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Common Best Proximity Points for Proximal Weak Commuting Mappings in Metric Spaces

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Abstract. In this paper, we initiate the concept of proximal weak commute mappings and establish some new existence of common best proximity point theorems for this class of mappings. Also, we provide interesting example to illustrate our main results.

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1. Introduction

In Mathematics, the fixed point theory is a key to identify the solutions of equations of the form $\Lambda\xi = \xi$ for a given mapping $\Lambda : X \to X$, where X is assumed as metric space with metric D or normed linear space. Suppose M and N are subsets of X and the map Λ from M to N, then the equation $\Lambda\xi = \xi$ need not have a solution. In such a situation, we seek an element ξ, which tends that $D(\xi, \Lambda\xi)$ is minimum. Hence the best proximity point theorems are helpful to find the sufficient conditions to minimize the quantity $D(\xi, \Lambda\xi)$. In other

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words, the results on best proximity point theorem give sufficient conditions to find an $\xi \in M$, such that $D(\xi, \Lambda\xi) = D(M, N)$, called best proximity point.

One can refer for such a existence results of best proximity point for different nature of contractions in [2, 3, 5, 7, 12, 15, 16, 17]. Recently, the existence of best proximity point for $\Lambda : M \cup N \to M \cup N$, relatively non expansive mappings were proved by Eldered *et al* [6]. In [4], Anuradha and Veeramani have given an existence proof of best proximity point for proximal pointwise contraction mappings.

Jungck [10], obtained an existence result of common fixed point for commuting mappings which is generalization of Banach's fixed point thoerem. Sessa [18] introduced weak commutativity and derived common fixed point theorem. Later, Jungck [11] defined compatibility and obtained sufficient conditions for existence of common fixed point. On the other hand, Renu Chugh and Sanjay Kumar [14] have established the results on common fixed point for compatible mappings in weak sense. For four mappings, Parvaneh Lo'lo' *et al* [13] have investigated common best proximity point theorems in metric type space.

Motivated by the work of Parvaneh Lo'lo' *et al* [13], in this article, we establish the concept of proximal weak commute mappings. Also, we obtain sufficient conditions and claimed existence of common best proximity point for proximal weak commute mappings. Further, our results are more general for non-self mappings to the corresponding results of self mappings in [13].

2. Preliminaries

We first give some tools which help to our work: We consider (X, D) is metric space and $M, N \subset X$.

$$P_M(\xi) = \{\eta \in M : D(\xi, \eta) = D(\xi, M)\};$$
$$D(M, N) = inf\{D(\xi, \eta) : \xi \in M, \eta \in N\};$$
$$M_0 = \{\xi \in M : D(\xi, \eta') = dist(M, N) \text{ for some } \eta' \in N\};$$
$$N_0 = \{\eta \in N : D(\xi', \eta) = dist(M, N) \text{ for some } \xi' \in M\}.$$

Here one can note that the pair (M_0, N_0) may be empty. For, if $M = (0, 1), N = (2, 3)$ in the metric space $(\mathbb{R}, D)$ where $D(\xi, \eta) = |\xi - \eta|$. Then there are no $\xi \in M$ and $\eta \in N$ such that $D(\xi, \eta) = 1 = dist(M, N)$. Suppose (M, N) is a bounded, closed and convex pair in a reflexive Banach space X, then (M_0, N_0) is also nonempty, closed and convex (see [8]), it enssures the pair (M_0, N_0) is nonempty.

Definition 2.1. An element $\xi \in M$ is called a common best proximity point of $\Phi_1, \Phi_2, ..., \Phi_n : M \to N$ if it satisfies,

$$D(\xi, \Phi_1\xi) = D(\xi, \Phi_2\xi) = ... = D(\xi, \Phi_n\xi) = D(M, N).$$

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Definition 2.2. ([16]) The mappings $\Gamma : M \to N$ and $\Lambda : M \to N$ are said to be commute proximally if they satisfy,

$$[D(\upsilon, \Gamma\xi) = D(\nu, \Lambda\xi) = D(M, N)] \Rightarrow \Gamma\nu = \Lambda\upsilon$$

where $\xi, \upsilon, \nu \in M$.

Here, we introduce the proximal weak commute mappings.

Definition 2.3. Let $\Gamma : M \to N$ and $\Lambda : M \to N$ be two nonself mappings. Then the pair $\{\Gamma, \Lambda\}$ is said to be proximal weak commute if they satisfy,

$$\begin{cases} \Gamma\xi = \Lambda\xi \\ D(\upsilon, \Gamma\xi) = D(M, N) \\ D(\nu, \Lambda\xi) = D(M, N) \end{cases} \Rightarrow \Gamma\nu = \Lambda\upsilon$$

where $\xi, \upsilon, \nu \in M$.

It is easy to observe that the notion of proximal weak commute reduces to weakly commute mappings when the mappings are self.

The following example shows that proximal weak commute mappings need not possess the condition of commute proximally mappings.

Example 2.4. Let $X = [0, 1] \times [2, 20]$. Define $D_1\big((\xi_1, \xi_2), (\eta_1, \eta_2)\big) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$. Let

$$M = \{(0, \xi) \in X : 2 \leq \xi \leq 20\}, \ N = \{(1, \eta) \in X : 2 \leq \eta \leq 20\}.$$

Then $D_1(M, N) = 1$. Let $\Phi, \Psi : M \to N$ defined by

$$\Phi(0, \xi) = \begin{cases} (1, \xi), & \xi = 2 \\ (1, 6), & 2 < \xi \leq 5 \\ (1, 2), & \xi > 5 \end{cases}, \Psi(0, \xi) = \begin{cases} (1, \xi), & \xi = 2 \\ (1, 12), & 2 < \xi \leq 5 \\ (1, \xi - 3), & \xi > 5. \end{cases}$$

Choose $(0, 5 + \frac{1}{2}) \in M$, then $D_1\big((0, 2), \Phi(0, 5 + \frac{1}{2})\big) = 1 = D_1\big((0, 2 + \frac{1}{2}), \Psi(0, 5 + \frac{1}{2})\big)$. But $\Phi(0, 2 + \frac{1}{2}) = (1, 6)$ and $\Psi(0, 2) = (1, 2)$. Therefore Φ and Ψ are not commute proximally. Now Φ and Ψ are coinciding at $(0, 2)$. And the point $(0, 2)$, which is the only point satisfying

$$D_1\big((0, 2), \Phi(0, 2)\big) = 1 = D_1\big((0, 2), \Psi(0, 2)\big)$$

Clearly, $\Phi(0, 2) = \Psi(0, 2)$. Then $\{\Phi, \Psi\}$ is proximal weak commute pair. Also, one can note that Φ and Ψ are fails in continuity at $(0, 2)$ and $(0, 5)$.

Definition 2.5. ([17]) Let $M_0 \neq \emptyset$ then the pair (M, N) is said to have P-property if for any $\xi_1, \xi_2 \in M_0$ and $\eta_1, \eta_2 \in N_0$

$$\begin{cases} D(\xi_1, \eta_1) = D(M, N) \\ D(\xi_2, \eta_2) = D(M, N) \end{cases} \Rightarrow D(\xi_1, \xi_2) = D(\eta_1, \eta_2).$$

Example 2.6. ([17]) Let M, N be two nonempty closed, and convex subsets of a Hilbert space X. Then (M, N) satisfies the P-property.

Example 2.7. ([1]) Let M, N be two nonempty, bounded, closed, and convex subsets of a uniformly convex Banach space X. Then (M, N) has the P-property.

3. Main Results

Let F be collection of all funtions $\chi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ such that χ is non-decreasing in all coordinate, upper semi-continuous and, for all $s > 0$,

$$\chi(s, s, 0, \alpha s, 0) \leq \beta s, \quad \chi(s, s, 0, 0, \alpha s) \leq \beta s,$$

with $\beta < 1$ for $\alpha < 2$, $\beta = 1$ for $\alpha = 2$,

$$\gamma(s) = \chi(s, s, \alpha_1 s, \alpha_2 s, \alpha_3 s) < s,$$

where $\gamma(s) : \mathbb{R}^+ \to \mathbb{R}^+$ and $\alpha_1 + \alpha_2 + \alpha_3 = 4$.

Lemma 3.1. [19] *For $s > 0, \gamma(s) < s$ iff $\lim_n \gamma^n(s) = 0$, where γ^n represents composition of γ at n times.*

Let M, N be two nonempty subsets of a metric space (X, D). Let Φ, Ψ, Γ and Λ be non-self mappings from M to N fulfilling the following conditions:

$$\Phi(M_0) \subset N_0 \text{ and } \Psi(M_0) \subset N_0, \tag{3.1}$$

$$\Phi(M_0) \subset \Lambda(M_0) \text{ and } \Psi(M_0) \subset \Gamma(M_0), \tag{3.2}$$

$$D(\Phi\xi, \Psi\eta) \leq \chi\big(D(\Gamma\xi, \Lambda\eta), D(\Phi\xi, \Gamma\xi), D(\Psi\eta, \Lambda\eta), D(\Phi\xi, \Lambda\eta), D(\Psi\eta, \Gamma\xi)\big) \tag{3.3}$$

for all $\xi, \eta \in M$, where $\chi \in F$.

Let $\xi_0 \in M_0$, since $\Phi(M_0) \subset \Lambda(M_0)$, then there exists $\xi_1 \in M_0$ such that $\Phi(\xi_0) = \Lambda(\xi_1)$. Similarly, we can pick an element $\xi_2 \in M_0$ such that $\Psi(\xi_1) = \Gamma(\xi_2)$. Continuing this procedure, we acquire a sequence $\{\xi_n\}$ in M_0 such that $\Phi(\xi_{2n}) = \Lambda(\xi_{2n+1})$ and $\Psi(\xi_{2n+1}) = \Gamma(\xi_{2n+2})$.
Since $\Phi(M_0) \subset N_0$ and $\Psi(M_0) \subset N_0$, there exists $\{\upsilon_n\} \subset M_0$ such that

$$D\big(\upsilon_{2n}, \Phi(\xi_{2n})\big) = D(M, N) \text{ and } D\big(\upsilon_{2n+1}, \Psi(\xi_{2n+1})\big) = D(M, N). \tag{3.4}$$

Therefore,

$$\begin{aligned} D\big(\upsilon_{2n}, \Phi\xi_{2n}\big) &= D\big(\upsilon_{2n}, \Lambda\xi_{2n+1}\big) = D\big(\upsilon_{2n+1}, \Psi\xi_{2n+1}\big) \\ &= D\big(\upsilon_{2n+1}, \Gamma\xi_{2n+2}\big) = D(M, N). \end{aligned} \tag{3.5}$$

Lemma 3.2. *Let (M, N) has the P-property. Then $\lim_n D(\upsilon_n, \upsilon_{n+1}) = 0$, where $\{\upsilon_n\}$ is the sequence as in (3.5).*

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Proof. Let $D_n = D(\upsilon_n, \upsilon_{n+1}), n = 0, 1, 2, ...$ First, we show that $\{D_n\}$ is non-increasing sequence in $\mathbb{R}^+$, that is, $D_n \leq D_{n-1}$ for $n = 1, 2, 3, ...$
From the inequality (3.3), and using P-property of (M, N), we have

$$\begin{aligned} D(\upsilon_{2n}, \upsilon_{2n+1}) &= D(\Phi\xi_{2n}, \Psi\xi_{2n+1}) \\ &\leq \chi\big(D(\Gamma\xi_{2n}, \Lambda\xi_{2n+1}), D(\Phi\xi_{2n}, \Gamma\xi_{2n}), D(\Lambda\xi_{2n+1}, \Psi\xi_{2n+1}), \\ &\quad D(\Phi\xi_{2n}, \Lambda\xi_{2n+1}), D(\Gamma\xi_{2n}, \Psi\xi_{2n+1})\big) \\ &= \chi(D_{2n-1}, D_{2n-1}, D_{2n}, 0, D_{2n-1} + D_{2n}). \end{aligned}$$

If $D_{n-1} < D_n$ for some n. Then we have, for some $\alpha < 2$, $D_{n-1} + D_n = \alpha D_n$. Since χ is non-increasing in every coordinate and since $\beta < 1$ for some $\alpha < 2$, so we have,

$$D_{2n} \leq \chi(D_{2n}, D_{2n}, D_{2n}, 0, \alpha D_{2n}) \leq \beta D_{2n} < D_{2n}.$$

Similarly, the inequality $D_{2n+1} < D_{2n+1}$ holds. Hence, for n, $D_n \leq \beta D_n < D_n$, which gives a contradiction. This implies that the sequence $\{D_n\}$ is nonincreasing in $\mathbb{R}^+$. Also, from (3.3) and P- property, we have

$$\begin{aligned} D_1 = D(\upsilon_1, \upsilon_2) &= D(\Phi\xi_2, \Psi\xi_1) \\ &\leq \chi\big(D(\Gamma\xi_2, \Lambda\xi_1), D(\Phi\xi_2, \Gamma\xi_2), D(\Psi\xi_1, \Lambda\xi_1), \\ &\quad D(\Phi\xi_2, \Lambda\xi_1), D(\Psi\xi_1, \Gamma\xi_2)\big) \\ &= \chi(D_0, D_1, D_0, D_0 + D_1, 0) \\ &\leq \chi(D_0, D_0, D_0, 2D_0, D_0) \\ &= \gamma(D_0). \end{aligned}$$

In general, we obtain $D_n \leq \gamma^n(D_0)$, which implies that, if $D_0 > 0$, by Lemma 3.1,

$$\lim_n D_n \leq \lim_n \gamma^n(D_0) = 0.$$

Therefore, we have $\lim_n D_n = 0$. □

Lemma 3.3. *The sequence $\{\upsilon_n\}$ defined by (3.5) is a Cauchy sequence in M_0.*

Proof. Suppose that $\{\upsilon_{2n}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for every even integer $2l$, there exist even integers $2m(l)$ and $2n(l)$ with $2m(l) > 2n(l) \geq 2l$ such that

$$D(\upsilon_{2m(l)}, \upsilon_{2n(l)}) > \epsilon. \tag{3.6}$$

For each $2l$, let $2m(l)$ be the smallest even integer which is exceeding $2n(l)$ and holds (3.6), that is,

$$D(\upsilon_{2n(l)}, \upsilon_{2m(l)-2}) \leq \epsilon \text{ and } D(\upsilon_{2n(l)}, \upsilon_{2m(l)}) > \epsilon. \tag{3.7}$$

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Therefore for every even integer $2l$, we derive

$$\begin{aligned}\epsilon &\leq D(\upsilon_{2n(l)}, \upsilon_{2m(l)}) \\ &\leq D(\upsilon_{2n(l)}, \upsilon_{2m(l)-2}) + D(\upsilon_{2m(l)-2}, \upsilon_{2m(l)-1}) + D(\upsilon_{2m(l)-1}, \upsilon_{2m(l)}).\end{aligned}$$

Because of Lemma 3.2 and 3.7, we obtain

$$D(\upsilon_{2n(l)}, \upsilon_{2m(l)}) \to \epsilon \text{ as } l \to \infty. \tag{3.8}$$

Since triangle inequality, we get

$$|D(\upsilon_{2n(l)}, \upsilon_{2m(l)-1}) - D(\upsilon_{2n(l)}, \upsilon_{2m(l)})| \leq D(\upsilon_{2m(l)-1}, \upsilon_{2m(l)})$$

and

$$\begin{aligned}|D(\upsilon_{2n(l)+1}, \upsilon_{2m(l)-1}) &- D(\upsilon_{2n(l)}, \upsilon_{2m(l)})| \\ &\leq D(\upsilon_{2m(l)-1}, \upsilon_{2m(l)}) + D(\upsilon_{2n(l)}, \upsilon_{2n(l)+1}).\end{aligned}$$

And by Lemma 3.2 and 3.8, as $l \to \infty$,

$$D(\upsilon_{2n(l)}, \upsilon_{2m(l)-1}) \to \epsilon \text{ and } D(\upsilon_{2n(l)+1}, \upsilon_{2m(l)-1}) \to \epsilon. \tag{3.9}$$

Therefore, by (3.3) and equation (3.5), with P-property, we obtain

$$\begin{aligned}D(\upsilon_{2n(l)}, \upsilon_{2m(l)}) &\leq D(\upsilon_{2n(l)}, \upsilon_{2n(l)+1}) + D(\upsilon_{2n(l)+1}, \upsilon_{2m(l)}) \\ &= D(\upsilon_{2n(l)}, \upsilon_{2n(l)+1}) + D(\Phi\xi_{2m(l)}, \Psi\xi_{2n(l)+1}) \\ &\leq D(\upsilon_{2n(l)}, \upsilon_{2n(l)+1}) + \chi\big(D(\Gamma\xi_{2m(l)}, \Lambda\xi_{2n(l)+1}), \\ &\quad D(\Phi\xi_{2m(l)}, \Gamma\xi_{2m(l)}), D(\Psi\xi_{2n(l)+1}, \Lambda\xi_{2n(l)+1}), \\ &\quad D(\Phi\xi_{2m(l)}, \Lambda\xi_{2n(l)+1}), D(\Psi\xi_{2n(l)+1}, \Gamma\xi_{2m(l)})\big) \\ &= D(\upsilon_{2n(l)}, \upsilon_{2n(l)+1}) + \chi\big(D(\upsilon_{2m(l)-1}, \upsilon_{2n(l)}), \\ &\quad D(\upsilon_{2m(l)}, \upsilon_{2m(l)-1}), D(\upsilon_{2n(l)+1}, \upsilon_{2n(l)}), \\ &\quad D(\upsilon_{2m(l)}, \upsilon_{2n(l)}), D(\upsilon_{2n(l)+1}, \upsilon_{2m(l)-1})\big).\end{aligned} \tag{3.10}$$

Letting $l \to \infty$, and since χ is upper semi continuous, Lemma 3.2, (3.8),(3.9) and (3.10), implies

$$\epsilon \leq \chi(\epsilon, 0, 0, \epsilon, \epsilon) < \gamma(\epsilon) < \epsilon,$$

shows a contradiction. Therefore, $\{\upsilon_{2n}\}$ must be Cauchy sequence in M_0 and this proves $\{\upsilon_n\}$ is also Cauchy. □

Theorem 3.4. *Let M, N be non-empty subsets of a complete metric space (X, D). Moreover, assume that M_0 is non-empty, closed set. Let the non-self mappings $\Phi, \Psi, \Gamma, \Lambda : M \to N$ satisfy:*

1. *$\{\Phi, \Gamma\}$ and $\{\Psi, \Lambda\}$ are proximal weak commute pairs;*
2. *the pair (M, N) has the P-property;*
3. *Φ, Ψ, Γ and Λ satisfy (3.2) and (3.3);*
4. *$\Gamma(M_0) = N_0$ and $\Lambda(M_0) = N_0$.*

Then Φ, Ψ, Γ and Λ have a unique common best proximity point.

Proof. By Lemma 3.3, the sequence $\{\upsilon_n\}$ is a Cauchy in M_0. And since M_0 is complete, there exists $\upsilon \in M_0$ such that $\lim_n \upsilon_n = \upsilon$. From (3.4), $D(\upsilon_{2n}, \Phi\xi_{2n}) = D(\upsilon_{2n+1}, \Psi\xi_{2n+1}) = D(M,N)$ and $D(\upsilon_{2n}, \Lambda\xi_{2n+1}) = D(\upsilon_{2n+1}, \Gamma\xi_{2n+2}) = D(M,N)$, then we have,

$$\begin{aligned} D(\upsilon_{2n}, \Phi\xi_{2n}) &= D(\upsilon_{2n+1}, \Psi\xi_{2n+1}) = D(\upsilon_{2n}, \Lambda\xi_{2n+1}) \\ &= D(\upsilon_{2n+1}, \Gamma\xi_{2n+2}) = D(M,N). \end{aligned}$$

As $n \to \infty$

$$\begin{aligned} \lim_n D(\upsilon, \Phi\xi_{2n}) &= \lim_n D(\upsilon, \Psi\xi_{2n+1}) = \lim_n D(\upsilon, \Lambda\xi_{2n+1}) \\ &= \lim_n D(\upsilon, \Gamma\xi_{2n+2}) = D(M,N). \end{aligned}$$

Since $\Gamma(M_0) = N_0$, we can choose a point $\tau \in M_0$ such that $D(\upsilon, \Gamma\tau) = D(M,N)$.
Then

$$D(\Phi\tau, \upsilon) \leq D(\Phi\tau, \Psi\xi_{2n+1}) + D(\Psi\xi_{2n+1}, \upsilon). \tag{3.11}$$

Suppose $\lim_n D(\Phi\tau, \Psi\xi_{2n+1}) \neq 0$, we have from (3.3),

$$\begin{aligned} D(\Phi\tau, \Psi\xi_{2n+1}) \leq\ & \chi\big(D(\Gamma\tau, \Lambda\xi_{2n+1}), D(\Phi\tau, \Gamma\tau), D(\Psi\xi_{2n+1}, \Lambda\xi_{2n+1}), \\ & D(\Phi\tau, \Lambda\xi_{2n+1}), D(\Psi\xi_{2n+1}, \Gamma\tau)\big). \end{aligned}$$

As $n \to \infty$, and by P-property, we have

$$\begin{aligned} D(\Phi\tau, \Gamma\tau) &= \chi\big(0, D(\Phi\tau, \Gamma\tau), 0, D(\Phi\tau, \Gamma\tau), 0\big) \\ &\leq \chi\big(D(\Phi\tau, \Gamma\tau), D(\Phi\tau, \Gamma\tau), D(\Phi\tau, \Gamma\tau), D(\Phi\tau, \Gamma\tau), D(\Phi\tau, \Gamma\tau)\big) \\ &< D(\Phi\tau, \Gamma\tau) \end{aligned}$$

which gives a contradiction. Then $\lim_n D(\Phi\tau, \Psi\xi_{2n+1}) = 0$. Applying limit to equation (3.11), we get $D(\upsilon, \Phi\tau) = D(M,N)$. Therefore, $D(\upsilon, \Gamma\tau) = D(M,N) = D(\upsilon, \Phi\tau)$. Similarly, since $\Lambda(M_0) = N_0$, there we find a point $\nu \in M_0$ such that $D(\upsilon, \Lambda\nu) = D(M,N)$. Then

$$D(\upsilon, \Psi\nu) \leq D(\upsilon, \Phi\tau) + D(\Phi\tau, \Psi\nu). \tag{3.12}$$

Suppose $D(\Phi\tau, \Psi\nu) \neq 0$, we have from (3.3) and by P-property,

$$\begin{aligned} D(\Phi\tau, \Psi\nu) &\leq \chi\big(D(\Gamma\tau, \Lambda\nu), D(\Phi\tau, \Gamma\tau), D(\Psi\nu, \Lambda\nu), \\ &\quad D(\Phi\tau, \Lambda\nu), D(\Psi\nu, \Gamma\tau)\big) \\ &= \chi\big(0, 0, D(\Psi\nu, \Phi\tau), 0, D(\Psi\nu, \Phi\tau)\big) \\ &\leq \chi\big(D(\Phi\tau, \Psi\nu), D(\Phi\tau, \Psi\nu), D(\Psi\nu, \Phi\tau), D(\Phi\tau, \Psi\nu), D(\Psi\nu, \Phi\tau)\big) \\ &< D(\Phi\tau, \Psi\nu) \end{aligned}$$

which gives again a contradiction. Then $D(\Phi\tau, \Psi\nu) = 0$. Therefore from (3.12), we get $D(\upsilon, \Psi\nu) = D(M,N)$. Therefore, $D(\upsilon, \Lambda\nu) = D(M,N) = D(\upsilon, \Psi\nu)$. Thus $D(\upsilon, \Gamma\tau) = D(\upsilon, \Phi\tau) = D(\upsilon, \Lambda\nu) = D(\upsilon, \Psi\nu) = D(M,N)$.

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By using P-property to the above equations, we have $\Gamma\tau = \Phi\tau = \Lambda\nu = \Psi\nu$.
Since $\{\Phi, \Gamma\}$ is proximal weak commute pair, which gives $\Phi\upsilon = \Gamma\upsilon$.
Now we claim, υ is a best proximity point of Φ.

$$D(\Phi\upsilon, \upsilon) \leq D(\Phi\upsilon, \Psi\nu) + D(\Psi\nu, \upsilon). \tag{3.13}$$

Suppose $D(\Phi\upsilon, \Psi\nu) \neq 0$, we have from (3.3) and by P-property,

$$\begin{aligned} D(\Phi\upsilon, \Psi\nu) &\leq \chi\big(D(\Gamma\upsilon, \Lambda\nu), D(\Phi\upsilon, \Gamma\upsilon), D(\Psi\nu, \Lambda\nu), D(\Phi\upsilon, \Lambda\nu), D(\Psi\nu, \Gamma\upsilon)\big) \\ &= \chi\big(D(\Phi\upsilon, \Psi\nu), 0, 0, D(\Phi\upsilon, \Psi\nu), D(\Psi\nu, \Phi\upsilon)\big) \\ &\leq \chi\big(D(\Phi\upsilon, \Psi\nu), D(\Phi\upsilon, \Psi\nu), D(\Phi\upsilon, \Psi\nu), D(\Phi\upsilon, \Psi\nu), D(\Psi\nu, \Phi\upsilon)\big) \\ &< D(\Phi\upsilon, \Psi\nu) \end{aligned}$$

which gives a contradiction. Then $D(\Phi\upsilon, \Psi\nu) = 0$. Therefore from (3.13), we get $D(\Phi\upsilon, \upsilon) = D(M, N)$. Also $D(\Phi\upsilon, \upsilon) = D(\Gamma\upsilon, \upsilon) = D(M, N)$.
Similarly, $\{\Psi, \Lambda\}$ is proximal weak commute pair, we have $\Psi\upsilon = \Lambda\upsilon$.
Now, we show that υ is a best proximity point of Ψ. Then

$$D(\upsilon, \Psi\upsilon) \leq D(\Phi\tau, \Psi\upsilon) + D(\Phi\tau, \upsilon). \tag{3.14}$$

Suppose $D(\Phi\tau, \Psi\upsilon) \neq 0$, we have from (3.3) and by P-property,

$$\begin{aligned} D(\Phi\tau, \Psi\upsilon) &\leq \chi\big(D(\Gamma\tau, \Lambda\upsilon), D(\Phi\tau, \Gamma\tau), D(\Psi\upsilon, \Lambda\upsilon), D(\Phi\tau, \Lambda\upsilon), D(\Psi\upsilon, \Gamma\tau)\big) \\ &= \chi\big(D(\Phi\tau, \Psi\upsilon), 0, 0, D(\Phi\tau, \Psi\upsilon), D(\Psi\upsilon, \Phi\tau)\big) \\ &\leq \chi\big(D(\Phi\tau, \Psi\upsilon), D(\Phi\tau, \Psi\upsilon), D(\Phi\tau, \Psi\upsilon), D(\Phi\tau, \Psi\upsilon), D(\Psi\upsilon, \Phi\tau)\big) \\ &< D(\Phi\tau, \Psi\upsilon) \end{aligned}$$

which is a contradiction. Then $D(\Phi\tau, \Psi\upsilon) = 0$. Therefore from (3.14), we get $D(\Psi\upsilon, \upsilon) = D(M, N)$. Also $D(\Psi\upsilon, \upsilon) = D(\Lambda\upsilon, \upsilon) = D(M, N)$.
Thus $D(\Phi\upsilon, \upsilon) = D(\Gamma\upsilon, \upsilon) = D(\Psi\upsilon, \upsilon) = D(\Lambda\upsilon, \upsilon) = D(M, N)$, and υ is a common best proximity point of Φ, Ψ, Γ and Λ.
Finally, for uniqueness of υ, suppose that ω is another common best proximity point of the mappings Φ, Ψ, Γ and Λ, so that

$$D(\Phi\omega, \omega) = D(\Gamma\omega, \omega) = D(\Psi\omega, \omega) = D(\Lambda\omega, \omega) = D(M, N).$$

By P-property,

$$D(\upsilon, \omega) = D(\Phi\upsilon, \Psi\omega). \tag{3.15}$$

Suppose $D(\Phi\upsilon, \Psi\omega) \neq 0$, we have from (3.3),

$$\begin{aligned} D(\Phi\upsilon, \Psi\omega) &\leq \chi\big(D(\Gamma\upsilon, \Lambda\omega), D(\Phi\upsilon, \Gamma\upsilon), D(\Psi\omega, \Lambda\omega), D(\Phi\upsilon, \Lambda\omega), D(\Psi\omega, \Gamma\upsilon)\big) \\ &= \chi\big(D(\Phi\upsilon, \Psi\omega), 0, 0, D(\Phi\upsilon, \Psi\omega), D(\Phi\upsilon, \Psi\omega)\big) \\ &\leq \chi\big(D(\Phi\upsilon, \Psi\omega), D(\Phi\upsilon, \Psi\omega), D(\Phi\upsilon, \Psi\omega), D(\Phi\upsilon, \Psi\omega), D(\Phi\upsilon, \Psi\omega)\big) \\ &< D(\Phi\upsilon, \Psi\omega) \end{aligned}$$

which gives a contradiction. Then $D(\Phi\upsilon, \Psi\omega) = 0$. Therefore from (3.15), we get $D(\upsilon, \omega) = 0$. This implies that $\upsilon = \omega$. □

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Corollary 3.5. *Let M, N be non-empty, closed, bounded and convex subsets of a uniformly convex Banach space X. Let the non-self mappings $\Phi, \Psi, \Gamma, \Lambda : M \to N$ satisfy:*

(1) $\{\Phi, \Gamma\}$ *and* $\{\Psi, \Lambda\}$ *are proximal weak commute pairs;*
(2) Φ, Ψ, Γ *and* Λ *satisfy (3.2) and (3.3);*
(3) $\Gamma(M_0) = N_0$ *and* $\Lambda(M_0) = N_0$.

Then Φ, Ψ, Γ and Λ have a unique common best proximity point.

From the following numerical example we illustrate our main result

Example 3.6. Let $X = [0, 1] \times [0, 1]$. Define $D_1\big((\xi_1, \xi_2), (\eta_1, \eta_2)\big) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$. Then (X, D_1) is complete metric space. Let

$$M = \{(0, \xi) : 0 \leq \xi \leq 1\},\ N = \{(1, \eta) : 0 \leq \eta \leq 1\}.$$

Then $D_1(M, N) = 1, M_0 = M$ and $N_0 = N$. Let Φ, Ψ, Γ and Λ be defined as $\Phi(0, \xi) = (1, \frac{\xi^2}{5}), \Psi(0, \xi) = (1, \frac{\sqrt{\xi}}{5}), \Gamma(0, \xi) = (1, \xi^2), \Lambda(0, \xi) = (1, \sqrt{\xi})$.
From $\Phi(0, \xi) = \Gamma(0, \xi)$, we have $(1, \frac{\xi^2}{5}) = (1, \xi^2)$, which implies that Φ and Γ are coinciding at $(0, 0)$. Also, $D_1\big((0, \upsilon), \Phi(0, 0)\big) = D_1\big((0, \nu), \Gamma(0, 0)\big) = D_1(M, N) = 1$, we get $\upsilon = 0, \nu = 0$. Then $\Phi(0, 0) = (1, 0) = \Gamma(0, 0)$. Therefore, $\{\Phi, \Gamma\}$ is proximal weak commute pair. Similarly, we can prove that $\{\Psi, \Lambda\}$ is proximal weak commute pair. Let $\chi(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = \frac{1}{5} \max\{\xi_1, \xi_2, \xi_3, \frac{1}{2}(\xi_4 + \xi_5)\}$. Clearly, $\chi \in F$. Since

$$D_1(\Phi\xi, \Psi\eta) = |\frac{\xi^2}{5} - \frac{\sqrt{\eta}}{5}| = \frac{1}{5}|\xi_2 - \sqrt{\eta}| = \frac{1}{5} D_1(\Gamma\xi, \Lambda\eta).$$

Therefore, by Theorem 3.4, the mappings Φ, Ψ, Γ, and Λ have a unique common best proximity point, that is $(0, 0)$.

We give another method to prove above theorem, by changing the construction of sequence.
For arbitrary point ξ_0 in M_0, since $\Phi(M_0) \subset \Lambda(M_0)$, then one can choose an element ξ_1 in M_0 such that $\Phi(\xi_0) = \Lambda(\xi_1)$. Similarly, from the condition $\Psi(M_0) \subset \Gamma(M_0)$, there is a point $\xi_2 \in M_0$ such that $\Psi(\xi_1) = \Gamma(\xi_2)$. Continuing this process and using (3.1), we can construct $\{\eta_n\} \subset N_0$ such that

$$\begin{aligned} \eta_{2n} &= \Phi(\xi_{2n}) = \Lambda(\xi_{2n+1}) \text{ and} \\ \eta_{2n+1} &= \Psi(\xi_{2n+1}) = \Gamma(\xi_{2n+2}),\ n = 0, 1, 2, 3, ... \end{aligned} \tag{3.16}$$

Lemma 3.7. [14] $\lim_n D(\eta_n, \eta_{n+1}) = 0$, *and* $\{\eta_n\}$ *is a Cauchy in* N_0.

Theorem 3.8. *Let M and N be subsets of a complete metric space (X, D). Assume that N_0 is non-empty and closed. Let $\{\Phi, \Gamma\}$ and $\{\Psi, \Lambda\}$ be proximal weak commute pairs of non-self maps from M to N satisfying (3.1), (3.2) and (3.3) with $\Gamma(M_0)$ and $\Lambda(M_0)$ are closed and assume (M, N) satisfies P-property. Then Φ, Ψ, Γ and Λ have a unique common best proximity point.*

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Proof. By Lemma 3.7, $\{\eta_n\}$ is a Cauchy sequence in N_0. Since N_0 is complete, there exists $\tau \in N_0$ such that $\lim_n \eta_n = \tau$. Therefore $\lim_n \Phi\xi_{2n} = \lim_n \Lambda\xi_{2n+1} = \tau$ and $\lim_n \Psi\xi_{2n+1} = \lim_n \Gamma\xi_{2n+2} = \tau$. Then

$$\lim_n \Phi\xi_{2n} = \lim_n \Lambda\xi_{2n+1} = \lim_n \Psi\xi_{2n+1} = \lim_n \Gamma\xi_{2n+2} = \tau.$$

Since $\Gamma(M_0)$ is a closed set, we get $\tau \in \Gamma(M_0)$. Then we have a point $\upsilon \in M_0$ such that $\Gamma\upsilon = \tau$. Using (3.3),

$$\begin{aligned} D(\Phi\upsilon, \tau) &\leq D(\Phi\upsilon, \Psi\xi_{2n+1}) + D(\Psi\xi_{2n+1}, \tau) \\ &\leq \chi\big(D(\Gamma\upsilon, \Lambda\xi_{2n+1}), D(\Phi\upsilon, \Gamma\upsilon), D(\Psi\xi_{2n+1}, \Lambda\xi_{2n+1}), \\ &\quad D(\Phi\upsilon, \Lambda\xi_{2n+1}), D(\Psi\xi_{2n+1}, \Gamma\upsilon)\big) + D(\Psi_{2n+1}, \tau). \end{aligned}$$

As $n \to \infty$,

$$\begin{aligned} D(\Phi\upsilon, \tau) &\leq \chi\big(0, D(\Phi\upsilon, \tau), 0, D(\Phi\upsilon, \tau), 0\big) \\ &\leq \beta D(\Phi\upsilon, \tau), \text{ where } \beta < 1. \end{aligned}$$

Therefore $\Phi\upsilon = \tau = \Gamma\upsilon$.
And again by $\Lambda(M_0)$ is closed, we obtain $\tau \in \Lambda(M_0)$. Then we have a point $\nu \in M_0$ such that $\Lambda\nu = \tau$. And again using (3.3),

$$\begin{aligned} D(\tau, \Psi\nu) &\leq D(\Phi\upsilon, \Psi\nu) \\ &\leq \chi\big(D(\Gamma\upsilon, \Lambda\nu), D(\Phi\upsilon, \Gamma\upsilon), D(\Psi\nu, \Lambda\nu), \\ &\quad D(\Phi\upsilon, \Lambda\nu), D(\Psi\nu, \Gamma\upsilon)\big) \\ &= \chi\big(0, 0, D(\Psi\nu, \tau), 0, D(\Psi\nu, \tau)\big) \\ &\leq \chi\big(D(\tau, \Psi\nu), D(\tau, \Psi\nu), D(\tau, \Psi\nu), 2D(\tau, \Psi\nu), 2D(\tau, \Psi\nu)\big) \\ &< D(\tau, \Psi\nu). \end{aligned}$$

Therefore $\Psi\nu = \tau = \Lambda\nu$. Finally we get, $\Phi\upsilon = \Gamma\upsilon = \Psi\nu = \Lambda\nu = \tau$. Since $\Phi(M_0) \subset N_0$, implies $\Phi\upsilon, \Gamma\upsilon \in N_0$. Therefore there exists $\upsilon_1, \upsilon_2 \in M_0$ such that $D(\upsilon_1, \Phi\upsilon) = D(M, N) = D(\upsilon_2, \Gamma\upsilon)$. Since $\{\Phi, \Gamma\}$ is proximal weak commute pair, we get

$$\Phi\upsilon_2 = \Gamma\upsilon_1. \tag{3.17}$$

Since (M, N) satisfies P-property, implies that $D(\upsilon_1, \upsilon_2) = D(\Phi\upsilon, \Gamma\upsilon) = 0$.
Then $\upsilon_1 = \upsilon_2 = \xi \in M_0$. Therefore (3.17) becomes $\Phi\xi = \Gamma\xi$.
We shall prove that ξ is a best proximity point of Φ.
Since we have $D(\upsilon_1, \Phi\upsilon) = D(\xi, \tau) = D(M, N)$ and $D(\upsilon_2, \Gamma\upsilon) = D(\xi, \Psi\nu) = D(M, N)$. Then

$$D(\Phi\xi, \xi) \leq D(\Phi\xi, \Psi\nu) + D(\Psi\nu, \xi). \tag{3.18}$$

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Suppose $D(\Phi\xi,\Psi\nu)\neq 0$, and using (3.3), we derive

$$\begin{aligned} D(\Phi\xi,\Psi\nu) &\leq \chi\big(D(\Gamma\xi,\Lambda\nu),D(\Phi\xi,\Gamma\xi),D(\Psi\nu,\Lambda\nu),\\ &\quad D(\Phi\xi,\Lambda\nu),D(\Psi\nu,\Gamma\xi)\big)\\ &= \chi\big(D(\Phi\xi,\tau),0,0,D(\Phi\xi,\tau),D(\tau,\Phi\xi)\big)\\ &\leq \chi\big(D(\Phi\xi,\tau),D(\Phi\xi,\tau),D(\Phi\xi,\tau),D(\Phi\xi,\tau),D(\tau,\Phi\xi)\big)\\ &< D(\Phi\xi,\tau)=D(\Phi\xi,\Psi\nu)\end{aligned}$$

which gives a contradiction. Then from (3.18), we obtain

$$D(\Phi\xi,\xi)=D(M,N)=D(\Gamma\xi,\xi).$$

Since $\Psi(M_0)\subset N_0$, implies that $\Psi\nu,\Lambda\nu\in N_0$. Therefore there exists $\nu_1,\nu_2\in M_0$ such that $D(\nu_1,\Psi\nu)=D(M,N)=D(\nu_2,\Lambda\nu)$. Since $\{\Psi,\Lambda\}$ is proximal weak commute pair, we get

$$\Psi\nu_2=\Lambda\nu_1. \tag{3.19}$$

Since (M,N) satisfies P-property, implies that $D(\nu_1,\nu_2)=D(\Psi\nu,\Lambda\nu)=0$. Then $\nu_1=\nu_2=\eta\in M_0$. Therefore (3.19) becomes $\Psi\eta=\Lambda\eta$.
We shall prove that η is a best proximity point of Ψ.
Since we have $D(\nu_1,\Psi\nu)=D(\eta,\Phi\upsilon)=D(M,N)$ and $D(\nu_2,\Lambda\nu)=D(\eta,\tau)=D(M,N)$. Then

$$D(\Psi\eta,\eta) \leq D(\Phi\upsilon,\Psi\eta)+D(\Phi\upsilon,\eta). \tag{3.20}$$

Suppose $D(\Phi\upsilon,\Psi\eta)\neq 0$, and using (3.3), we derive

$$\begin{aligned} D(\Phi\upsilon,\Psi\eta) &\leq \chi\big(D(\Gamma\upsilon,\Lambda\eta),D(\Phi\upsilon,\Gamma\upsilon),D(\Psi\eta,\Lambda\eta),\\ &\quad D(\Phi\upsilon,\Lambda\eta),D(\Psi\eta,\Gamma\upsilon)\big)\\ &= \chi\big(D(\Phi\upsilon,\Psi\eta),0,0,D(\Phi\upsilon,\Psi\eta),D(\Psi\eta,\Phi\upsilon)\big)\\ &\leq \chi\big(D(\Phi\upsilon,\Psi\eta),D(\Phi\upsilon,\Psi\eta),D(\Phi\upsilon,\Psi\eta),D(\Phi\upsilon,\Psi\eta),D(\Phi\upsilon,\Psi\eta)\big)\\ &< D(\Phi\upsilon,\Psi\eta)\end{aligned}$$

which gives a contradiction. Then from (3.20), we get $D(\Psi\eta,\eta)=D(M,N)=D(\Lambda\eta,\eta)$. For common best proximity point, we claim $\xi=\eta$. Now, by (M,N) has P-property,

$$D(\xi,\eta)=D(\Phi\xi,\Psi\eta). \tag{3.21}$$

Suppose $D(\Phi\xi,\Psi\eta)\neq 0$, and using (3.3), we obtain

$$\begin{aligned} D(\Phi\xi,\Psi\eta) &\leq \chi\big(D(\Gamma\xi,\Lambda\eta),D(\Phi\xi,\Gamma\xi),D(\Psi\eta,\Lambda\eta),\\ &\quad D(\Phi\xi,\Lambda\eta),D(\Psi\eta,\Gamma\xi)\big)\\ &= \chi\big(D(\Phi\xi,\Psi\eta),0,0,D(\Phi\xi,\Psi\eta),D(\Psi\eta,\Phi\xi)\big)\\ &\leq \chi\big(D(\Phi\xi,\Psi\eta),D(\Phi\xi,\Psi\eta),D(\Phi\xi,\Psi\eta),D(\Phi\xi,\Psi\eta),D(\Phi\xi,\Psi\eta)\big)\\ &< D(\Phi\xi,\Psi\eta).\end{aligned}$$

which gives a contradiction. So $D(\xi,\eta)=0$.
Thus $D(\Phi\xi,\xi)=D(\Gamma\xi,\xi)=D(\Psi\xi,\xi)=D(\Lambda\xi,\xi)=D(M,N)$.
It is clear to show the uniqueness of best proximity point. □

Corollary 3.9. *Let M,N be two subsets of a complete metric space (X,D). Assume that N_0 is non-empty, closed set. Let $\{\Phi,\Gamma\}$ and $\{\Psi,\Lambda\}$ be proximal weak commute pairs of non-self maps from M to N satisfying (3.1), (3.2) and (3.22),*

$$\begin{aligned} D(\Phi\xi,\Psi\eta) &\leq hM(\xi,\eta),\ 0\leq h<1,\ \xi,\eta\in M_0,\ \textit{where}\\ M(\xi,\eta) &= max\{D(\Gamma\xi,\Lambda\eta),D(\Phi\xi,\Gamma\xi),D(\Psi\eta,\Lambda\eta),\\ &\quad [D(\Phi\xi,\Lambda\eta)+D(\Psi\eta,\Gamma\xi)]/2\}. \end{aligned} \tag{3.22}$$

Suppose $\Gamma(M_0)$ and $\Lambda(M_0)$ are closed and assume (M,N) satisfies P-property. Then Φ,Ψ,Γ and Λ have a unique common best proximity point.

Corollary 3.10. *Let M,N be two subsets of a complete metric space (X,D). Assume that N_0 is non-empty and closed. Let $\{\Phi,\Gamma\}$ and $\{\Psi,\Lambda\}$ be proximal weak commute pairs of non-self maps from M to N satisfying (3.1), (3.2) and (3.23),*

$$\begin{aligned} D(\Phi\xi,\Psi\eta) &= h\,max\{D(\Phi\xi,\Gamma\xi),D(\Psi\eta,\Lambda\eta),[D(\Phi\xi,\Lambda\eta)]/2,\\ &\quad [D(\Psi\eta,\Gamma\xi)]/2,D(\Gamma\xi,\Lambda\eta)\}. \end{aligned} \tag{3.23}$$

for all $\xi,\eta\in M_0$, where $0\leq h<1$.
Suppose $\Gamma(M_0)$ and $\Lambda(M_0)$ are closed and assume (M,N) satisfies P-property. Then Φ,Ψ,Γ and Λ have a unique common best proximity point.

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