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On Some Differential Inequalities for Certain Analytic Functions

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Abstract. The purpose of the paper is to derive some interesting implications associated with some differential inequalities, for certain analytic functions in the open unit disk. Connections to previously well known results are also established.

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1. Introduction and Preliminaries

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. An analytic function p in Δ with $p(0) = 1$ is said to be a Carathéodory function of order α if it satisfies

$$Re[p(z)] > \alpha \quad (0 \leq \alpha < 1, z \in \Delta).$$

We denote by $\mathcal{P}(\alpha)$ the class of all Carathéodory functions of order α in Δ and $\mathcal{P} = \mathcal{P}(0)$. Let $\mathcal{A}$ denote the class of analytic functions f defined in Δ normalized by $f(0) = 0$ and $f'(0) = 1$. Further we denote by $S^*(\alpha)$ and $\mathcal{K}(\alpha)$

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the subclasses of $\mathcal{A}$ consisting of starlike and convex functions of order α in Δ, respectively. That is, a function $f \in \mathcal{A}$ belongs to the classes $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ if f satisfies $Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$, respectively in Δ.

Let f and g be analytic in Δ, then we say that f is subordinate to g in Δ written $f \prec g$, if there exists an analytic function w in Δ, such that $w(0) = 0$, $|w(z)| < 1$, $(z \in \Delta)$ and $f(z) = g(w(z))$, $(z \in \Delta)$.

If g is univalent in Δ then the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Let $\mathcal{Q}$ denote the set of all functions q that are analytic and injective on $\overline{\Delta} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial\Delta : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \backslash E(q)$. Further, let the subclass of $\mathcal{Q}$ for $\mathcal{Q}(0) \equiv a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q} \equiv \mathcal{Q}_1$.

As of late various geometric properties of Carathéodory functions are being studied by researchers in the field of geometric function theory. Several implications have been put up by various authors [5, 7, 1, 2, 6, 3, 11, 12, 13, 14, 15] for functions to be starlike and convex of certain order or belong to the class of Carathéodory functions etc.,.In this article we intend to study more about the Carathéodory class of functions with the aid of differential subordination and Jack's Lemma. And the consequently obtained results, further simplifies and supplements the already existing and well known results of [1, 2]. The following results will be needed for the planned study.

Lemma 1.1. [4] *Suppose that function w is analytic for $|z| \leq r$, $w(0) = 0$ and*

$$|w(z_0)| = \max_{|z|=r} |w(z)|,$$

then $z_0 w'(z_0) = k w(z_0)$, where k is a real number with $k \geq 1$.

We recall the following Lemma from the theory of differential subordinations, developed by Miller and Mocanu[9, 10],

Lemma 1.2. [9, 10] *Let Ω be a set in the complex plane $\mathbb{C}$ and let q be a univalent function on $\overline{\Delta}$. Suppose that the function $H : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ satisfies the condition*

$$H[q(\zeta), m\zeta q'(\zeta); z] \notin \Omega, \quad \textit{whenever } z \in \Delta, |\zeta| = 1 \textit{ and } m \geq n.$$

If the function $p \in H[q(0), n]$ satisfies $H[p(z), zp'(z); z] \in \Omega$ for all $z \in \Delta$, then $p(z) \prec q(z)$.

Lemma 1.3. [8] *Let q be univalent in Δ and let θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$, $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z)), h(z) = \theta(q(z)) + Q(z)$, and suppose that*

(1) *Q is starlike in Δ.*

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(2) $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0, \ \ (z \in \Delta).$

If p is analytic in Δ and satisfies $\theta(p(z))+zp'(z)\phi(p(z)) \prec \theta(q(z))+zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.

2. Main Results

In the following Theorems we try to explore the Lemma's due to Jack[4] and Miller and Mocanu[6] to its fullest capacity and check the effects on the sufficient conditions for an analytic function, $f(z)$ to belong to the Carathéodory class and also $Re(af(z))$ to be positive, for $a \in \mathbb{C}$ with $Re(a) > 0$.

Theorem 2.1. *Let p be analytic in Δ with $p(0) = 1$. If*

$$Re\{(1-\alpha)p(z) + \alpha p^2(z) + \beta z p'(z)\} > (1-\alpha)\delta + \alpha(\delta^2 - y^2) - \frac{Re(\beta)}{2(1-\delta)}(1 - 2\delta + |p(z)|^2) \tag{2.1}$$

for $0 \le \delta < 1$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$, with $Re(\beta) \ge 0$, then $p \in P(\delta)$.

Proof. Let w be an analytic function in Δ with $w(0) = 0$ and

$$p(z) = \frac{1 + (1-2z)w(z)}{1 - w(z)}, (z \in \Delta).$$

Suppose there exist a point z_0 in Δ such that $Re\, p(z) > \delta$ for $|z| < |z_0|$ and $Re\ p(z_0) = \delta$, then we have $w(z) < 1$ for $|z| < |z_0|$ and $w(z_0) = 1$. By using Lemma 1.1, we have

$$z_0 w'(z_0) = k w(z_0)$$

where k is a real number and $k \ge 1$. Now,

$$z_0 p'(z_0) = \frac{2k(1-\delta)w(z_0)}{(1-w(z_0))^2}.$$

$$\text{Hence,} \qquad \frac{1}{2k(1-\delta)} z_0 p'(z_0) = \frac{2(Re\ w(z_0) - 1)}{|1 - w(z_0)|^4}.$$

As $Re\{w(z_0)\} \le 1$, taking $p(z_0) = \delta + iy$, we observe $z_0 p'(z_0)$ to be a non positive real number, since

$$w(z_0) = 1 - \frac{2(1-\delta)^2}{(1-\delta)^2 + y^2} + i\frac{2(1-\delta)y}{(1-\delta)^2 + y^2}$$

and

$$z_0 p'(z_0) = -k\frac{(1-\delta)^2 + y^2}{2(1-\delta)}. \tag{2.2}$$

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From (2.2), we obtain

$$\begin{aligned}
&Re\{(1-\alpha)p(z_0)+\alpha p^2(z_0)+\beta z_0 p'(z_0)\} \\
&\quad = (1-\alpha)\delta + \alpha(\delta^2 - y^2) - \frac{k((1-\delta)^2+y^2)Re\{\beta\}}{2(1-\delta)} \\
&\quad \leq (1-\alpha)\delta + \alpha(\delta^2 - y^2) - \frac{(1-2\delta+|p(z_0)|^2)}{2(1-\delta)}Re\{\beta\},
\end{aligned}$$

which is a contradiction to the assumption (2.1) and hence we conclude the proof. □

Theorem 2.2. *If p is an analytic function in Δ with $p(0)=1$. If for $0 \leq \alpha < 1$ and $a, \beta \in \mathbb{C}$*

$$Re((1-\alpha)p(z)+\alpha p^2(z)+\beta z p'(z)) > \frac{(Im(a))^2(1-\alpha+L)^2 - L(L|a|^2+2\alpha[(Re(a))^2-(Im(a))^2])}{2(L|a|^2+2\alpha[(Re(a))^2-(Im(a))^2]} \tag{2.3}$$

where $L = \frac{Re\{\bar{a}\beta\}}{Re\{a\}}$ with $Re(\bar{a}\beta) > 0$ and $Re(a) > 0$, then $Re\{ap(z)\} > 0$.

Proof. Let $q(z) = ap(z)$ and $h(z) = \frac{a+\bar{a}z}{1-z}$, we observe that the functions $q(z)$ and $h(z)$ are analytic functions in Δ with $q(0) = h(0) = a \in \mathbb{C}$ with $h(\Delta) = \{w; Re\{w\} > 0\}$. Now suppose that $q(z) \not\prec h(z)$, then by using Lemma 1.2, there exist points $z_0 \in \Delta$ and $\zeta_0 \in \partial\Delta \backslash \{1\}$, such that, $q(z_0) = h(\zeta_0)$ and $z_0 q'(z_0) = m\zeta_0 h'(\zeta_0)$, $m \geq n \geq 1$, with

$$\zeta_0 = h^{-1}(q(z_0)) = \frac{q(z_0)-a}{q(z_0)-\bar{a}} \quad \text{and} \quad \zeta_0 h'(\zeta_0) = -\frac{|q(z_0)-a|^2}{2Re((a)-q(z_0))}.$$

Since we have $h(\zeta_0) = \rho i$ $(\rho \in \mathbb{R})$, we observe that

$$\begin{aligned}
&Re\{(1-\alpha)p(z_0)+\alpha p^2(z_0)+\beta z_0 p'(z_0)\} \\
&\quad = Re\left\{(1-\alpha)\frac{h(\zeta_0)}{a} + \alpha\left(\frac{h(\zeta_0)}{a}\right)^2 + \beta m \frac{\zeta_0 h'(\zeta_0)}{a}\right\} \\
&\quad = Re\left\{(1-\alpha)\frac{\rho i}{a}\right\} + Re\left\{\alpha\left(\frac{\rho i}{a}\right)^2\right\} - \frac{m(\rho i - a)^2}{2Re(a)} Re\left\{\frac{\beta}{a}\right\} \\
&\quad \leq (1-\alpha)Re\left\{\frac{\rho i}{a}\right\} + \alpha Re\left\{\left(\frac{\rho i}{a}\right)^2\right\} - \frac{|\rho i - a|^2}{2Re(a)} Re\left\{\frac{\beta}{a}\right\} \\
&\quad = A\rho^2 + B\rho + C \\
&\quad = g(\rho)
\end{aligned} \tag{2.4}$$

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$$\text{where}\quad A = -\frac{1}{|a|^2}\left[\frac{L}{2} + \frac{\alpha}{|a|^2}[(Re(a))^2 - (Im(a))^2]\right],$$
$$B = \frac{Im(a)}{|a|^2}(1-\alpha+L) \text{ and}$$
$$C = -\frac{L}{2}$$

here L is given by the quantity $Re(\bar{a}\beta)/Re(a)$.The function $g(\rho)$ takes the maximum value at ψ, given by

$$\psi = \frac{Im(a)(1-\alpha+L)}{L + \frac{2\alpha}{|a|^2}[(Re(a))^2 - (Im(a))^2]}$$

which results in,

$$\begin{aligned}
&Re\{(1-\alpha)p(z_0) + \alpha p^2(z_0) + \beta z_0 p'(z_0)\} \leq g(\psi) \\
&\quad = \frac{(Im(a))^2(1-\alpha+L)^2 - L(L|a|^2 + 2\alpha[(Re(a))^2 - (Im(a))^2])}{2(L|a|^2 + 2\alpha[(Re(a))^2 - (Im(a))^2])}
\end{aligned}$$

contradicting the fact (2.3) holds, therefore $Re(ap(z)) > 0$. □

Theorem 2.3. *Let $p(z)$ be a nonzero analytic function in Δ with $p(0) = 1$. For $\beta, \gamma \in \mathbb{R}$, if*

$$t_1 < Im\left(1 - \beta(1-p(z)) + (1-\gamma)\frac{zp'(z)}{p(z)}\right) < t_2, \tag{2.5}$$

where

$$t_1 = \frac{-\sqrt{(1-\gamma)}(\sqrt{(1-\gamma)|a|^2 + 2\beta(Re(a))^2}) - (1-\gamma)Im(a)}{Re(a)}$$

and

$$t_2 = \frac{\sqrt{(1-\gamma)}(\sqrt{(1-\gamma)|a|^2 + 2\beta(Re(a))^2}) + (1-\gamma)Im(a)}{Re(a)}$$

then $Re(ap(z)) > 0$ where $Re(a) > 0$.

Proof. Proceeding similarly as the proof of the above Theorem we observe that,

$$\begin{aligned}
Im\left(1 - \beta(1-p(z_0) + (1-\gamma)\frac{zp'(z_0)}{p(z_0)}\right) &= Im\left(1 - \beta(1-q(z_0)) + (1-\gamma)\frac{z_0 q'(z_0)}{q(z_0)}\right) \\
&= Im\left(1 - \beta\left(1 - \frac{\rho i}{a}\right) + (1-\gamma)\frac{m|\rho i - a|^2}{2\rho Re(a)}\right)
\end{aligned}$$

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For the case $\rho > 0$, we obtain

$$\begin{aligned} & Im\left[1-\beta(1-p(z_0))+(1-\gamma)\frac{z_0p'(z_0)}{p(z_0)}\right] \\ & \quad \geq \frac{\beta\rho}{|a|^2}Re(a)+\frac{(1-\gamma)|\rho i-a|^2}{2\rho Re(a)} \\ & \quad = \frac{1}{2\rho Re(a)}\left[\left(\frac{2\beta(Re(a))^2}{|a|^2}+(1-\gamma)\right)\rho^2-2(1-\gamma)Im(a)\rho+(1-\gamma)|a|^2\right] \\ & \quad = g(\rho). \end{aligned} \tag{2.6}$$

The function $g(\rho)$ in (2.6) takes the minimum value at η_1 given by

$$\eta_1 = \frac{\sqrt{(1-\gamma)}|a|^2}{\sqrt{(1-\gamma)|a|^2+2\beta(Re(a))^2}},$$

which yields,

$$Im\left[1-\beta(1-p(z_0))+(1-\gamma)\frac{zp'(z_0)}{p(z_0)}\right] \geq g(\eta_1) = t_2,$$

a contradiction.Therefore we have $Re(ap(z)) > 0$.
Now for the case where $\rho < 0$, we obtain

$$\begin{aligned} & Im\left(1-\beta(1-p(z_0))+(1-\gamma)\frac{zp'(z_0)}{p(z_0)}\right] \\ & \quad \leq \frac{\beta\rho}{|a|^2}Re(a)+\frac{(1-\gamma)|\rho i-a|^2}{2\rho Re(a)} \\ & \quad = \frac{1}{2\rho Re(a)}\left[\left(\frac{2\beta(Re(a))^2}{|a|^2}+(1-\gamma)\right)\rho^2-2(1-\gamma)Im(a)\rho+(1-\gamma)|a|^2\right] \\ & \quad = g(\rho). \end{aligned} \tag{2.7}$$

The function $g(\rho)$ in (2.7), takes the maximum value at η_2 given by

$$\eta_2 = -\frac{\sqrt{(1-\gamma)}|a|^2}{\sqrt{|a|^2(1-\gamma)+2\beta(Re(a))^2}},$$

which implies

$$Im\left[1-\beta(1-p(z_0))+\gamma\frac{z_0p'(z_0)}{p(z_0)}\right] \leq g(\eta_2) = t_1,$$

a contradiction. Therefore we have $Re(ap(z)) > 0$. □

The proof of the next theorem is much akin to the Theorem 2.3, hence omitted.

Theorem 2.4. *Let $p(z)$ be an analytic function in Δ and $p(0) = 1$. If*

$$t_1 < Im\left(1+\gamma\frac{zp'(z)}{p(z)}+\beta p(z)\right) < t_2, (\gamma, \beta \in \mathbb{R}) \tag{2.8}$$

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where

$$t_1 = \frac{-\sqrt{\gamma}\sqrt{\gamma|a|^2 + 2\beta(Re(a))^2} - \gamma Im(a)}{Re(a)}$$

and

$$t_2 = \frac{\sqrt{\gamma}\sqrt{\gamma|a|^2 + 2\beta(Re(a))^2} - \gamma Im(a)}{Re(a)},$$

then

$$Re(ap(z)) > 0 \quad (Re(a) > 0).$$

Theorem 2.5. *Let p be non zero analytic function in Δ with $p(0) = 1$. If*

$$\left| arg \left(\lambda p(z) + \frac{zp'(z)}{p(z) + \mu} \right) \right| < \frac{\pi}{2} \chi(\lambda, A, B, C), \tag{2.9}$$

for $-1 \le B < A \le 1, \lambda > 0,\ \mu \ge 0, |A + B\mu| < |1 + \mu|$ and $\chi(\lambda, A, B, C)$ is given by

$$\chi(\lambda, A, B, C) = 1 + \frac{2}{\pi} \arctan \left[\frac{\begin{array}{l} \lambda + \lambda(B + C) + -\lambda C(A + B) + (\lambda(A + C) + (C - B))(B + C) \\ +\lambda ABC^2 \end{array}}{[\lambda(B + C)(1 - AC) - (\lambda(A + C) + (C - B))(1 - BC)]} \right] \tag{2.10}$$

$$\textit{then} \qquad |\arg\ p(z)| < \arctan \left[\frac{A - B}{1 + AB + A + B} \right].$$

Proof. Choosing

$$q(z) = \frac{1 + Az}{1 + Bz},\ \theta(w) = \lambda w \text{ and } \phi(w) = \frac{1}{w + \mu}. \tag{2.11}$$

we observe that q is univalent (convex) in Δ and $Re(q(z)) > 0,\ (z \in \Delta)$. Further, θ and ϕ are analytic in $q(\Delta)$ and $\phi(w) \neq 0$ ($w \in q(\Delta)$).The function

$$\begin{aligned} Q(z) &= zq'(z)\phi(q(z)) \\ &= \frac{(C - B)z}{(1 + Bz)(1 + Cz)}, \quad \text{where } C = \frac{A + B\mu}{1 + \mu} \end{aligned}$$

is univalent and starlike in Δ, because

$$\begin{aligned} Re\left(\frac{zQ'(z)}{Q(z)}\right) &= Re\left[1 - \left(\frac{Cz}{1 + Cz} + \frac{Bz}{1 + Bz}\right)\right] \\ &> \left(\frac{1}{1 + |B|}\right) + \left(\frac{1}{1 + |C|}\right) - 1 = \frac{1 - |B||C|}{(1 + |B|)(1 + |C|)} > 0. \end{aligned} \tag{2.12}$$

Further, we have

$$h(z) = \theta(q(z)) + Q(z) = \lambda \left(\frac{1 + Az}{1 + Bz}\right) + \frac{(C - B)z}{(1 + Bz)(1 + Cz)}$$

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and

$$
\begin{aligned}
Re\left\{\frac{zh'(z)}{Q(z)}\right\} &= Re\left[\lambda(1+\mu)\left(\frac{1+Cz}{1+Bz}\right)+\frac{zQ'(z)}{Q(z)}\right] \\
&= \lambda(1+\mu)Re\left(\frac{1+Cz}{1+Bz}\right)+Re\left(\frac{zQ'(z)}{Q(z)}\right) \\
&\geq \lambda(1+\mu)\left(\frac{1-|C|}{1-|B|}\right) > 0. \qquad (2.13)
\end{aligned}
$$

Now, it follows from (2.9) to (2.11) that

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z).$$

We also note that $h(0) = \lambda$ and

$$
h(e^{i\theta}) = \lambda\left(\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\right)+\frac{(C-B)e^{i\theta}}{(1+Be^{i\theta})(1+Ce^{i\theta})}
$$

$$
= \frac{\begin{array}{r}\Big\{e^{i\pi/2}\lambda[(A-B)(1+C^2)]\sin\theta+(C-B)\sin\theta+2\lambda(A-B)C\sin\theta\cos\theta \\ -i\Big(\lambda[(A+B)(1+C^2)+2C(1+AB)]\cos\theta \\ +2\lambda(A+B)\cos^2\theta+\lambda[(1+C^2)(1+AB)+(A-B)(1-C)]\Big)\Big\}\end{array}}{(1+(B+C)\cos\theta+BC(\cos 2\theta)^2+(B+C)\sin\theta+(BC\sin 2\theta)^2}.
$$

Therefore,

$$
\begin{aligned}
arg(he^{i\theta}) &= \frac{\pi}{2}+\arctan\left[\frac{\begin{array}{l}\lambda+\lambda(B+C)+[(\lambda(A+C)+(C-B))(1+BC)]\cos\theta \\ +\lambda C(A+B)\cos 2\theta+(\lambda(A+C)+(C-B))(B+C)+\lambda ABC^2\end{array}}{\begin{array}{l}[\lambda(B+C)(1-AC)-(\lambda(A+C)+(C-B))(1-BC)]\sin\theta \\ -\lambda C(A-C)\sin 2\theta\end{array}}\right] \\
&\geq 1+\frac{2}{\pi}\arctan\left[\frac{\begin{array}{l}\lambda+\lambda(B+C)+-\lambda C(A+B)+(\lambda(A+C)+(C-B))(B+C) \\ +\lambda ABC^2\end{array}}{[\lambda(B+C)(1-AC)-(\lambda(A+C)+(C-B))(1-BC)]}\right] \\
&= \frac{\pi}{2}\chi(\lambda,A,B,C),
\end{aligned}
$$

where $\chi(\lambda, A, B, C)$ is given by (2.10). Now, it follows from equations (2.12) and (2.13) that

$$\lambda p(z) + \frac{zp'(z)}{p(z)+\mu} \prec h(z).$$

Then by virtue of Lemma 1.3, we have $p(z) \prec q(z)$ or equivalently

$$|arg\ p(z)| < \arctan\left[\frac{(A-B)}{1+AB+(A+B)}\right] \ (z \in \Delta).$$

□

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3. Corollaries and Consequences

Taking $\alpha = 0$ and $\beta \in \mathbb{R}$ in Theorem 2.1, we obtain the following result due to Kim et al.[6, Theorem 2.3],

Corollary 3.1. *Let p be analytic in Δ with $p(0) = 1$. If*

$$Re\{p(z) + \beta z p'(z)\} > \delta - \frac{\beta}{2(1-\delta)}(1 - 2\delta + |p(z)|^2), (0 \leq \delta < 1, \beta \geq 0)$$

then $p \in P(\delta)$.

Remarks 3.2. (i) Letting $\alpha = \delta = 0$ and $\beta = 1$ in the above Corollary, we obtain the result by Nunokawa et al.[12]. Also assuming $p(z_0)$ to be a real number, $\alpha = 0$ and $\beta \in \mathbb{R}$ in Theorem 2.1, we have the result due to Kim et al.[6, Theorem 2.6], additionally letting $\beta = 1$ leads to [6, Corollary 2.7].

(ii) On taking $p(z) = f'(z)$, $\alpha = 0 = \delta$ in Theorem 2.1, we have the following result. If $f \in \mathcal{A}$,

$$Re\{f'(z) + \beta z f''(z)\} > -\frac{Re(\beta)}{2}(1 + |f'(z)|^2), Re(\beta) > 0,$$

then $Re\{f'(z)\} > 0$.

(iii) Letting $p(z) = \frac{f(z)}{z}$ in Theorem 2.1, with $\alpha = \delta = 0$ and $\beta = 1$ we get the following result. If $f \in \mathcal{A}$ and

$$Re\{f'(z)\} > -\frac{1}{2}\left(1 + \left|\frac{f(z)}{z}\right|^2\right),$$

then $Re\left\{\frac{f(z)}{z}\right\} > 0$.

Further for the choice of $\alpha = 1$ and $\beta = 1$ in Theorem 2.2, we obtain the following result,

Corollary 3.3. *Let $p(z)$ be a function analytic in Δ with $p(0) = 1$ and*

$$Re(p^2(z) + zp'(z)) > \frac{-3(Re(a))^2 + 2(Im(a))^2}{6(Re(a))^2 - 2(Im(a))^2}$$

then

$$Re(ap(z)) > 0.$$

Taking $p(z) = \frac{zf'(z)}{f(z)}$ and $\beta = \alpha$ in Theorem 2.2, we have the following corollary

Corollary 3.4. *Let $f \in \mathcal{A}$ and*

$$Re\left(\frac{zf'(z)}{f(z)} + \alpha z^2 \frac{f''(z)}{f(z)}\right) > \frac{(Im(a))^2\alpha^2|a|^2}{2\alpha(3(Re(a))^2 - (Im(a))^2)}$$

then

$$Re\left(\frac{azf'(z)}{f(z)}\right) > 0.$$

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with $Re(a) > 0$.

By taking $a = 1$ in the above Corollary, we get the sufficient condition for $f \in \mathcal{A}$ to be starlike. And when we take $p(z) = f'(z)$, we have the following

Corollary 3.5. *Let* $f \in \mathcal{A}$ *and if*

$$Re\left\{f'(z)\left[(1-\alpha)+\alpha\left(f'(z)+\frac{zf''(z)}{f'(z)}\right)\right]\right\} > -\frac{\alpha}{2}$$

then $Re(f'(z)) > 0$*, or equivalently* $f'(z) \prec \frac{1+z}{1-z}$.

For example $f(z) = z + \frac{z^2}{4}, \alpha = 1$ in Corollary 3.5 gives $Re(1 + \frac{3z}{2} + \frac{z^2}{4}) > -\frac{1}{2}$ which implies $Re(1 + \frac{z}{2}) > 0$. The image of the unit disk under the above mentioned function is as follows,

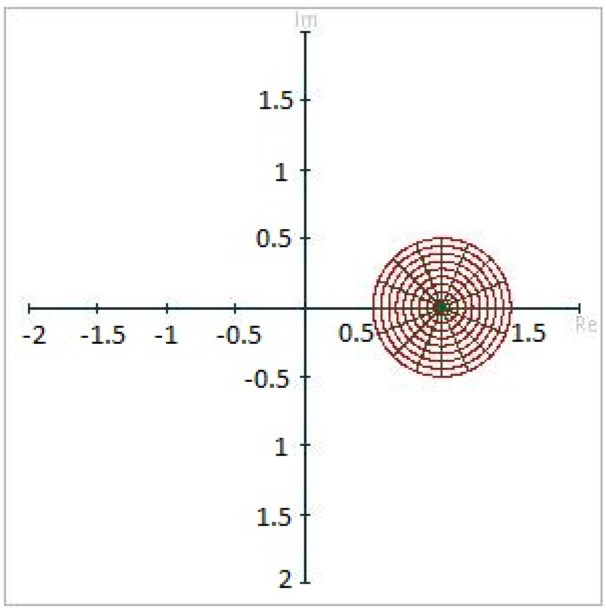

FIGURE 1. Image of $Re(1 + \frac{z}{2})$.

Now taking the values $\beta = 1$ and $\gamma = 0$ in Theorem 2.3, we have the following result by Attiya and Nasr[1],

Corollary 3.6. *Let* $p(z)$ *be a nonzero analytic function in* Δ *and* $p(0) = 1$*. If*

$$\gamma_1 < Im\left(p(z) + \frac{zp'(z)}{p(z)}\right) < \gamma_2$$

where

$$\gamma_1 = -\frac{\sqrt{|a|^2 + 2(Re(a))^2} - Im(a)}{Re(a)}$$

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and

$$\gamma_2 = \frac{\sqrt{|a|^2 + 2(Re(a))^2} + Im(a)}{Re(a)}$$

then $Re(ap(z)) > 0$*, where* $Re(a) > 0$*.*

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