho_U\langle f, f\rangle}\ \sqrt{\rho_U\langle \Lambda_j^*\Lambda_j f, \Lambda_j^*\Lambda_j f\rangle} \\
&\leq \overline{\rho_U}(f)\ \overline{\rho_U}(Q\{\Lambda_j f\}) \\
&\leq \overline{\rho_U}(f)\hat{\rho_U}(Q)\ \overline{\rho}\{\Lambda_j f\}_{j\in J_1}.
\end{aligned}$$

It follows that $\overline{\rho}_U\{\Lambda_j f\}_{j\in J_1}\leq \hat{\rho}_U(Q)\ \overline{\rho}_U(f)$. On the other hand, since for any $g = \{g_j\}_{j\in J}\in\bigoplus_{j\in J}V_j$ and $f\in U$,

$$\begin{aligned}
\langle Qg, f\rangle &= \langle \sum_{j\in J}\Lambda_j^* g, f\rangle \\
&= \sum_{j\in J}\langle g_j, \Lambda_j f\rangle \\
&= \langle g, \{\Lambda_j f\}_{j\in J}\rangle \\
&= \langle g, Q^* f\rangle,
\end{aligned}$$

it follows that $Q^* f = \{\Lambda_j f\}_{j\in J}$. Now, since Q is onto, therefore by [1, Proposition 2.2] Q^* is bounded below, so Q^* is invertible. Then, by [2, Proposition 3.2], we have:

$$\|(Q^*)^{-1}\|_\infty^{-2}\ \langle f, f\rangle \leq \langle Q^* f, Q^* f\rangle \leq \|Q^*\|_\infty^2\langle f, f\rangle.$$

It is easy to check that :

$$\langle Q^* f, Q^* f\rangle = \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle.$$

Hence,

$$\|(Q^*)^{-1}\|_\infty^{-2}\ \langle f, f\rangle \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq \|Q^*\|_\infty^2\langle f, f\rangle. \tag{3.7}$$

On the other hand, since K^* is an invertible element in $End_A^*(U)$, therefore:

$$\|(K^*)^{-1}\|_\infty^{-2}\langle f, f\rangle \leq \langle K^* f, K^* f\rangle \leq \|K^*\|_\infty^2\langle f, f\rangle. \tag{3.8}$$

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Hence by (3.8) we have:

$$\begin{aligned}\|(Q^*)^{-1}\|_\infty^{-2}\|K^*\|_\infty^{-2}\langle K^*f, K^*f\rangle &\le \|(Q^*)^{-1}\|_\infty^{-2}\langle f, f\rangle \\ &\le \langle Q^*f, Q^*f\rangle \\ &= \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \\ &\le \|Q^*\|_\infty^2\langle f, f\rangle.\end{aligned}$$

Therefore $\{\Lambda_j\}_{j\in J}$ is a K-g-frame.

□

The following theorem characterizes K-g-frame by frame operator

Theorem 3.12. *Let $\{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ be a g-Bessel for U with respect to $\{V_j\}_{j\in J}$. Then $\{\Lambda_j\}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$ if and only if there exists a constant $C > 0$ such that $S \ge CKK^*$, where S is the frame operator for $\{\Lambda_j\}_{j\in J}$.*

Proof. . We know $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for U with bounds C and D if and only if

$$C\ \langle K^*f, K^*f\rangle \le \textstyle\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \le D\langle f, f\rangle,$$

if and only if

$$C\ \langle KK^*f, f\rangle \le \textstyle\sum_{j\in J}\langle \Lambda_j^*\Lambda_j f, f\rangle \le D\langle f, f\rangle,$$

if and only if

$$C\ \langle KK^*f, f\rangle \le \langle Sf, f\rangle \le D\langle f, f\rangle,$$

where S is the K-g-frame operator for $\{\Lambda_j\}_{j\in J}$. Therefore, the conclusion holds. □

4. g-Atomic systems in Hilbert pro-C^*-modules

First, the following definition of a generalized atomic system (or g-atomic system) is presented, then main theorems are stated.

Definition 4.1. . A sequence $\{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ is a generalized atomic system (or g-atomic system) for $K \in End_A^*(U, V_j)$, if

(1) For any $a_x = \{a_j\}_{j\in J} \in \bigoplus_{j\in J} V_j$, $\sum_{j\in J}\Lambda_j^* a_j$ is convergent;
(2) There exists $C > 0$ such that for any $x \in U$, there is $a_x = \{a_j\}_{j\in J} \in \bigoplus_{j\in J} V_j$ such that:

$$\langle a_x, a_x\rangle \le C\langle x, x\rangle, \qquad Kx = \sum_{j\in J}\Lambda_j^* a_j.$$

Lemma 4.2. . *Let $K \in End_A^*(U)$. Then there exists a g-atomic system for $K \in End_A^*(U)$ with respect to $End_A^*(U)$.*

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Theorem 4.3. . *Let $\{\Lambda_j \in End_A^*(U, V_j\}_{j\in J}$ be a g-atomic system for $K \in End_A^*(U)$. Then $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for U with respect to $\{V_j\}_{j\in J}$.*

Proof. . For any $f \in U$ we have:

$$\begin{aligned}\overline{\rho}(K^*x) &= sup_{\overline{\rho}(y)\leq 1}(\rho\langle K^*x, y\rangle)\\ &= sup_{\overline{\rho}(y)\leq 1}(\rho\langle x, Ky\rangle).\end{aligned}$$

By definition there exists $C > 0$ such that for any $y \in U$,

$$\langle a_y, a_y\rangle \leq C\langle y, y\rangle, \qquad a_y \in \bigoplus_{j\in J} V_j,$$

and $Ky = \sum_{j\in J}\Lambda_j^* a_j$. Thus:

$$\begin{aligned}(\overline{\rho}(K^*x))^2 &= (sup_{\overline{\rho}(y)\leq 1}(\rho\ \langle x, \sum_{j\in J}\Lambda_j^* a_j\rangle))^2\\ &= (sup_{\overline{\rho}(y)\leq 1}(\rho\sum_{j\in J}\langle\Lambda_j x, a_j\rangle))^2\\ &\leq sup_{\overline{\rho}(y)\leq 1}(\rho\sum_{j\in J}\langle\Lambda_j x, \Lambda_j x\rangle)(\rho\sum_{j\in J}\langle a_j, a_j\rangle)\\ &\leq C(\rho\sum_{j\in J}\langle\Lambda_j x, \Lambda_j x\rangle).\end{aligned}$$

Thus:

$$\frac{1}{C}(\overline{\rho}(K^*x))^2 \leq \rho(\textstyle\sum_{j\in j}\langle\Lambda_j x, \Lambda_j x\rangle).$$

The proof is complete. □

Theorem 4.4. . *The sequence $\{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ is a g-atomic system for $K \in End_A^*(U)$ with respect to $\{V_j\}_{j\in J}$ if and only if $\{\Lambda_j L\}_{j\in J}$ is a g-atomic system for L^*K, where $L \in End_A^*(U)$ is surjective.*

Proof. . Let $\{\Lambda_j \in End_A^*(U, V_j)\}_{j\in J}$ be a g-atomic system for $K \in End_A^*(U)$ with respect to $\{V_j\}_{j\in J}$. Thus $Kx = \sum_{j\in J}\Lambda_j^* a_j$ and

$$L^*K = \textstyle\sum_{j\in J} L^*\Lambda_j^* a_j = \sum_{j\in J}(\Lambda_j L)^* a_j$$

Therefore $\{\Lambda_j L\}_{j\in J}$, is a g-atomic system.

For the converse, Let $\{\Lambda_j L\}_{j\in J}$ be a g-atomic system such that:

$$L^*K = \textstyle\sum_{j\in J}(\Lambda_j L)^* b_j = \sum_{j\in J} L^*\Lambda_j^* b_j.$$

Therefore $L^*(Kx - \sum_{j\in J}\Lambda_j^* b_j) = 0$, so $Kx = \sum_{j\in J}\Lambda_j^* b_j$. Hence $\{\Lambda_j\}_{j\in J}$ is a g-atomic system for $K \in End_A^*(U)$. □

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