Iranian Journal of Mathematical Sciences and Informatics
Vol. 18, No. 2 (2023), pp 31-50
DOI: 10.52547/ijmsi.18.2.31

Projectively Flat Finsler Spaces with Transformed Metrics

Gauree Shanker, Sarita Rani*, Kirandeep Kaur

Department of Mathematics and Statistics, School of Basic Sciences,
Central University of Punjab, Bathinda, Punjab-151 401, India

E-mail: grshnkr2007@gmail.com
E-mail: saritas.ss92@gmail.com
E-mail: kirandeepiitd@gmail.com

Abstract. In this paper, we consider some special Finsler spaces obtained via Randers-β change. First, we find the fundamental metric tensor and Cartan tensor of these metrics. Next, we establish a general formula for the inverse tensor of fundamental metric tensors of these metrics. Finally, we find the necessary and sufficient conditions under which these metrics are projectively flat, and we give examples to support our results.

Keywords: (α, β)-metric, Randers-β change, Fundamental metric tensor, Cartan tensor, Projective flatness.

2000 Mathematics subject classification: 53B40, 53C60.

1. Introduction

According to Chern [5], Finsler geometry is just Riemannian geometry without quadratic restriction. Finsler geometry is an interesting and active area of research for both pure and applied reasons [2]. Though there has been a lot of development in this area, still there is huge scope of research work in Finsler geometry. The concept of $(\alpha, \beta)-$metric was introduced by Matsumoto [9] in 1972. For a general Finsler metric F, Shibata [20] introduced the notion of β-change in F, i.e., $\bar{F} = f(F, \beta)$ in 1984. Recall [6] that a Finsler metric F on

*Corresponding Author

Received 07 March 2019; Accepted 09 May 2021

an open subset $\mathcal{U}$ of $\mathbb{R}^n$ is called projectively flat if and only if all the geodesics are straight lines in $\mathcal{U}$.
Current paper is organized as follows:
In section 2, we give basic definitions and examples of Finsler spaces with some special metrics obtained by Randers-β change. In section 3, we find fundamental metric tensors $\bar{g}_{ij}$ and Cartan tensors $\bar{C}_{ijk}$ for these metrics. In section 4, we find a generalized formula for the inverse tensor $\bar{g}^{ij}$ of fundamental metric tensor $\bar{g}_{ij}$. In sections 5, we find necessary and sufficient conditions for aforesaid metrics to be projectively flat, and we give examples to support our results.

2. Preliminaries

Though there is vast literature available for Riemann-Finsler geometry, here we give some basic definitions, examples and results required for subsequent sections.

Definition 2.1. Let M be an n-dimensional smooth manifold, T_xM the tangent space at $x \in M$, and $TM := \bigsqcup_{x\in M} T_xM$ be the tangent bundle of M whose elements are denoted by (x, y), where $x \in M$ and $y \in T_xM$. A Finsler structure on M is a function $F : TM \longrightarrow [0, \infty)$ with the properties:

- Regularity: F is C^∞ on the slit tangent bundle $TM\backslash\{0\}$.
- Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)\ \forall\ \lambda > 0$.
- Strong convexity: The matrix $(g_{ij}) = \left(\left[\frac{1}{2}F^2\right]_{y^iy^j}\right)_{n\times n}$ is positive-definite at every point of $TM\backslash\{0\}$.

A smooth manifold M together with the Finsler structure F, i.e., (M, F) is called Finsler space and the corresponding geometry is called Finsler geometry.

An (α, β)-metric on a connected smooth manifold M is a Finsler metric F constructed from a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and a one-form $\beta = b_i(x)y^i$ on M and is of the form $F = \alpha\phi\left(\frac{\beta}{\alpha}\right)$, where ϕ is a smooth function on M. (α, β)-metrics are the generalization of Randers metrics. A lot of work [3, 13, 14, 17, 18, 22, 23] has been carried out on (α, β)-metrics. Let us recall Shen's lemma [4, 6] which provides necessary and sufficient condition for a function of α and β to be a Finsler metric.

Lemma 2.2. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, where ϕ is a smooth function on $(-b_0, b_0)$, α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < b_0$. Then F is a Finsler metric if and only if the following conditions are satisfied:*

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + \left(b^2 - s^2\right)\phi''(s) > 0, \quad \forall \quad |s| \le b < b_0.$$

There are so many classical examples of (α, β)-metrics, below we mention few of them:

Randers metric, Kropina metric, generalized Kropina metric, Z. Shen's square metric, Matsumoto metric, exponential metric, infinite series metric.
Recall following definitions:

Definition 2.3. [10] Let (M, F) be an n-dimensional Finsler space and $\beta = b_i(x)y^i$ be a 1-form on M. Then the metric

$$\bar{F} = F + \beta \tag{2.1}$$

is called Randers changed Finsler metric, and the change defined in (2.1) is called Randers change.

Definition 2.4. [20] Let (M, F) be an n-dimensional Finsler space and $\beta = b_i(x)y^i$ be a 1-form on M. Then the change $F \longrightarrow \bar{F} = f(F, \beta)$ is called β-change of metric F, where $f(F, \beta)$ is a positively homogeneous function of F and β of degree one.

Following are some examples of β-change of Finsler metric F:

$$F \longrightarrow \bar{F} = F + \beta \tag{2.2}$$

$$F \longrightarrow \bar{F} = F^2/\beta \tag{2.3}$$

$$F \longrightarrow \bar{F} = F^{m+1}/\beta^m, \ (m \neq 0. \ -1) \tag{2.4}$$

$$F \longrightarrow \bar{F} = (F + \beta)^2/F \tag{2.5}$$

Changes (2.2), (2.3), (2.4) and (2.5) are called Randers change, Kropina change, generalized Kropina change and square change respectively.
Next, we construct some special Finsler metrics via Randers-β change. Our further studies will be based on these metrics. Let (M, F) be an n-dimensional Finsler space and $\beta = b_i(x)y^i$ be a 1-form on M. Then, we construct the following:

(1) Kropina-Randers change of F:
Applying Kropina change and Randers change simultaneously to F, we obtain a new metric $\bar{F} = \dfrac{F^2}{\beta} + \beta$, which we call Kropina-Randers change of the metric F.

(2) Generalized Kropina-Randers change of F:
Applying Generalized Kropina change and Randers change simultaneously to F, we obtain a new metric $\bar{F} = \dfrac{F^{m+1}}{\beta^m} + \beta \ (m \neq 0, -1)$, which we call generalized Kropina-Randers change of the metric F.

(3) Square-Randers change of F:
Applying Square change and Randers change simultaneously to F, we obtain a new metric $\bar{F} = \dfrac{(F+\beta)^2}{F} + \beta$, which we call square-Randers change of the metric F.

Recall [11, 21] the following definition:

[DOI: 10.61186/ijmsi.18.2.31]

Definition 2.5. Let (M, F) be an n-dimensional Finsler space. If $F = \sqrt[m]{a_{i_1 i_2 \ldots i_m} y^{i_1} y^{i_2} \ldots y^{i_m}}$, with $A := a_{i_1 i_2 \ldots i_m} y^{i_1} y^{i_2} \ldots y^{i_m}$ symmetric in all the indices, then F is called m^{th}-root Finsler metric.

m^{th}-root Finsler metric has applications in Ecology [1]. This metric is generalization of Riemannian metric as the square root metric is a Riemannian metric. We will focus on Randers-β change of square root Finsler metrics in this paper. We use following notations in the subsequent sections:

$$\frac{\partial \bar{F}}{\partial x^i} = \bar{F}_{x^i},\ \frac{\partial \bar{F}}{\partial y^i} = \bar{F}_{y^i},\ \frac{\partial A}{\partial x^i} = A_{x^i},\ \frac{\partial A}{\partial y^i} = A_i,\ A_{x^i} y^i = A_0,\ A_{x^i y^j} y^i = A_{0j},$$
$$\frac{\partial \beta}{\partial x^i} = \beta_{x^i},\ \frac{\partial \beta}{\partial y^i} = b_i \text{ or } \beta_i,\ \beta_{x^i} y^i = \beta_0,\ \beta_{x^i y^j} y^i = \beta_{0j},\ m_i = b_i - \frac{\beta}{F^2} y_i \text{ etc.}$$

3. Fundamental Metric Tensors and Cartan Tensors

Definition 3.1. Let (M, F) be an n-dimensional Finsler space. The function

$$g_{ij} := \left(\frac{1}{2} F^2\right)_{y^i y^j} = F F_{y^i y^j} + F_{y^i} F_{y^j} = h_{ij} + \ell_i \ell_j$$

is called fundamental metric tensor of the metric F.

Definition 3.2. Let (M, F) be an n-dimensional Finsler space. Then its Cartan tensor is defined as

$$C_{ijk}(y) = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \left(F^2\right)_{y^i y^j y^k},$$

which is symmetric in all the three indices i, j, k.

For any Finsler metric F, we find fundamental metric tensor and Cartan tensor for all the metrics constructed in the previous section.
Kropina-Randers change of F is

$$\bar{F} = \frac{F^2}{\beta} + \beta. \tag{3.1}$$

Differentiating (3.1) w.r.t. y^i, we get

$$\bar{F}_{y^i} = \frac{2F}{\beta} F_{y^i} - \frac{F^2 - \beta^2}{\beta^2} b_i. \tag{3.2}$$

Differentiation of (3.2) further w.r.t. y^j gives

$$\bar{F}_{y^i y^j} = \frac{2}{\beta}\, g_{ij} - \frac{2F}{\beta^2} \left(b_i F_{y^j} + b_j F_{y^i}\right) + \frac{2F^2}{\beta^3}\, b_i b_j. \tag{3.3}$$

[DOI: 10.61186/ijmsi.18.2.31]

Now,

$$\begin{aligned}\bar{g}_{ij} =&\bar{F}\bar{F}_{y^iy^j} + \bar{F}_{y^i}\bar{F}_{y^j}\\ =&\left\{\frac{F^2}{\beta}+\beta\right\}\left\{\frac{2}{\beta}\, g_{ij} - \frac{2F}{\beta^2}\left(b_iF_{y^j}+b_jF_{y^i}\right)+\frac{2F^2}{\beta^3}\, b_ib_j\right\}\\ &+\left\{\frac{2F}{\beta}F_{y^i}-\frac{F^2-\beta^2}{\beta^2}b_i\right\}\left\{\frac{2F}{\beta}F_{y^j}-\frac{F^2-\beta^2}{\beta^2}b_j\right\}.\end{aligned}$$

Simplifying, we get

$$\bar{g}_{ij} = \frac{2(F^2+\beta^2)}{\beta^2}\, g_{ij} - \frac{4F^3}{\beta^3}\left(b_iF_{y^j}+b_jF_{y^i}\right)+\frac{4F^2}{\beta^2}F_{y^i}F_{y^j}+\left(\frac{3F^4}{\beta^4}+1\right)b_ib_j. \tag{3.4}$$

Hence, we have following:

Proposition 3.3. *Let $(M,\bar{F})$ be an n-dimensional Finsler space with $\bar{F} = \dfrac{F^2}{\beta}+\beta$ as a Kropina-Randers change of F. Then its fundamental metric tensor is given by equation (3.4).*

Next, we find Cartan tensor for Kropina-Randers changed metric.
By definition, we have

$$2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^k}.$$

From the equation (3.4), we get

$$\begin{aligned}2\bar{C}_{ijk} =&\frac{\partial}{\partial y^k}\left(\bar{g}_{ij}\right)\\ =&\frac{2(F^2+\beta^2)}{\beta^2}\left(2C_{ijk}\right)+\frac{4\left(\beta y_k - F^2b_k\right)}{\beta^3}\left(h_{ij}+\frac{y_iy_j}{F^2}\right)-\frac{4F^3}{\beta^3}\left(\frac{b_ih_{jk}+b_jh_{ik}}{F}\right)\\ &-\frac{12\left(\beta Fy_k - F^3b_k\right)}{\beta^4}\left(\frac{b_iy_j+b_jy_i}{F}\right)+\frac{4F^2}{\beta^2}\left(\frac{y_ih_{jk}+y_jh_{ik}}{F^2}\right)\\ &+\frac{8\left(\beta y_k - F^2b_k\right)}{\beta^3}\frac{y_iy_j}{F^2}+\frac{12\left(\beta F^2y_k - F^4b_k\right)}{\beta^5}b_ib_j.\end{aligned}$$

After simplification, we get

$$\begin{aligned}2\bar{C}_{ijk} =&\frac{4(F^2+\beta^2)}{\beta^2}C_{ijk}+h_{ij}\left(\frac{-4F^2}{\beta^3}b_k+\frac{4}{\beta^2}y_k\right)+h_{jk}\left(\frac{-4F^2}{\beta^3}b_i+\frac{4}{\beta^2}y_i\right)\\ &+h_{ki}\left(\frac{-4F^2}{\beta^3}b_j+\frac{4}{\beta^2}y_j\right)-\frac{12F^4}{\beta^5}\left\{b_ib_jb_k-\frac{\beta}{F^2}\left(b_ib_ky_j+b_ib_jy_k+b_jb_ky_i\right)\right.\\ &\left.-\frac{\beta^3}{F^6}y_iy_jy_k+\frac{\beta^2}{F^4}\left(y_iy_kb_j+y_iy_jb_k+y_jy_kb_i\right)\right\}\end{aligned}$$

$$\therefore\ \bar{C}_{ijk} = \frac{2(F^2+\beta^2)}{\beta^2}C_{ijk}-\frac{2F^2}{\beta^3}\left(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j\right)-\frac{6F^4}{\beta^5}\, m_im_jm_k. \tag{3.5}$$

Above discussion leads to the following:

Proposition 3.4. *Let $(M, \bar{F})$ be an n-dimensional Finsler space, with $\bar{F} = \dfrac{F^2}{\beta} + \beta$ as a Kropina-Randers change of F. Then its Cartan tensor is given by equation (3.5).*

Next, generalized Kropina-Randers change of F is

$$\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \quad (m \neq 0, -1). \tag{3.6}$$

Differentiating (3.6) w.r.t. y^i, we get

$$\bar{F}_{y^i} = (m+1)\frac{F^m}{\beta^m} F_{y^i} + \left(1 - m\frac{F^{m+1}}{\beta^{m+1}}\right) b_i. \tag{3.7}$$

Differentiation of (3.7) further w.r.t. y^j gives

$$\begin{aligned}\bar{F}_{y^i y^j} =&(m+1)\frac{F^m}{\beta^m} F_{y^i y^j} + m(m+1)\frac{F^{m-1}}{\beta^m} F_{y^i} F_{y^j} \\ &- m(m+1)\frac{F^m}{\beta^{m+1}} \left(b_i F_{y^j} + b_j F_{y^i}\right) + m(m+1)\frac{F^{m+1}}{\beta^{m+2}} b_i b_j.\end{aligned} \tag{3.8}$$

Now,

$$\begin{aligned}\bar{g}_{ij} =&\bar{F}\bar{F}_{y^i y^j} + \bar{F}_{y^i}\bar{F}_{y^j} \\ =&\left\{\frac{F^{m+1}}{\beta^m} + \beta\right\}\left\{(m+1)\frac{F^m}{\beta^m} F_{y^i y^j} + m(m+1)\frac{F^{m-1}}{\beta^m} F_{y^i} F_{y^j}\right. \\ &\left. - m(m+1)\frac{F^m}{\beta^{m+1}} \left(b_i F_{y^j} + b_j F_{y^i}\right) + m(m+1)\frac{F^{m+1}}{\beta^{m+2}} b_i b_j\right\} \\ &+ \left\{(m+1)\frac{F^m}{\beta^m} F_{y^i} + \left(1 - m\frac{F^{m+1}}{\beta^{m+1}}\right) b_i\right\}\left\{(m+1)\frac{F^m}{\beta^m} F_{y^j}\right. \\ &\left. + \left(1 - m\frac{F^{m+1}}{\beta^{m+1}}\right) b_j\right\}.\end{aligned}$$

After simplification, we get

$$\begin{aligned}\bar{g}_{ij} =&(m+1)\left(\frac{F^{2m}}{\beta^{2m}} + \frac{F^{m-1}}{\beta^{m-1}}\right) g_{ij} \\ &- (m+1)\left(2m\frac{F^{2m+1}}{\beta^{2m+1}} + (m-1)\frac{F^m}{\beta^m}\right)\left(b_i F_{y^j} + b_j F_{y^i}\right) \\ &+ (m+1)\left(2m\frac{F^{2m}}{\beta^{2m}} + (m-1)\frac{F^{m-1}}{\beta^{m-1}}\right) F_{y^i} F_{y^j} \\ &+ \left(1 + m(2m+1)\frac{F^{2m+2}}{\beta^{2m+2}} + m(m-1)\frac{F^{m+1}}{\beta^{m+1}}\right) b_i b_j.\end{aligned} \tag{3.9}$$

Hence, we have the following proposition:

Proposition 3.5. *Let $(M,\bar{F})$ be an n-dimensional Finsler space with $\bar{F} = \dfrac{F^{m+1}}{\beta^m} + \beta \quad (m \neq 0,-1)$ as generalized Kropina-Randers change of F. Then its fundamental metric tensor is given by equation (3.9).*

Next, we find Cartan tensor for generalized Kropina-Randers changed metric.
By definition, we have

$$2C_{ijk}(y) = \frac{\partial g_{ij}}{\partial y^k}.$$

From the equation (3.9), we get

$$\begin{aligned}
2\bar{C}_{ijk} &= \frac{\partial}{\partial y^k}\left(\bar{g}_{ij}\right) \\
&= 2(m+1)\left(\frac{F^{2m}}{\beta^{2m}} + \frac{F^{m-1}}{\beta^{m-1}}\right) C_{ijk} \\
&+ (m+1)\left\{2m\frac{\beta F^{2m-2}y_k - F^{2m}b_k}{\beta^{2m+1}} + (m-1)\frac{\beta F^{m-3}y_k - F^{m-1}b_k}{\beta^m}\right\} \\
&\quad \times \left(h_{ij} + \frac{y_i y_j}{F^2}\right) \\
&- (m+1)\left(2m\frac{F^{2m+1}}{\beta^{2m+1}} + (m-1)\frac{F^m}{\beta^m}\right)\left(\frac{b_i h_{jk} + b_j h_{ik}}{F}\right) \\
&- m(m+1)\left\{2(2m+1)\frac{\beta F^{2m-1}y_k - F^{2m+1}b_k}{\beta^{2m+2}}\right. \\
&\quad \left. + (m-1)\frac{\beta F^{m-2}y_k - F^m b_k}{\beta^{m+1}}\right\}\left(\frac{b_i y_j + b_j y_i}{F}\right) \\
&+ (m+1)\left(2m\frac{F^{2m}}{\beta^{2m}} + (m-1)\frac{F^{m-1}}{\beta^{m-1}}\right)\left(\frac{y_i h_{jk} + y_j h_{ik}}{F^2}\right) \\
&+ (m+1)\left\{4m^2\frac{\beta F^{2m-2}y_k - F^{2m}b_k}{\beta^{2m+1}} + (m-1)^2\frac{\beta F^{m-3}y_k - F^{m-1}b_k}{\beta^m}\right\}\frac{y_i y_j}{F^2} \\
&+ m(m+1)\left\{2(2m+1)\frac{\beta F^{2m}y_k - F^{2m+2}b_k}{\beta^{2m+3}}\right. \\
&\quad \left. + (m-1)\frac{\beta F^{m-1}y_k - F^{m+1}b_k}{\beta^{m+2}}\right\} b_i b_j.
\end{aligned}$$

After simplification, we get

$$\begin{aligned}
\bar{C}_{ijk} =& (m+1)\left(\frac{F^{2m}}{\beta^{2m}} + \frac{F^{m-1}}{\beta^{m-1}}\right) C_{ijk} \\
&- \frac{(m+1)}{2}\frac{F^{m-1}}{\beta^m}\left(2m\frac{F^{m+1}}{\beta^{m+1}} + (m-1)\right)\left(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j\right) \\
&- \frac{m}{2}\frac{F^{m+1}}{\beta^{m+2}}\left((2m+1)(2m+2)\frac{F^{m+1}}{\beta^{m+1}} + (m^2-1)\right) m_i m_j m_k.
\end{aligned} \tag{3.10}$$

[DOI: 10.61186/ijmsi.18.2.31]

The above discussion leads to the following proposition:

Proposition 3.6. *Let $(M,\bar{F})$ be an n-dimensional Finsler space with $\bar{F}=\dfrac{F^{m+1}}{\beta^m}+\beta$ $(m\neq 0,-1)$ as generalized Kropina-Randers change of F. Then its Cartan tensor is given by equation (3.10).*

Next, square-Randers change of F is

$$\bar{F}=\frac{(F+\beta)^2}{F}+\beta. \tag{3.11}$$

Differentiating (3.11) w.r.t. y^i, we get

$$\bar{F}_{y^i}=\left(1-\frac{\beta^2}{F^2}\right)F_{y^i}+\left(\frac{2\beta}{F}+3\right)b_i. \tag{3.12}$$

Differentiation of (3.12) further w.r.t. y^j gives

$$\bar{F}_{y^iy^j}=\left(1-\frac{\beta^2}{F^2}\right)F_{y^iy^j}-\frac{2\beta}{F^2}\left(b_iF_{y^j}+b_jF_{y^i}\right)+\frac{2\beta^2}{F^3}F_{y^i}F_{y^j}+\frac{2}{F}b_ib_j. \tag{3.13}$$

Now,

$$\begin{aligned}\bar{g}_{ij}&=\bar{F}\bar{F}_{y^iy^j}+\bar{F}_{y^i}\bar{F}_{y^j}\\&=\left\{F+\frac{\beta^2}{F}+3\beta\right\}\left\{\left(1-\frac{\beta^2}{F^2}\right)F_{y^iy^j}-\frac{2\beta}{F^2}\left(b_iF_{y^j}+b_jF_{y^i}\right)\right.\\&\qquad\left.+\frac{2\beta^2}{F^3}F_{y^i}F_{y^j}+\frac{2}{F}b_ib_j\right\}\\&\quad+\left\{\left(1-\frac{\beta^2}{F^2}\right)F_{y^i}+\left(\frac{2\beta}{F}+3\right)b_i\right\}\left\{\left(1-\frac{\beta^2}{F^2}\right)F_{y^j}+\left(\frac{2\beta}{F}+3\right)b_j\right\}.\end{aligned}$$

Simplifying, we get

$$\begin{aligned}\bar{g}_{ij}=&\left(1+\frac{3\beta}{F}-\frac{3\beta^3}{F^3}-\frac{\beta^4}{F^4}\right)g_{ij}+\left(3-\frac{4\beta^3}{F^3}-\frac{9\beta^2}{F^2}\right)\left(b_iF_{y^j}+b_jF_{y^i}\right)\\&+\left(-\frac{3\beta}{F}+\frac{9\beta^3}{F^3}+\frac{4\beta^4}{F^4}\right)F_{y^i}F_{y^j}+\left(11+\frac{18\beta}{F}+\frac{6\beta^2}{F^2}\right)b_ib_j.\end{aligned} \tag{3.14}$$

Hence, we have following:

Proposition 3.7. *Let $(M,\bar{F})$ be an n-dimensional Finsler space with $\bar{F}=\dfrac{(F+\beta)^2}{F}+\beta$ as a square-Randers change of F. Then its fundamental metric tensor is given by equation (3.14).*

Next, we find Cartan tensor for square-Randers changed metric.
By definition, we have

$$2C_{ijk}(y)=\frac{\partial g_{ij}}{\partial y^k}.$$

[DOI: 10.61186/ijmsi.18.2.31]

From the equation (3.14), we get

$$
\begin{aligned}
2\bar{C}_{ijk} &= \frac{\partial}{\partial y^k}\left(\bar{g}_{ij}\right) \\
&= 2\left(1+\frac{3\beta}{F}-\frac{3\beta^3}{F^3}-\frac{\beta^4}{F^4}\right)C_{ijk} \\
&+\left\{\frac{3(F^2b_k-\beta y_k)}{F^3}-\frac{9(\beta^2F^2b_k-\beta^3y_k)}{F^5}-\frac{4(\beta^3F^2b_k-\beta^4y_k)}{F^6}\right\}\left(h_{ij}+\frac{y_iy_j}{F^2}\right) \\
&-\left(3-\frac{4\beta^3}{F^3}-\frac{9\beta^2}{F^2}\right)\left(\frac{b_ih_{jk}+b_jh_{ik}}{F}\right) \\
&-\left\{-\frac{12(\beta^2F^2b_k-\beta^3y_k)}{F^5}-\frac{18(\beta F^2b_k-\beta^2y_k)}{F^4}\right\}\left(\frac{b_iy_j+b_jy_i}{F}\right) \\
&+\left(-\frac{3\beta}{F}+\frac{9\beta^3}{F^3}+\frac{4\beta^4}{F^4}\right)\left(\frac{y_ih_{jk}+y_jh_{ik}}{F^2}\right) \\
&+\left\{\frac{16(\beta^3F^2b_k-\beta^4y_k)}{F^6}+\frac{27(\beta^2F^2b_k-\beta^3y_k)}{F^5}-\frac{3(F^2b_k-\beta y_k)}{F^3}\right\}\frac{y_iy_j}{F^2} \\
&+\left\{\frac{18(F^2b_k-\beta y_k)}{F^3}+\frac{12(\beta F^2b_k-\beta 2y_k)}{F^4}\right\}b_ib_j.
\end{aligned}
$$

After simplification, we get

$$
\begin{aligned}
\bar{C}_{ijk} &= \left(1+\frac{3\beta}{F}-\frac{3\beta^3}{F^3}-\frac{\beta^4}{F^4}\right)\ C_{ijk} \\
&+\frac{1}{2}\left(\frac{3}{F}-\frac{9\beta^2}{F^3}-\frac{4\beta^3}{F^4}\right)(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j)+\left(\frac{9}{F}+\frac{6\beta}{F^2}\right)m_im_jm_k.
\end{aligned}
\tag{3.15}
$$

The above discussion leads to the following proposition:

Proposition 3.8. *Let $(M,\bar{F})$ be an n-dimensional Finsler space with $\bar{F}=\dfrac{(F+\beta)^2}{F}+\beta$ as a square-Randers change of F. Then its Cartan tensor is given by equation (3.15).*

4. General formula for inverse metric tensor $\bar{g}^{ij}$ of fundamental metric tensor $\bar{g}_{ij}$:

Let us put

$$
\frac{2(F^2+\beta^2)}{\beta^2}=\rho_0,\ -\frac{4F^3}{\beta^3}=\rho_1,\ \frac{4F^2}{\beta^2}=\rho_2,\ \frac{3F^4}{\beta^4}+1=\rho_3
$$

[DOI: 10.61186/ijmsi.18.2.31]

in (3.4). Then the fundamental metric tensor $\bar{g}_{ij}$ for Kropina-Randers changed Finsler metric $\bar{F} = \dfrac{F^2}{\beta} + \beta$ takes following form

$$\begin{aligned}\bar{g}_{ij} &= \rho_0 g_{ij} + \rho_1\left(b_i\frac{y_j}{F} + b_j\frac{y_i}{F}\right) + \rho_2\frac{y_i}{F}\frac{y_j}{F} + \rho_3 b_i b_j\\ &= \rho_0\left\{g_{ij} + \frac{\rho_1}{\rho_0}\left(b_i + \frac{y_i}{F}\right)\left(b_j + \frac{y_j}{F}\right) + \left(\frac{\rho_2-\rho_1}{\rho_0}\right)\frac{y_i}{F}\frac{y_j}{F} + \left(\frac{\rho_3-\rho_1}{\rho_0}\right)b_i b_j\right\},\end{aligned}$$

i.e.,

$$\bar{g}_{ij} = \rho_0\left\{g_{ij} + \lambda\left(b_i + \frac{y_i}{F}\right)\left(b_j + \frac{y_j}{F}\right) + \mu\frac{y_i}{F}\frac{y_j}{F} + \nu\ b_i b_j\right\}, \tag{4.1}$$

where

$$\lambda = \frac{\rho_1}{\rho_0},\ \mu = \frac{\rho_2-\rho_1}{\rho_0} \text{ and } \nu = \frac{\rho_3-\rho_1}{\rho_0}. \tag{4.2}$$

Similarly, for other two metrics constructed in section two, the fundamental metric tensors obtained in (3.9) and (3.14) respectively, can be written in the form of (4.1), where the values of λ, μ, and ν can be computed from (3.9) and (3.14) respectively.

Next, we find the inverse metric tensor $\bar{g}^{ij}$ of fundamental metric tensor $\bar{g}_{ij}$ for the metric $\bar{F}$.
Let

$$\begin{aligned}m_{ij} &= g_{ij} + \lambda\left(b_i + \frac{y_i}{F}\right)\left(b_j + \frac{y_j}{F}\right)\\ &= g_{ij} + \lambda c_i c_j, \qquad c_i = b_i + \frac{y_i}{F}.\end{aligned}$$

Define

$$\begin{aligned}c^2 &= g^{ij}c_i c_j\\ &= g^{ij}\left(b_i + \frac{y_i}{F}\right)\left(b_j + \frac{y_j}{F}\right)\\ &= b^2 + \frac{2\beta}{F} + 1.\end{aligned}$$

Next,

$$\begin{aligned}\det\, m_{ij} &= \left(1 + \lambda c^2\right)\det\, g_{ij}\\ &= X\det\, g_{ij},\end{aligned}$$

where

$$X = 1 + \lambda c^2 = 1 + \lambda\left(b^2 + \frac{2\beta}{F} + 1\right). \tag{4.3}$$

Also,

$$m^{ij} = g^{ij} - \frac{\lambda}{1+\lambda c^2}c^i c^j = g^{ij} - \frac{\lambda}{X}A^{ij},$$

[DOI: 10.61186/ijmsi.18.2.31]

where

$$\begin{aligned} A^{ij} &= c^i c^j \\ &= g^{ih} c_h g^{jk} c_k \\ &= g^{ih}\left(b_h + \frac{y_h}{F}\right) g^{jk}\left(b_k + \frac{y_k}{F}\right) \\ &= \left(b_i + \frac{y_i}{F}\right)\left(b_j + \frac{y_j}{F}\right). \end{aligned}$$

Then (4.1) becomes

$$\bar{g}_{ij} = \rho_0 \left\{ m_{ij} + \mu \frac{y_i}{F}\frac{y_j}{F} + \nu\ b_i b_j \right\}. \tag{4.4}$$

Further, assume that

$$n_{ij} = m_{ij} + \mu d_i d_j, \quad d_i = \frac{y_i}{F}. \tag{4.5}$$

Define

$$\begin{aligned} d^2 &= m^{ij} d_i d_j \\ &= \left\{ g^{ij} - \frac{\lambda}{X}\left(b^i b^j + b^i \frac{y^j}{F} + b^j \frac{y^i}{F} + \frac{y^i y^j}{F^2} \right) \right\} \frac{y^i}{F}\frac{y^j}{F} \\ &= 1 - \frac{\lambda}{X}\left(1 + \frac{\beta}{F}\right)^2. \end{aligned}$$

Next,

$$\begin{aligned} \det\ n_{ij} &= \left(1 + \mu d^2\right) \det\ m_{ij} \\ &= YX \det\ g_{ij}, \end{aligned}$$

where

$$Y = 1 + \mu d^2 = 1 + \mu \left\{ 1 - \frac{\lambda}{X}\left(1 + \frac{\beta}{F}\right)^2 \right\}. \tag{4.6}$$

Also,

$$\begin{aligned} n^{ij} =& m^{ij} - \frac{\mu}{1 + \mu d^2} d^i d^j \\ &= g^{ij} - \frac{\lambda}{X} A^{ij} - \frac{\mu}{Y} B^{ij} \\ &= g^{ij} - \frac{\lambda}{X}\left(b_i + \frac{y_i}{F}\right)\left(b_j + \frac{y_j}{F}\right) - \frac{\mu}{Y} B^{ij}, \end{aligned} \tag{4.7}$$

[DOI: 10.61186/ijmsi.18.2.31]

where

$$
\begin{aligned}
B^{ij} &= d^i d^j \\
&= m^{ih} d_h m^{jk} d_k \\
&= \frac{1}{F^2}\left\{1-\frac{\lambda}{X}\left(1+\frac{\beta}{F}\right)\right\}^2 y^i y^j \\
&+ \frac{1}{F}\left\{-\frac{\lambda}{X}\left(1+\frac{\beta}{F}\right)+\frac{\lambda^2}{X^2}\left(1+\frac{\beta}{F}\right)^2\right\}\left(y^i b^j + y^j b^i\right)+\frac{\lambda^2}{X^2}\left(1+\frac{\beta}{F}\right)^2 b^i b^j.
\end{aligned} \tag{4.8}
$$

From (4.4) and (4.5), we get

$$\bar{g}_{ij} = \rho_0 \left\{n_{ij} + \nu\, b_i b_j\right\}, \tag{4.9}$$

Define

$$
\begin{aligned}
\tilde{b}^2 &= n^{ij} b_i b_j \\
&= b^2 - \frac{\lambda}{X}\left(b^2+\frac{\beta}{F}\right)^2 - \frac{\mu}{Y}\frac{\beta^2}{F^2}\left\{1-\frac{\lambda}{X}\left(1+\frac{\beta}{F}\right)\right\}^2 \\
&\quad - 2b^2\frac{\beta}{F}\frac{\mu}{Y}\left\{-\frac{\lambda}{X}\left(1+\frac{\beta}{F}\right)+\frac{\lambda^2}{X^2}\left(1+\frac{\beta}{F}\right)^2\right\} - b^4\frac{\mu}{Y}\frac{\lambda^2}{X^2}\left(1+\frac{\beta}{F}\right)^2.
\end{aligned} \tag{4.10}
$$

From (4.9), we have

$$\det\, \bar{g}_{ij} = \rho_0^n\left(1+\nu\tilde{b}^2\right)\det\, n_{ij},$$

and the inverse metric tensor $\bar{g}^{ij}$ of fundamental metric tensor $\bar{g}_{ij}$ is given by

$$\bar{g}^{ij} = \frac{1}{\rho_0}\left(n^{ij} - \frac{\nu}{1+\nu\tilde{b}^2} b^i b^j\right). \tag{4.11}$$

From (4.7), (4.8) and (4.11), we get

$$
\begin{aligned}
\bar{g}^{ij} =& \frac{1}{\rho_0}\Bigg\{g^{ij} + \frac{1}{F}\left[-\frac{\lambda}{X} - \frac{\mu}{Y}\left(-\frac{\lambda}{X}\left(1+\frac{\beta}{F}\right)+\frac{\lambda^2}{X^2}\left(1+\frac{\beta}{F}\right)^2\right)\right]\left(y^i b^j + y^j b^i\right) \\
&+ \frac{1}{F^2}\left[-\frac{\lambda}{X} - \frac{\mu}{Y}\left(1-\frac{\lambda}{X}\left(1+\frac{\beta}{F}\right)\right)^2\right] y^i y^j \\
&- \left[\frac{\lambda}{X} + \frac{\mu}{Y}\frac{\lambda^2}{X^2}\left(1+\frac{\beta}{F}\right)^2 + \frac{\nu}{1+\nu\tilde{b}^2}\right] b^i b^j\Bigg\},
\end{aligned} \tag{4.12}
$$

where the values of λ, μ and ν are obtained by (4.2), X by (4.3), Y by (4.6) and $\tilde{b}$ by (4.10).

Above discussion leads to the following theorem:

[DOI: 10.61186/ijmsi.18.2.31]

Theorem 4.1. *Let (M,F) be an n-dimensional Finsler space. Let $\bar{F}=f(F,\beta)+\beta$ be a Finsler metric obtained by Randers-β change of a Finsler metric $F=f(\alpha,\beta)$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form. If $\bar{g}_{ij}$ is the fundamental metric tensor of $\bar{F}$ given by (4.1), then its inverse fundamental metric tensor $\bar{g}^{ij}$ is given by (4.12).*

5. Projective flatness of Finsler metrics

The notion of projective flatness is one of the important topics in differential geometry. The real starting point of the investigation of projectively flat metrics is Hilbert's fourth problem [8]. During the International Congress of Mathematicians, held in Paris (1900), Hilbert asked about the spaces in which the shortest curves between any pair of points are straight lines. The first answer was given by Hilbert's student Hamel in 1903. In [7], Hamel found the necessary and sufficient conditions in order that a space satisfying a system of axioms, which is a modification of Hilbert's system of axioms for Euclidean geometry, be projectively flat. After Hamel, many authors (see [3, 15, 16, 19, 24]) have worked on this topic. Here, we find necessary and sufficient conditions for Randers-β changed (α,β)-metrics, namely Kropina-Randers changed, generalized Kropina-Randers changed, and square-Randers changed metrics to be projectively flat. For this, first we discuss some definitions and results related to projective flatness.

Definition 5.1. Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively equivalent if they have same geodesics as point sets,
i.e., for any geodesic $\bar{\sigma}(\bar{t})$ of $\bar{F}$, there is a geodesic $\sigma(t):=\bar{\sigma}(\bar{t}(t))$ of F, where $\bar{t}=\bar{t}(t)$ is oriented re-parametrization, and vice-versa.

Theorem 5.2. [6, 7] *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. Then F is projectively equivalent to $\bar{F}$ if and only if*

$$F_{x^iy^j}y^i-F_{x^j}=0.$$

Here, the spray coefficients of both the metrics are related by $G^i=\bar{G}^i+Py^i$, where $P=\dfrac{F_{x^k}y^k}{2F}$ is called projective factor of F.

It is well known that if $\bar{F}$ is a standard Euclidean norm on $\mathbb{R}^n$, then spray coefficients of $\bar{F}$ vanish identically, i.e., $\bar{G}^i=0$. As geodesics are straight lines in $\mathbb{R}^n$, we have the following:

Definition 5.3. For a Finsler metric F on an open subset $\mathcal{U}$ of $\mathbb{R}^n$, the geodesics of F are straight lines if and only if the spray coefficients satisfy

$$G^i=Py^i,$$

where P is same as defined in Theorem 5.2.

[DOI: 10.61186/ijmsi.18.2.31]

Definition 5.4. [6] A Finsler metric F on an open subset $\mathcal{U}$ of $\mathbb{R}^n$ is called projectively flat if and only if all the geodesics are straight lines in $\mathcal{U}$, and a Finsler metric F on a manifold M is called locally projectively flat, if at any point, there is a local co-ordinate system (x^i) in which F is projectively flat.

Therefore, by Theorem 5.2 and Definition 5.4, we have

Theorem 5.5. *A Finsler metric F on an open subset $\mathcal{U}$ of $\mathbb{R}^n$ is projectively flat if and only if it satisfies the following system of differential equations*

$$F_{x^i y^j} y^i - F_{x^j} = 0.$$

Firstly, we find necessary and sufficient conditions for $\bar{F}$, Kropina-Randers change of square root Finsler metric F, i.e.,

$$\bar{F} = \frac{F^2}{\beta} + \beta$$

to be projectively flat.
Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = \frac{A}{\beta} + \beta. \tag{5.1}$$

Differentiating (5.1) w.r.t. x^i, we get

$$\bar{F}_{x^i} = \frac{A_{x^i}}{\beta} - \frac{A}{\beta^2}\beta_{x^i} + \beta_{x^i}. \tag{5.2}$$

Differentiation of (5.2) further w.r.t. y^j gives

$$\bar{F}_{x^i y^j} = \frac{A_{x^i y^j}}{\beta} - \frac{A_{x^i}}{\beta^2}\beta_j - \frac{A}{\beta^2}\beta_{x^i y^j} - \frac{A_j}{\beta^2}\beta_{x^i} + \frac{2A}{\beta^3}\beta_{x^i}\beta_j + \beta_{x^i y^j}. \tag{5.3}$$

Contracting (5.3) with y^i, we get

$$\begin{aligned}\bar{F}_{x^i y^j} y^i &= \frac{A_{0j}}{\beta} - \frac{A_0\beta_j}{\beta^2} - \frac{A\beta_{0j}}{\beta^2} - \frac{A_j\beta_0}{\beta^2} + \frac{2A\beta_0\beta_j}{\beta^3} + \beta_{0j} \\ &= \frac{1}{\beta^3}\left\{\beta^2 A_{0j} - \beta A_0\beta_j - \beta A\beta_{0j} - \beta A_j\beta_0 + 2A\beta_0\beta_j + \beta^3\beta_{0j}\right\}.\end{aligned}$$

From the equation (5.2), we have

$$\bar{F}_{x^j} = \frac{A_{x^j}}{\beta} - \frac{A}{\beta^2}\beta_{x^j} + \beta_{x^j} = \frac{1}{\beta^3}\left\{\beta^2 A_{x^j} - A\beta\beta_{x^j} + \beta^3\beta_{x^j}\right\}.$$

We know that $\bar{F}$ is projectively flat if and only if

$$\bar{F}_{x^i y^j} y^i - \bar{F}_{x^j} = 0. \tag{5.4}$$

Substituting the values of $\bar{F}_{x^i y^j} y^i$ and $\bar{F}_{x^j}$ in (5.4), and simplifying, we get

$$A\left\{2\beta_0\beta_j + \beta\left(\beta_{x^j} - \beta_{0j}\right)\right\} + \left\{\beta^3\left(\beta_{0j} - \beta_{x^j}\right) + \beta^2\left(A_{0j} - A_{x^j}\right) - \beta\left(A_0\beta_j + A_j\beta_0\right)\right\} = 0.$$

From the above equation, we conclude that $\bar{F}$ is projectively flat if and only if following two equations are satisfied:

$$2\beta_0\beta_j + \beta\left(\beta_{x^j} - \beta_{0j}\right) = 0. \tag{5.5}$$

$$\beta^3\left(\beta_{0j} - \beta_{x^j}\right) + \beta^2\left(A_{0j} - A_{x^j}\right) - \beta\left(A_0\beta_j + A_j\beta_0\right) = 0. \tag{5.6}$$

Above discussion leads to the following theorem:

Theorem 5.6. *Let $(M, \bar{F})$ be an n-dimensional Finsler space with $\bar{F} = \dfrac{F^2}{\beta} + \beta$ as a Kropina-Randers change of square root Finsler metric $F = \sqrt{A}$. Then $\bar{F}$ is projectively flat if and only if equations (5.5) and (5.6) are satisfied.*

Example 5.7. Let M be a surface (plane) embedded in the Euclidean space $\mathbb{R}^3$, i.e., $M \longrightarrow \mathbb{R}^3$ by

$$x = (x^1,\ x^2) \mapsto (x^1,\ x^2,\ z = f(x^1,\ x^2)), \tag{5.7}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function given by

$$f(x^1,\ x^2) = p\ x^1 + q\ x^2 + r,$$

p, q, r are constants. M is the graph of $z = f(x^1,\ x^2)$ with Riemannian metric [12] $\alpha = \sqrt{A} = \sqrt{a_{ij}(x)y^iy^j}$ induced from $\mathbb{R}^3$ as follows :

$$(a_{\mathrm{ij}}) = \begin{pmatrix} 1 + f_{x^1}^2 & f_{x^1}f_{x^2} \\ 1 + f_{x^2}^2 & f_{x^1}f_{x^2} \end{pmatrix}. \tag{5.8}$$

Choose a 1-form $\beta = f_{x^1}y^1 + f_{x^2}y^2$ on M.

We obtain Kropina-Randers change of square root Finsler metric $F = \sqrt{A}$ as follows:

$$\bar{F} = \frac{F^2}{\beta} + \beta = \frac{A}{\beta} + \beta = \frac{\alpha^2}{\beta} + \beta,$$

where

$$\alpha = \sqrt{(1 + (f_{x^1})^2)(y^1)^2 + 2f_{x^1}f_{x^2}y^1y^2 + (1 + (f_{x^2})^2)(y^2)^2},$$
$$\beta = f_{x^1}y^1 + f_{x^2}y^2,$$

i.e.,

$$\alpha = \sqrt{(1 + p^2)(y^1)^2 + 2\ p\ q\ y^1y^2 + (1 + q^2)(y^2)^2},$$
$$\beta = p\ y^1 + q\ y^2.$$

Now, we calculate entities required to check projective flatness of $\bar{F}$ as follows:

$$A = \alpha^2 = (1 + p^2)(y^1)^2 + 2\ p\ q\ y^1y^2 + (1 + q^2)(y^2)^2,\ \beta = p\ y^1 + q\ y^2.$$

$$\begin{aligned}
&A_{x^j} = 0, && A_0 = A_{x^i}y^i = 0, && A_{0j} = A_{x^iy^j}y^i = 0,\\
&\beta_{x^j} = 0, && \beta_0 = \beta_{x^i}y^i = 0, && \beta_{0j} = A_{x^iy^j}y^i = 0,\\
&A_j = A_{y^j}, && \beta_j = \beta_{y^j}, &&\\
&\beta_1 = \beta_{y^1} = p, && \beta_2 = \beta_{y^2} = q, &&
\end{aligned} \tag{5.9}$$

[DOI: 10.61186/ijmsi.18.2.31]

$$\begin{aligned} A_1 &= A_{y^1} = 2(1+p^2)\ y^1 + 2\ p\ q\ y^2, \\ A_2 &= A_{y^2} = 2(1+q^2)\ y^2 + 2\ p\ q\ y^1. \end{aligned} \tag{5.10}$$

The conditions (5.5) and (5.6) are clearly satisfied by these entities. Therefore, the space M with the metric under consideration is projectively flat.

Next, we find necessary and sufficient conditions for $\bar{F}$, generalized Kropina-Randers change of square root Finsler metric F, i.e.,

$$\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta \ \ (m \neq 0, -1)$$

to be projectively flat.
Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = \frac{A^{(m+1)/2}}{\beta^m} + \beta. \tag{5.11}$$

Differentiating (5.11) w.r.t. x^i, we get

$$\bar{F}_{x^i} = \frac{m+1}{2\beta^m} A^{(m-1)/2} A_{x^i} - \frac{m}{\beta^{m+1}} A^{(m+1)/2} \beta_{x^i} + \beta_{x^i}. \tag{5.12}$$

Differentiation of (5.12) further w.r.t. y^j gives

$$\begin{aligned} \bar{F}_{x^i y^j} =& \frac{m+1}{2\beta^m} A^{(m-1)/2} A_{x^i y^j} + \frac{m^2-1}{4\beta^m} A^{(m-3)/2} A_{x^i} A_j \\ & - \frac{m(m+1)}{2\beta^{m+1}} A^{(m-1)/2} A_{x^i} \beta_j - \frac{m}{\beta^{m+1}} A^{(m+1)/2} \beta_{x^i y^j} \\ & - \frac{m(m+1)}{2\beta^{m+1}} A^{(m-1)/2} \beta_{x^i} A_j + \frac{m(m+1)}{\beta^{m+2}} \beta_j A^{(m+1)/2} \beta_{x^i} + \beta_{x^i y^j}. \end{aligned} \tag{5.13}$$

Contracting (5.13) with y^i, we get

$$\begin{aligned} \bar{F}_{x^i y^j} y^i =& \frac{m+1}{2\beta^{m+2}} A^{(m-3)/2} \Big\{ \beta^2 A A_{0j} + \frac{m-1}{2} \beta^2 A_0 A_j - m\beta A A_0 \beta_j \\ & - \frac{2m}{m+1} \beta A^2 \beta_{0j} - m\beta A \beta_0 A_j + 2m\beta_j A^2 \beta_0 + \frac{2}{m+1} \beta^{m+2} A^{(-m+3)/2} \beta_{0j} \Big\}. \end{aligned}$$

From the equation (5.12), we get

$$\bar{F}_{x^j} = \frac{m+1}{2\beta^{m+2}} A^{(m-3)/2} \Big\{ \beta^2 A A_{x^j} - \frac{2m}{m+1} \beta A^2 \beta_{x^j} + \frac{2}{m+1} \beta^{m+2} A^{(-m+3)/2} \beta_{x^j} \Big\}.$$

Substituting the above values in equation (5.4) and simplifying, we get

$$\begin{aligned} & \frac{m-1}{2} \beta^2 A_0 A_j + A \left\{ \beta^2 \left(A_{0j} - A_{x^j} \right) - m\beta \left(A_0 \beta_j + \beta_0 A_j \right) \right\} \\ & + 2mA^2 \left\{ \frac{\beta}{m+1} \left(\beta_{x^j} - \beta_{0j} \right) + \beta_j \beta_0 \right\} + \frac{2}{m+1} \beta^{m+2} A^{(-m+3)/2} \left(\beta_{0j} - \beta_{x^j} \right) = 0. \end{aligned}$$

[DOI: 10.61186/ijmsi.18.2.31]

From the above equation, we conclude that $\bar{F}$ is projectively flat if and only if following four equations are satisfied:

$$\frac{m-1}{2}\beta^2 A_0 A_j = 0. \tag{5.14}$$

$$\beta^2 \left(A_{0j} - A_{x^j}\right) - m\beta \left(A_0\beta_j + \beta_0 A_j\right) = 0. \tag{5.15}$$

$$\frac{\beta}{m+1}\left(\beta_{x^j} - \beta_{0j}\right) + \beta_j\beta_0 = 0. \tag{5.16}$$

$$\beta_{0j} = \beta_{x^j}. \tag{5.17}$$

Further, from the equation (5.14), we see that
either $m = 1$ or $A_0 A_j = 0$.
Now, if $m = 1$, then (5.16) reduces to (5.5) and (5.15), (5.17) reduce to (5.6).
But the equations (5.5) and (5.6) are necessary and sufficient conditions for Kropina-Randers change of square root Finsler metric to be projectively flat.
Therefore, we exclude the case $m = 1$ for general case.
Then

$$A_0 A_j = 0. \tag{5.18}$$

Also from the equations (5.16) and (5.17), we get

$$\beta_0\beta_j = 0. \tag{5.19}$$

Above discussion leads to the following theorem:

Theorem 5.8. *Let $(M, \bar{F})$ be an n-dimensional Finsler space with $\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta$ $(m \neq -1, 0, 1)$ as generalized Kropina-Randers changed of square root Finsler metric $F = \sqrt{A}$. Then $\bar{F}$ is projectively flat if and only if equations (5.15), (5.17), (5.18) and (5.19) are satisfied.*

Example 5.9. Take surface M, Riemannian metric α, and 1-form β as defined in Example 5.7. We obtain generalized Kropina-Randers change of square root Finsler metric $F = \sqrt{A}$ as follows:

$$\bar{F} = \frac{F^{m+1}}{\beta^m} + \beta = \frac{A^{(m+1)/2}}{\beta^m} + \beta = \frac{\alpha^{m+1}}{\beta^m} + \beta, \quad (m \neq -1, 0, 1).$$

The conditions (5.15), (5.17), (5.18) and (5.19) are clearly satisfied by all the entities in (5.9) and (5.10). Therefore, the surface M with generalized Kropina-Randers change of Finsler square root metric is projectively flat.

Next, we find necessary and sufficient conditions for $\bar{F}$, square-Randers change of square root Finsler metric F, i.e.,

$$\bar{F} = \frac{(F+\beta)^2}{F} + \beta$$

[DOI: 10.61186/ijmsi.18.2.31]

to be projectively flat.
Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = A^{1/2} + \frac{\beta^2}{A^{1/2}} + 3\beta. \tag{5.20}$$

Differentiating (5.20) w.r.t. x^i, we get

$$\bar{F}_{x^i} = \frac{1}{2}A^{-1/2}A_{x^i} + 2\beta A^{-1/2}\beta_{x^i} - \frac{1}{2}\beta^2 A^{-3/2}A_{x^i} + 3\beta_{x^i}. \tag{5.21}$$

Differentiation of (5.21) w.r.t. y^j gives

$$\begin{aligned}\bar{F}_{x^i y^j} =& \frac{1}{2}A^{-1/2}A_{x^i y^j} - \frac{1}{4}A^{-3/2}A_j A_{x^i} + 2\beta A^{-1/2}\beta_{x^i y^j} + 2\beta_j A^{-1/2}\beta_{x^i} \\ & - \beta A^{-3/2}A_j\beta_{x^i} - \frac{1}{2}\beta^2 A^{-3/2}A_{x^i y^j} - \beta\beta_j A^{-3/2}A_{x^i} + \frac{3}{4}\beta^2 A^{-5/2}A_j A_{x^i} \\ & + 3\beta_{x^i y^j}.\end{aligned} \tag{5.22}$$

Contracting (5.22) with y^k, we get

$$\begin{aligned}\bar{F}_{x^i y^j}y^i =& \frac{1}{4}A^{-5/2}\Big[2A^2A_{0j} - AA_jA_0 + 8\beta A^2\beta_{0j} + 8\beta_j A^2\beta_0 - 4\beta AA_j\beta_0 \\ & - 2\beta^2 AA_{0j} - 4\beta\beta_j AA_0 + 3\beta^2 A_jA_0 + 12A^{5/2}\beta_{0j}\Big].\end{aligned}$$

From the equation (5.21), we get

$$\bar{F}_{x^j} = \frac{1}{4}A^{-5/2}\left[2A^2A_{x^j} + 8\beta A^2\beta_{x^j} - 2\beta^2 AA_{x^j} + 12A^{5/2}\beta_{x^j}\right].$$

Substituting the values $\bar{F}_{x^i y^j}y^i$ and $\bar{F}_{x^j}$ in (5.4) and simplifying, we get

$$\begin{aligned}&3\beta^2 A_jA_0 + 2A\left\{\beta^2\left(A_{x^j} - A_{0j}\right) - 2\beta\left(A_j\beta_0 + \beta_jA_0\right) - A_jA_0\right\} \\ &+ 2A^2\left\{A_{0j} - A_{x^j} + 4\beta_j\beta_0 + 4\beta\left(\beta_{0j} - \beta_{x^j}\right)\right\} + 12A^{5/2}\left\{\beta_{0j} - \beta_{x^j}\right\} = 0.\end{aligned}$$

From the above equation, we conclude that $\bar{F}$ is projectively flat if and only if following four equations are satisfied:

$$A_jA_0 = 0. \tag{5.23}$$

$$\beta^2\left(A_{x^j} - A_{0j}\right) - 2\beta\left(A_j\beta_0 + \beta_jA_0\right) - A_jA_0 = 0. \tag{5.24}$$

$$4\beta\left(\beta_{0j} - \beta_{x^j}\right) + A_{0j} - A_{x^j} + 4\beta_j\beta_0 = 0. \tag{5.25}$$

$$\beta_{0j} = \beta_{x^j}. \tag{5.26}$$

Above discussion leads to the following theorem:

Theorem 5.10. *Let $(M, \bar{F})$ be an n-dimensional Finsler space with $\bar{F} = \frac{(F+\beta)^2}{F} + \beta$ as a square-Randers change of square root Finsler metric $F = \sqrt{A}$. Then $\bar{F}$ is projectively flat if and only if the equations (5.23), (5.24), (5.25), and (5.26) are satisfied.*

Example 5.11. Take surface M, Riemannian metric $\alpha,$ and 1-form β as defined in Example 5.7. We obtain square-Randers change of square root Finsler metric $F=\sqrt{A}$ as follows:

$$\bar{F}=\frac{(F+\beta)^2}{F}+\beta=A^{1/2}+\frac{\beta^2}{A^{1/2}}+3\beta=\alpha+\frac{\beta^2}{\alpha}+3\beta.$$

One can easily see that the conditions (5.23), (5.24), (5.25), and (5.26) are satisfied by all the entities in (5.9) and (5.10). Therefore, the surface M with metric under consideration is projectively flat.

Acknowledgements

Authors are thankful to the referees for their valuable comments and suggestions. First author is thankful to Central University of Punjab, Bathinda for providing financial assistance as a Research Seed Money grant vide the letter no. CUPB/CC/17/369. Second author is very much thankful to UGC for providing financial assistance in terms of JRF fellowship vide letter with Sr. No. 2061641032 and Ref. No. 19/06/2016(i)EU-V.

References

1. S. I. Amari, H. Nagaoka, *Methods of Information Geometry*, Translations of Mathematical Monographs, AMS, 191, Oxford Univ. Press, 2000.
2. P. L. Antonelli, R. S. Ingarden, M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Vol. 58, Springer Science & Business Media, 2013.
3. S. A. Baby, G. Shanker, On Randers-conformal Change of Finsler Space with Special (α,β)-metrics, *International Journal of Contemporary Mathematical Sciences*, **11**(9), (2016), 415-423.
4. D. Bao, S. S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, New York, 2000.
5. S. S. Chern, Finsler Geometry is just Riemannian Geometry without Quadratic Restriction, *Notices of AMS*, **43**(9), (1996), 959-963.
6. S. S. Chern, Z. Shen, *Riemann-Finsler Geometry, Nankai Tracts in Mathematics*, Vol. 6, World Scientific, Singapore, 2005.
7. G. Hamel, Über die Geometrieen, in denen die Geraden die Kürzesten sind, *Mathematische Annalen*, **57**(2), (1903), 231-264.
8. D. Hilbert, *Mathematische Probleme, In Gesammelte Abhandlungen*, Springer, Berlin, Heidelberg, 1970, 290-329.
9. M. Matsumoto, On C-reducible Finsler-spaces, *Tensor N. S.*, **24**, (1972), 29-37.
10. M. Matsumoto, On Finsler Spaces with Randers Metric and Special Forms of Important Tensors, *J. Math. Kyoto Univ.*, **14**(3), (1974), 477-498.
11. M. Matsumoto, On Finsler Spaces with 1-form Metric II, Berwald-Moor's metric $L=(y^1y^2...y^n)^{1/n}$, *Tensor N. S.*, **32**, (1978), 275-278.
12. M. Matsumoto, A Slope of Mountain is a Finsler Surface with Respect to a Time Measure, *Journal of Mathematics of Kyoto University*, **29**(1), (1989), 17-25.
13. S. Rani, G. Shanker, On S-curvature of a Homogeneous Finsler Space with Randers Changed Square Metric, *Facta Universitatis, Series: Mathematics and Informatics*, **35**(3), (2020), 673-691.

[DOI: 10.61186/ijmsi.18.2.31]

14. S. Rani, G. Shanker, On CW-translations of a Homogeneous Finsler Space with (α, β)-metrics, *To appear in Balkan Journal of Geometry and Its Applications*.
15. G. Shanker, S. A. Baby, On the Riemann and Ricci Curvature of Finsler Spaces with Special (α, β)-metric, *International Journal of Mathematical Combinatorics*, **4**, (2016), 61-69.
16. G. Shanker, P. Gupta, Projectively Flat Finsler Spaces with Special (α, β)-metric, *Journal of International Academy of Physical Sciences*, **17**(4), (2013), 369-376.
17. G. Shanker, S. Rani, On S-curvature of a Homogeneous Finsler Space with Square Metric, *International Journal of Geometric Methods in Modern Physics*, **17**(2), (2020), 2050019 (16 pages).
18. G. Shanker, S. Rani, K. Kaur, Dually Flat Finsler Spaces Associated with Randers-β Change, *Journal of Rajasthan Academy of Physical Sciences*, **18**(1-2), (2019), 95-106.
19. G. Shanker, R. K. Sharma, R. D. S. Kushwaha, On the Matsumoto Change of a Finsler Space with m^{th} Root Metric, *Acta Matematica Academie Paedagogiace Nyiregyháziensis*, **32**(2), (2017), 255-264.
20. C. Shibata, On Invariant Tensors of β-change of Finsler Metrics, *Journal of Mathematics of Kyoto University*, **24**(1), (1984), 163-188.
21. H. Shimada, On Finsler Spaces with the Metric $L = \sqrt[m]{a_{i_1 i_2 \ldots i_m} y^{i_1} y^{i_2} \ldots y^{i_m}}$, *Tensor N. S.*, **33**, (1979), 365-372.
22. A. Tayebi, E. Peyghan, On $C3$-Like Finsler Metrics, *Iranian Journal of Mathematical Sciences and Informatics*, **7**(1), (2012), 1-6.
23. B. Tiwari, M. Kumar, On Randers Change of a Finsler Space with m^{th}-root Metric, *International Journal of Geometric Methods in Modern Physics*, **11**(10), (2014), 1450087 (13 pages).
24. R. Yadav, G. Shanker, On Some Projectively Flat (α, β)-metrics, *Gulf Journal of Mathematics*, **1**, (2013), 72-77.

[DOI: 10.61186/ijmsi.18.2.31]