

Horadam Polynomials Estimates for λ -Pseudo-Starlike Bi-Univalent Functions

Hammadi Alaa Hussein^a, Abbas Kareem Wanas^{b*}

^aDepartment of Computer Information Systems College of Computer Science
and Information Technology University of Al-Qadisiyah, Iraq

^bDepartment of Mathematics College of Science University of Al-Qadisiyah,
Iraq

E-mail: alaa.hammadi@qu.edu.iq

E-mail: abbas.kareem.w@qu.edu.iq

ABSTRACT. In the present investigation, we use the Horadam Polynomials to establish upper bounds for the second and third coefficients of functions belongs to a new subclass of analytic and λ -pseudo-starlike bi-univalent functions defined in the open unit disk U . Also, we discuss Fekete-Szegő problem for functions belongs to this subclass.

Keywords: Bi-univalent functions, Coefficient bounds, Horadam polynomials, λ -Pseudo-starlike functions, Fekete-Szegő problem, Subordination.

2010 Mathematics subject classification: 30C45, 30C50.

1. INTRODUCTION

Let $\mathcal{A}$ stand for the family of functions f which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

*Corresponding Author

Further, let S indicate the subclass of $\mathcal{A}$ consisting of the form (1.1) which are univalent in U . It is well known (see [3]) that every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$, ($z \in U$) and $f(f^{-1}(w)) = w$, ($|w| < r_0(f)$), $r_0(f) \geq \frac{1}{4}$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U . Let Σ stands for the class of bi-univalent functions in U given by (1.1). In fact, Srivastava et al. [13] has apparently revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [4], Goyal and Goswami [5], Caglar et al. [2] and others (see, for example [9, 10, 11, 12, 14]).

A function $f \in S$ is said to be λ -pseudo-starlike function of order β $0 \leq \beta < 1$ in U , if it satisfies (see [1]): This class of functions was denoted by $\mathcal{L}_\lambda(\beta)$. It is observed that for $\lambda = 1$, we have the class of starlike functions.

With a view to recalling the principal of subordination between analytic functions, let the functions f and g be analytic in U . We say that the function f is said to be subordinate to g , if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ ($z \in U$). It is well known that, if the function g is univalent in U , then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [6]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N}),$$

with

$$h_1(r) = a \quad \text{and} \quad h_2(r) = br, \quad (1.3)$$

for some real constant a, b, p and q .

The generating function of the Horadam polynomials $h_n(r)$ (see [7]) is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \quad (1.4)$$

2. MAIN RESULTS

We begin this section by defining the subclass $\mathcal{L}_\Sigma(\delta, \lambda, r)$ as follows:

Definition 2.1. For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{L}_\Sigma(\delta, \lambda, r)$ if it satisfies the subordinations:

$$1 + \frac{1}{\delta} \left(\frac{z(f'(z))^\lambda}{f(z)} - 1 \right) \prec \Pi(r, z) + 1 - a$$

and

$$1 + \frac{1}{\delta} \left(\frac{w (f'(w))^\lambda}{f(w)} - 1 \right) \prec \Pi(r, w) + 1 - a,$$

where a is real constant and the function $g = f^{-1}$ is given by (1.2).

Theorem 2.2. For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{L}_\Sigma(\delta, \lambda, r)$. Then

$$|a_2| \leq \frac{\sqrt{2\delta} |br| \sqrt{|br|}}{\sqrt{\left| \left[\delta (4\lambda^2 - 5\lambda + 1) b - 2p(\lambda - 1)^2 \right] br^2 - 2qa(\lambda - 1)^2 \right|}}$$

and

$$|a_3| \leq \frac{\delta |br|}{3\lambda - 1} + \frac{\delta^2 b^2 r^2}{(\lambda - 1)^2}.$$

Proof. Let $f \in \mathcal{L}_\Sigma(\delta, \lambda, r)$. Then there are two analytic functions $u, v : U \rightarrow U$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \dots \quad (z \in U) \quad (2.1)$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \dots \quad (w \in U), \quad (2.2)$$

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in U$ such that

$$1 + \frac{1}{\delta} \left(\frac{z (f'(z))^\lambda}{f(z)} - 1 \right) = \Pi(r, u(z)) + 1 - a$$

and

$$1 + \frac{1}{\delta} \left(\frac{w (f'(w))^\lambda}{f(w)} - 1 \right) = \Pi(r, v(w)) + 1 - a.$$

Or, equivalently

$$1 + \frac{1}{\delta} \left(\frac{z (f'(z))^\lambda}{f(z)} - 1 \right) = 1 + h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \dots \quad (2.3)$$

and

$$1 + \frac{1}{\delta} \left(\frac{w (f'(w))^\lambda}{f(w)} - 1 \right) = 1 + h_1(r) + h_2(r)v(w) + h_3(r)v^2(w) + \dots \quad (2.4)$$

Combining (2.1), (2.2), (2.3) and (2.4) yields

$$1 + \frac{1}{\delta} \left(\frac{z (f'(z))^\lambda}{f(z)} - 1 \right) = 1 + h_2(r)u_1 z + [h_2(r)u_2 + h_3(r)u_1^2] z^2 + \dots \quad (2.5)$$

and

$$1 + \frac{1}{\delta} \left(\frac{w (f'(w))^\lambda}{f(w)} - 1 \right) = 1 + h_2(r)v_1 w + [h_2(r)v_2 + h_3(r)v_1^2] w^2 + \dots \quad (2.6)$$

It is quite well-known that if $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in U$, then

$$|u_i| \leq 1 \quad \text{and} \quad |v_i| \leq 1 \quad \text{for all } i \in \mathbb{N}. \quad (2.7)$$

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

$$\frac{2\lambda - 1}{\delta} a_2 = h_2(r)u_1, \quad (2.8)$$

$$\frac{3\lambda - 1}{\delta} a_3 + \frac{2\lambda(\lambda - 2) + 1}{\delta} a_2^2 = h_2(r)u_2 + h_3(r)u_1^2, \quad (2.9)$$

$$-\frac{2\lambda - 1}{\delta} a_2 = h_2(r)v_1 \quad (2.10)$$

and

$$\frac{3\lambda - 1}{\delta} (2a_2^2 - a_3) + \frac{2\lambda(\lambda - 2) + 1}{\delta} a_2^2 = h_2(r)v_2 + h_3(r)v_1^2. \quad (2.11)$$

It follows from (2.8) and (2.10) that

$$u_1 = -v_1 \quad (2.12)$$

and

$$\frac{2(\lambda - 1)^2}{\delta^2} a_2^2 = h_2^2(r)(u_1^2 + v_1^2). \quad (2.13)$$

If we add (2.9) to (2.11), we find that

$$\frac{4\lambda^2 - 5\lambda + 1}{\delta} a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \quad (2.14)$$

Substituting the value of $u_1^2 + v_1^2$ from (2.13) in the right hand side of (2.14), we deduce that

$$a_2^2 = \frac{\delta^2 h_2^3(r)(u_2 + v_2)}{\delta(4\lambda^2 - 5\lambda + 1)h_2^2(r) - 2h_3(r)(\lambda - 1)^2}. \quad (2.15)$$

Further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_2| \leq \frac{\sqrt{2}\delta |br| \sqrt{|br|}}{\sqrt{\left| \left[\delta(4\lambda^2 - 5\lambda + 1)b - 2p(\lambda - 1)^2 \right] br^2 - 2qa(\lambda - 1)^2 \right|}}.$$

Next, if we subtract (2.11) from (2.9), we can easily see that

$$\frac{2(3\lambda - 1)}{\delta} (a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2). \quad (2.16)$$

In view of (2.12) and (2.13), we get from (2.16)

$$a_3 = \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + \frac{\delta^2 h_2^2(r)(u_1^2 + v_1^2)}{2(\lambda - 1)^2}.$$

Thus applying (1.3), we obtain

$$|a_3| \leq \frac{\delta |br|}{3\lambda - 1} + \frac{\delta^2 b^2 r^2}{(\lambda - 1)^2}.$$

This completes the proof of Theorem 2.2 $\square$

In the next theorem, we discuss the Fekete-Szegő problem for the subclass $\mathcal{L}_\Sigma(\delta, \lambda, r)$.

Theorem 2.3. For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r, \mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{L}_\Sigma(\delta, \lambda, r)$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta |br|}{3\lambda - 1}; \\ \text{for } |\mu - 1| \leq \frac{|\delta(4\lambda^2 - 5\lambda + 1)b - 2p(\lambda - 1)^2|br^2 - 2qa(\lambda - 1)^2|}{2\delta b^2 r^2(3\lambda - 1)}, \\ \frac{2\delta^2 |br|^3 |\mu - 1|}{|\delta(4\lambda^2 - 5\lambda + 1)b - 2p(\lambda - 1)^2|br^2 - 2qa(\lambda - 1)^2|}; \\ \text{for } |\mu - 1| \geq \frac{|\delta(4\lambda^2 - 5\lambda + 1)b - 2p(\lambda - 1)^2|br^2 - 2qa(\lambda - 1)^2|}{2\delta b^2 r^2(3\lambda - 1)}. \end{cases}$$

Proof. It follows from (2.15) and (2.16) that

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + (1 - \mu)a_2^2 \\ &= \frac{\delta h_2(r)(u_2 - v_2)}{2(3\lambda - 1)} + \frac{\delta^2 h_2^3(r)(u_2 + v_2)(1 - \mu)}{\delta(4\lambda^2 - 5\lambda + 1)h_2^2(r) - 2h_3(r)(\lambda - 1)^2} \\ &= h_2(r) \left[\left(\psi(\mu, r) + \frac{\delta}{2(3\lambda - 1)} \right) u_2 + \left(\psi(\mu, r) - \frac{\delta}{2(3\lambda - 1)} \right) v_2 \right], \end{aligned}$$

where

$$\psi(\mu, r) = \frac{\delta^2 h_2^2(r)(1 - \mu)}{\delta(4\lambda^2 - 5\lambda + 1)h_2^2(r) - 2h_3(r)(\lambda - 1)^2}.$$

According to (1.3), we find that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta |br|}{3\lambda - 1}, & 0 \leq |\psi(\mu, r)| \leq \frac{\delta}{2(3\lambda - 1)}, \\ 2|br| |\psi(\mu, r)|, & |\psi(\mu, r)| \geq \frac{\delta}{2(3\lambda - 1)}. \end{cases}$$

After some computations, we obtain

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta |br|}{3\lambda - 1}; \\ \text{for } |\mu - 1| \leq \frac{|\delta(4\lambda^2 - 5\lambda + 1)b - 2p(\lambda - 1)^2|br^2 - 2qa(\lambda - 1)^2|}{2\delta b^2 r^2(3\lambda - 1)}, \\ \frac{2\delta^2 |br|^3 |\mu - 1|}{|\delta(4\lambda^2 - 5\lambda + 1)b - 2p(\lambda - 1)^2|br^2 - 2qa(\lambda - 1)^2|}; \\ \text{for } |\mu - 1| \geq \frac{|\delta(4\lambda^2 - 5\lambda + 1)b - 2p(\lambda - 1)^2|br^2 - 2qa(\lambda - 1)^2|}{2\delta b^2 r^2(3\lambda - 1)}. \end{cases}$$

□

Putting $\mu = 1$ in Theorem 2.3, we obtain the following result:

Corollary 2.4. For $\delta \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{L}_\Sigma(\delta, \lambda, r)$. Then

$$|a_3 - a_2^2| \leq \frac{\delta |br|}{3\lambda - 1}.$$

Remark 2.5. If we put $\lambda = 1$ in our Theorems, we have the result for well-known class $S_\Sigma^*(r)$ of bi-starlike functions which was considered recently by Srivastava et al. [8].

3. ACKNOWLEDGMENT

The authors are thankful to the reviewers for their valuable comments and suggestions which improved the quality of the paper.

REFERENCES

1. K. O. Babalola, On λ -Pseudo-Starlike Functions, *J. Class. Anal.*, **3**(2), (2013), 137-147.
2. M. Caglar, H. Orhan, N. Yagmur, Coefficient Bounds for New Subclasses of Bi-Univalent Functions, *Filomat*, **27**, (2013), 1165-1171.
3. P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.
4. B. A. Frasin, M. K. Aouf, New Subclasses of Bi-Univalent Functions, *Appl. Math. Lett.*, **24**, (2011), 1569-1573.
5. S. P. Goyal, P. Goswami, Estimate for Initial Maclaurin Coefficients of Bi-Univalent Functions for a Class Defined by Fractional Derivatives, *J. Egyptian Math. Soc.*, **20**, (2012), 179-182.
6. A. F. Horadam, J. M. Mahon, Pell and Pell-Lucas polynomials, *Fibonacci Quart.*, **23**, (1985), 7-20.
7. T. Hörcum, E. G. Kocer, On some Properties of Horadam polynomials, *Int. Math. Forum*, **4**, (2009), 1243-1252.
8. H. M. Srivastava, S. Altinkaya, S. Yalin, Certain Subclasses of Bi-Univalent Functions Associated with the Horadam Polynomials, *Iran. J. Sci. Technol. Trans. Sci.*, (2018), 1-7.
9. H. M. Srivastava, D. Bansal, Coefficient Estimates for a Subclass of Analytic and Bi-Univalent Functions, *J. Egyptian Math. Soc.*, **23**, (2015), 242-246.
10. H. M. Srivastava, S. S. Eker, R. M. Ali, Coefficient Bounds for a Certain Class of Analytic and Bi-Univalent Functions, *Filomat*, **29**, (2015), 1839-1845.
11. H. M. Srivastava, S. S. Eker, S. G. Hamidi, J. M. Jahangiri, Faber Polynomial Coefficient Estimates for Bi-Univalent Functions Defined by the Tremblay Fractional Derivative Operator, *Bull. Iranian Math. Soc.*, **44**(1), (2018), 149-157.
12. H. M. Srivastava, S. Gaboury, F. Ghanim, Coefficient Estimates for some General Subclasses of Analytic and Bi-Univalent Functions, *Africa Math.*, **28**, (2017), 693-706.
13. H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain Subclasses of Analytic and Bi-Univalent Functions, *Appl. Math. Lett.*, **23**, (2010), 1188-1192.
14. A. K. Wanas, A. H. Majeed, Certain New Subclasses of Analytic and m-Fold Symmetric Bi-Univalent Functions, *Applied Mathematics E-Notes*, **18**, (2018), 178-188.