Iranian Journal of Mathematical Sciences and Informatics

Vol. 17, No. 2 (2022), pp 97-108 DOI: 10.52547/ijmsi.17.2.97

A Note on Absolute Central Automorphisms of Finite p-Groups

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ABSTRACT. Let G be a finite group. The automorphism σ of a group G is said to be an absolute central automorphism, if for all $x \in G$, $x^{-1}x^{\sigma} \in L(G)$, where L(G) be the absolute centre of G. In this paper, we study some properties of absolute central automorphisms of a given finite p-group.

Keywords: Absolute centre, Absolute central automorphisms, Finite p-groups.

2000 Mathematics subject classification: 20D45, 20D25, 20D15.

1. Introduction

Let G be a finite group and N a characteristic subgroup of G. Suppose σ is an automorphism of G. If $(Ng)^{\sigma} = Ng$ for all g in G or equivalently σ induces the identity automorphism on G/N, we shall say σ centralizes G/N. We let $\operatorname{Aut}^N(G)$ denote the group of all automorphisms of G centralizing G/N. Clearly $\sigma \in \operatorname{Aut}^N(G)$ if and only if $x^{-1}x^{\sigma} \in N$ for all $x \in G$. Now let M be a normal subgroup of G. Let us denote by $C_{\operatorname{Aut}^N(G)}(M)$ the group of all automorphisms of $\operatorname{Aut}^N(G)$ centralizing M. Various authors have studied the groups $\operatorname{Aut}^Z(G)$, the central automorphisms of G, where G stands for the commutator subgroup of G, and $\operatorname{Aut}^{\Phi}(G)$, where Φ denote the Frattini subgroup of G, the intersection of all maximal subgroups of G, see for example [14, 17, 19, 20]. For any

Received 30 September 2018; Accepted 15 August 2019 ©2022 Academic Center for Education, Culture and Research TMU

element $g \in G$ and $\sigma \in \text{Aut}(G)$, the element $[g, \sigma] = g^{-1}g^{\sigma}$ is called the autocommutator of g and σ . Also inductively, for all $\sigma_1, \sigma_2, ..., \sigma_n \in \text{Aut}(G)$, define $[g, \sigma_1, \sigma_2, ..., \sigma_n] = [[g, \sigma_1, \sigma_2, ..., \sigma_{n-1}], \sigma_n]$. Hegarty [7], generalized the concept of centre into absolute centre L(G) of a group G as

$$L(G) = \{g \in G \mid [g, \sigma] = 1, \forall \sigma \in Aut(G)\}.$$

One can easily check that the absolute centre is a characteristic subgroup contained in the centre of G. Also he introduced the concept of the absolute central automorphism. An automorphism σ of G is called an absolute central automorphism if σ centralizes G/L(G). We denote the set of all absolute central automorphisms of G by $\operatorname{Aut}^L(G)$. Singh and Gumber [18], Kaboutari Farimani [9], also Shabani-Attar [17] have given some necessary and sufficient conditions for a finite non-abelian p-group such that all absolute central automorphisms are inner. In this paper, we will characterize the finite non-abelian p-groups G such that $\operatorname{Aut}^L(G) = \operatorname{Aut}^{G'}(G)$. Then, we determine the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$. Finally, we classify all finite p-groups G of order $p^n(3 \leq n \leq 5)$, such that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$.

Throughout this paper all groups are assumed to be finite and p always denotes a prime number. Most of our notation is standard, and can be found in [5], for example. In particular, a p-group G is said to be extraspecial if $G' = Z(G) = \Phi(G)$ is of order p. Let $L_1(G) = L(G)$ and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{ g \in G \mid [g, \sigma_1, \sigma_2, ..., \sigma_n] = 1, \forall \sigma_1, \sigma_2, ..., \sigma_n \in Aut(G) \}.$$

A group G is called autonilpotent of class at most n if $L_n(G) = G$, for some $n \in \mathbb{N}$. If σ is an automorphism of G and X is an element of G, we write X^{σ} for the image of X under σ and o(X) for the order of X. For a finite group G, $\exp(G)$, d(G) and $\operatorname{cl}(G)$, denote the exponent of G, minimal number of generators of G and the nilpotency class of G, respectively. Recall that a group G is called a central product of its subgroups $G_1, ..., G_n$ if $G = G_1 \cdots G_n$ and $[G_i, G_j] = 1$, for all $1 \leq i < j \leq n$. In this situation, we shall write $G = G_1 * \cdots * G_n$. For $s \geq 1$, we use the notation G^{*s} for the iterated central product defined by $G^{*s} = G * G^{*(s-1)}$ with $G^{*1} = G$, where G is a finite p-group. We also make the convention $G^{*0} = 1$. Finally, we use X^n for the direct product of n-copies of a group X, C_n for the cyclic group of order n where $n \geq 1$, as usual, D_8 for the dihedral group, Q_8 for the quaternion group, of order 8, respectively and $M_p(n,m)$ and $M_p(n,m,1)$ for the minimal non-abelian p-groups of order p^{n+m} and p^{n+m+1} defined respectively by

$$\langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle,$$

where $n \ge 2$, $m \ge 1$ and

$$\langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle,$$

where $n \ge m \ge 1$ and if p = 2, then m + n > 2.

2. Preliminary results

In this section we give some results which will be used in the rest of the paper.

Let G and H be any two groups. We denote by $\operatorname{Hom}(G,H)$ the set of all homomorphisms from G into H. Clearly, if H is an abelian group, then $\operatorname{Hom}(G,H)$ forms an abelian group under the following operation (fg)(x)=f(x)g(x), for all $f,g\in\operatorname{Hom}(G,H)$ and $x\in G$.

The following lemma is a well-known.

Lemma 2.1. Let A, B and C be finite abelian groups. Then

- (i) $\operatorname{Hom}(A \times B, C) \cong \operatorname{Hom}(A, C) \times \operatorname{Hom}(B, C)$;
- (ii) $\operatorname{Hom}(A, B \times C) \cong \operatorname{Hom}(A, B) \times \operatorname{Hom}(A, C)$;
- (iii) $\operatorname{Hom}(C_m, C_n) \cong C_e$, where e is the greatest common divisor of m and n.

We have the following theorem due to Müller [14].

Theorem 2.2. [14, Theorem] If G is a finite p-group which is neither elementary abelian nor extraspecial, then $Aut^{\Phi}(G)/\text{Inn}(G)$ is a non-trivial normal p-subgroup of the group of outer automorphisms of G.

The following preliminary lemma is well-known result [19, Lemma 2.2].

Lemma 2.3. Let G be a group and M, N be normal subgroups of G with $N \leq M$ and $C_N(M) \leq Z(G)$. Then $C_{\operatorname{Aut}^N(G)}(M) \cong \operatorname{Hom}(G/M, C_N(M))$.

Corollary 2.4. If G is a finite group, then

$$C_{\operatorname{Aut}^L(G)}(Z(G)) \cong \operatorname{Hom}(G/Z(G), L(G)),$$

where L = L(G).

Moghaddam and Safa [12], proved that for a finite group G,

$$\operatorname{Aut}^{L}(G) \cong \operatorname{Hom}(G/L(G), L(G)).$$

The following theorem states a useful result for finite p-groups.

Theorem 2.5. Let G be a finite p-group different from C_2 . Then $\operatorname{Aut}^L(G) \cong \operatorname{Hom}(G, L(G))$.

Proof. Let $\theta \in \operatorname{Aut}^L(G)$. We define the map $f_\theta : G \to L(G)$ by $f_\theta(g) = g^{-1}g^\theta$. It is easy to see that f_θ is a homomorphism, and $\theta \mapsto f_\theta$ is an injective map from $\operatorname{Aut}^L(G)$ to $\operatorname{Hom}(G, L(G))$. Conversely, assume that $f \in \operatorname{Hom}(G, L(G))$. Then we define $\theta = \theta_f : G \to G$ by $g^\theta = gf(g)$. Since by [11, Corollary 3.7], $g^{-1}g^\theta \in L(G) \leq \Phi(G)$, for every element $g \in G$, we may write G as the product of the image of θ and the Frattini subgroup of G and so the image of θ must be G itself. Hence θ is an automorphism of G. Now $\theta = \theta_f \in \operatorname{Aut}^L(G)$ and $f_{\theta_f} = f$. Finally, suppose that $\alpha, \beta \in \operatorname{Aut}^L(G)$. Then for any $x \in G$,

$$f_{\alpha\beta}(x) = x^{-1}x^{\alpha\beta} = x^{-1}(xx^{-1}x^{\alpha})^{\beta} = x^{-1}x^{\beta}x^{-1}x^{\alpha} = x^{-1}x^{\alpha}x^{-1}x^{\beta},$$

since $x^{-1}x^{\alpha} \in L(G)$. Thus $f_{\alpha\beta}(x) = f_{\alpha}(x)f_{\beta}(x)$ and so $\theta \mapsto f_{\theta}$ is a homomorphism, which completes the proof.

We next give a necessary and sufficient condition on a finite p-group G for the group $\operatorname{Aut}^L(G)$ to be elementary abelian.

Corollary 2.6. Let G be a finite p-group. Then $\operatorname{Aut}^L(G)$ is elementary abelian if and only if $\exp(G/G') = p$ or $\exp(L(G)) = p$.

Proof. It is straightforward by Lemma 2.1 and Theorem 2.5. \Box

3. Main results

For a finite abelian p-group G, |L(G)| = 1, 2 by [11, Lemma 4.4] and so $|\operatorname{Aut}^L(G)| = 1$ or $\operatorname{Aut}^L(G) \cong C_2^d$, with d = d(G). Thus we may assume that G is a non-abelian p-group. In this section, first we characterize the finite non-abelian p-groups G such that $\operatorname{Aut}^L(G) = \operatorname{Aut}^{G'}(G)$. Then, we determine the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$.

In [9], Kaboutari Farimani proved the following two results giving some information of absolute central automorphisms of a finite p-group.

Lemma 3.1. Let G be a finite non-abelian p-group. Then $C_{\operatorname{Aut}^L(G)}(Z(G)) = \operatorname{Inn}(G)$ if and only if G/L(G) is abelian and L(G) is cyclic.

Theorem 3.2. Let G be a finite non-abelian p-group. Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if G/L(G) is abelian, L(G) is cyclic and $Z(G) = L(G)G^{p^n}$ where $\exp(L(G)) = p^n$.

Note that the Theorem 3.2 yields the following corollary that is the Corollary 1 of Singh and Gumber [18].

Let G be a finite non-abelian p-group such that $G' \leq L(G)$. Let $G/Z(G) = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_r}}$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 1$. Also let $G/L(G) = C_{p^{\beta_1}} \times C_{p^{\beta_2}} \times \cdots \times C_{p^{\beta_s}}$, where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_s \geq 1$ and $L(G) = C_{p^{\gamma_1}} \times C_{p^{\gamma_1}} \times C_{p^{\gamma_1}} \times C_{p^{\gamma_2}} \times \cdots \times C_{p^{\gamma_s}}$

 $C_{p^{\gamma_2}} \times \cdots \times C_{p^{\gamma_t}}$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_t \geq 1$. Since G/Z(G) is a quotient group of G/L(G) by [2, Section 25], $r \leq s$ and $\alpha_i \leq \beta_i$ for all $1 \leq i \leq r$.

By the above notation, we prove the following corollary:

Corollary 3.3. [18, Corollary 1] Let G be a finite non-abelian p-group. Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if $G' \leq L(G)$, L(G) is cyclic and either L(G) = Z(G) or d(G/L(G)) = d(G/Z(G)), $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, where k is the largest integer such that $\beta_k > \gamma_1$.

Proof. First assume that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. Hence by Theorem 3.2, $G' \leq L(G)$ and L(G) is cyclic. If $\exp(G/L(G)) \leq \exp(L(G))$, then

$$G/Z(G) \cong \operatorname{Aut}^{L}(G) \cong \operatorname{Hom}(G/L(G), L(G)) \cong G/L(G),$$

because L(G) is cyclic and by [12, Proposition 1]. Therefore L(G) = Z(G). Next, let $\exp(G/L(G)) > \exp(L(G))$ and k is the largest integer such that $\beta_k > \gamma_1$. Since L(G) and G/L(G) are abelian,

$$d(G/Z(G))=d(\operatorname{Hom}(G/L(G),L(G)))=d(G/L(G))d(L(G))=d(G/L(G)).$$

Now we have $\operatorname{Hom}(G/L(G),L(G)) \cong C_{p^{\gamma_1}} \times C_{p^{\gamma_1}} \times \cdots \times C_{p^{\gamma_1}} \times C_{p^{\beta_{k+1}}} \times \cdots \times C_{p^{\beta_s}}$ and $\operatorname{Hom}(G/L(G),L(G)) \cong G/Z(G) = C_{p^{\alpha_1}} \times C_{p^{\alpha_2}} \times \cdots \times C_{p^{\alpha_r}}$. Hence $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \gamma_1$ and $\alpha_i = \beta_i$ for $k+1 \leq i \leq r$, as required.

Conversely if L(G) = Z(G), then $\exp(G/Z(G)) = \exp(G')|\exp(Z(G))$, since $G' \leq L(G)$ and by [13, Lemma 0.4]. Now

$$\operatorname{Hom}(G/L(G), L(G)) = \operatorname{Hom}(G/Z(G), Z(G)) \cong G/Z(G),$$

because Z(G) is cyclic and so $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. Next assume that L(G) < Z(G), s = d(G/L(G)) = d(G/Z(G)) = r, $\alpha_i = \gamma_1$ for $1 \le i \le k$ and $\alpha_i = \beta_i$ for $k+1 \le i \le r$, where k is the largest integer such that $\beta_k > \gamma_1$. We claim that $Z(G) = L(G)G^{p^{\gamma_1}}$. Since $\exp(G/Z(G)) = \exp(L(G))$, we have $L(G) \le L(G)G^{p^{\gamma_1}} \le Z(G)$. It follows that G/Z(G) is a quotient group of $G/L(G)G^{p^{\gamma_1}}$. Now let $G/L(G)G^{p^{\gamma_1}} = C_{p^{\gamma_1}} \times C_{p^{\delta_2}} \times \cdots \times C_{p^{\delta_r}}$, where $\delta_1 = \gamma_1 \ge \delta_2 \ge \cdots \ge \delta_r \ge 1$, since $d(G/L(G)) = d(G/L(G)G^{p^{\gamma_1}})$ and $\exp(G/L(G)G^{p^{\gamma_1}}) = p^{\gamma_1}$. Therefore $\gamma_1 = \alpha_i \le \delta_i \le \gamma_1$ for $1 \le i \le k$, whence we have $\delta_i = \gamma_1 = \alpha_i$ for $1 \le i \le k$. As $\beta_i = \alpha_i \le \delta_i \le \beta_i$ for $k+1 \le i \le r$, it follows that $\delta_i = \alpha_i = \beta_i$ for $k+1 \le i \le r$. Hence $G/Z(G) = G/L(G)G^{p^{\gamma_1}}$ and consequently $Z(G) = L(G)G^{p^{\gamma_1}}$. Therefore by Theorem 3.2, $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. This completes the proof.

As an application of Theorem 3.2, we get another proof of the main result of [15].

Theorem 3.4. [15, Theorem 3.2] Let G be a non-abelian autonilpotent finite p-group of class 2. Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if L(G) = Z(G) and L(G) is cyclic.

Proof. Suppose that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. Hence L(G) is cyclic and $Z(G) = L(G)G^{p^n}$, where $\exp(L(G)) = p^n$. Now by [15, Proposition 2.13], $\exp(G/L(G))$ divides $\exp(L(G))$ and so $Z(G) = L(G)G^{p^n} = L(G)$. Conversely, assume that L(G) = Z(G) and L(G) is cyclic. Since G be a non-abelian autonilpotent p-group of class 2, $\operatorname{Aut}^L(G) = \operatorname{Aut}(G)$, by [15, Lemma 2.11]. Therefore $\operatorname{Inn}(G) \leq \operatorname{Aut}^L(G)$, $G' \leq L(G)$ and G/L(G) is abelian. Obviously, $Z(G) = L(G) = L(G)G^{p^n}$, where $\exp(L(G)) = p^n$, and so $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$, by Theorem 3.2, as required.

Corollary 3.5. Let G be an extraspecial p-group.

- (i) If p > 2, then L(G) and $Aut^{L}(G)$ is trivial.
- (ii) If p = 2, then $L(G) \cong C_2$ and $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$.

Proof. Let G be an extraspecial p-group. First assume that p > 2. By [10, Theorem 3], L(G) is trivial and so $\operatorname{Aut}^{L}(G) = 1$.

To prove (ii), since |G'|=2, and G' is a characteristic subgroup of G, we have $G' \leq L(G) \leq Z(G)$. Thus $G' = L(G) = Z(G) = \Phi(G)$ is cyclic of order 2. Now by Theorem 3.2, $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$.

Let G be a finite non-abelian p-group such that G/L(G) is abelian. Then G is of class 2 and $\operatorname{Aut}^{G'}(G) \leq \operatorname{Aut}^{L}(G)$. Let $G/G' = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$, where $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. Also let $L(G) = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_l}}$, where $b_1 \geq b_2 \geq \cdots \geq b_l \geq 1$ and $G' = C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_n}}$, where $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$. Since $G' \leq L(G)$, by [2, Section 25] we have $n \leq l$ and $e_j \leq b_j$ for all $1 \leq j \leq n$. By the above notation, we prove the following theorem:

Theorem 3.6. Let G be a finite non-abelian p-group. Then $\operatorname{Aut}^L(G) = \operatorname{Aut}^{G'}(G)$ if and only if G' = L(G) or G' < L(G), d(G') = d(L(G)) and $a_1 = e_t$, where t is the largest integer between 1 and n such that $b_t > e_t$.

Proof. Suppose that $\operatorname{Aut}^L(G) = \operatorname{Aut}^{G'}(G)$ and $G' \neq L(G)$. By Theorem 2.5 and Lemma 2.3, we have $|\operatorname{Hom}(G/G', L(G))| = |\operatorname{Hom}(G/G', G')|$. First, we claim that d(G') = d(L(G)). Suppose, for a contradiction, that d(G') = n < l = d(L(G)). Since $b_j \geq e_j$ for all j such that $1 \leq j \leq n$, by Lemma 2.1,

$$\begin{split} |\mathrm{Aut}^{G'}(G)| &= |\mathrm{Hom}(G/G',G')| = |\mathrm{Hom}(G/G',C_{p^{e_1}}\times C_{p^{e_2}}\times \cdots \times C_{p^{e_n}})| \\ &\leq |\mathrm{Hom}(G/G',C_{p^{b_1}}\times C_{p^{b_2}}\times \cdots \times C_{p^{b_n}})| < |\mathrm{Hom}(G/G',C_{p^{b_1}}\times C_{p^{b_2}}\times \cdots \times C_{p^{b_n}}) \\ &\times |\mathrm{Hom}(G/G',C_{p^{b_{n+1}}}\times \cdots \times C_{p^{b_l}})| = |\mathrm{Hom}(G/G',C_{p^{b_1}}\times C_{p^{b_2}}\times \cdots \times C_{p^{b_l}})| \\ &= |\mathrm{Hom}(G/G',L(G))| = |\mathrm{Aut}^L(G)|, \end{split}$$

which is a contradiction. So n=l, as required. Next, since $|\operatorname{Aut}^L(G)|=|\operatorname{Aut}^{G'}(G)|$, we have

$$\prod_{1 \le i \le k, 1 \le j \le l} p^{\min\{a_i, b_j\}} = \prod_{1 \le i \le k, 1 \le j \le l} p^{\min\{a_i, e_j\}}.$$

Since $b_j \geq e_j$ for all j such that $1 \leq j \leq l$, we have $\min\{a_i, b_j\} \geq \min\{a_i, e_j\}$, where $1 \leq i \leq k, 1 \leq j \leq l$. Thus $\min\{a_i, b_j\} = \min\{a_i, e_j\}$, for all $1 \leq i \leq k, 1 \leq j \leq l$. Next, since G' < L(G), there exists some $1 \leq j \leq l$ such that $e_j < b_j$. Let t be the largest integer between 1 and n such that $e_t < b_t$. We show that $a_1 \leq e_t$. Suppose, on the contrary, that $a_1 > e_t$. Then by the above equality, we must have $\min\{a_1, b_t\} = \min\{a_1, e_t\} = e_t$, which is impossible. Hence $a_1 \leq e_t$. Let $\exp(G/Z(G)) = p^f$, where $f \in \mathbb{N}$. Since $\operatorname{cl}(G) = 2$, by [13, Lemma 0.4], $f = e_1$. But $a_1 \leq e_t \leq e_{t-1} \leq \cdots \leq e_1 = f \leq a_1$. Whence $a_1 = e_t$.

Conversely, if G' = L(G), then $\operatorname{Aut}^{G'}(G) = \operatorname{Aut}^{L}(G)$. Assume that G' < L(G), d(G') = n = d(L(G)) = l and $a_1 = e_t$, where t is the largest integer between 1 and n such that $b_t > e_t$. Now by Lemma 2.3,

$$|\mathrm{Aut}^{G'}(G)|=|\mathrm{Hom}(G/G',G')|=\prod_{1\leq i\leq k, 1\leq j\leq l}p^{\min\{a_i,e_j\}},$$

and by Theorem 2.5,

$$|\mathrm{Aut}^L(G)| = |\mathrm{Hom}(G/G', L(G))| = \prod_{1 \leq i \leq k, 1 \leq j \leq l} p^{\min\{a_i, b_j\}}.$$

Since $a_1 = e_t$, we have $1 \le a_k \le \cdots \le a_2 \le a_1 = e_t \le e_{t-1} \le \cdots \le e_2 \le e_1$. Thus $b_j \ge e_j \ge a_i$ for all $1 \le i \le k$ and $1 \le j \le t$, which shows that $\min\{a_i, e_j\} = a_i = \min\{a_i, b_j\}$ for $1 \le i \le k$ and $1 \le j \le t$. Since $e_j = b_j$ for all $j \ge t+1$, we have $\min\{a_i, e_j\} = \min\{a_i, b_j\}$ for all $1 \le i \le k$ and $t+1 \le j \le l$. Thus $\min\{a_i, e_j\} = \min\{a_i, b_j\}$ for all $1 \le i \le k$ and $1 \le j \le l$. Therefore $|\operatorname{Aut}^{G'}(G)| = |\operatorname{Aut}^L(G)|$. Since G' < L(G) we have $\operatorname{Aut}^{G'}(G) = \operatorname{Aut}^L(G)$, which completes the proof.

In [11], Meng and Guo proved that for a finite group G, if C_2 is not a direct factor of G, then $L(G) \leq \Phi(G)$. We end this section by characterizing the finite non-abelian p-groups G with cyclic Frattini subgroup for which $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$.

First, we give some basic results about the finite non-abelian p-groups G with cyclic Frattini subgroup.

Let n > 1. Following [1], we denote by $D_{2^{n+3}}^+$ and $Q_{2^{n+3}}^+$ the 2-groups of order 2^{n+3} defined by the following presentations.

$$D_{2^{n+3}}^+ = \langle a,b,c \mid a^{2^{n+1}} = b^2 = c^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, [b,c] = 1 \rangle,$$

 $\begin{aligned} Q_{2^{n+3}}^+ &= \langle a,b,c \mid a^{2^{n+1}} = b^2 = 1, a^b = a^{-1+2^n}, a^c = a^{1+2^n}, a^{2^n} = c^2, [b,c] = 1 \rangle. \\ \text{Note that if G is either $D_{2^{n+3}}^+$ or $Q_{2^{n+3}}^+$, then $\operatorname{cl}(G) = n+1$.} \end{aligned}$

In [1], Berger, Kovács and Newman proved the following result.

Theorem 3.7. [1, Theorem 2] If G is a finite p-group with $Z(\Phi(G))$ cyclic, then

$$G = E \times (G_0 * G_1 * \cdots * G_s),$$

where E is an elementary abelian, $G_1, ..., G_s$ are non-abelian of order p^3 , of exponent p for p odd and dihedral for p = 2, while $G_0 > 1$ if E > 1, $|G_0| > 2$ if s > 0, and G_0 is one of the following types: cyclic, non-abelian with a cyclic maximal subgroup, $D_{2^{n+2}} * \mathbb{Z}_4, S_{2^{n+2}} * \mathbb{Z}_4, D_{2^{n+3}}^+, Q_{2^{n+3}}^+, D_{2^{n+3}}^+ * \mathbb{Z}_4$, all with n > 1. Conversely, every such group has cyclic Frattini subgroup.

Theorem 3.8. [20, Theorem 2.3] Let G be a finite non-abelian p-group with cyclic Frattini subgroup $\Phi(G)$.

- (i) If p > 2, or p = 2 and cl(G) = 2, then $\Phi(G) \leq Z(G)$.
- (ii) If cl(G) > 2, then $G' = \Phi(G)$.

Lemma 3.9. [20, Lemma 2.4] Let G be a finite group with $\Phi(G) \leq Z(G)$. Then there is a bijection from $\operatorname{Hom}(G/G', \Phi(G))$ onto $\operatorname{Aut}^{\Phi}(G)$ associating to every homomorphism $f: G \to \Phi(G)$ the automorphism $x \mapsto xf(x)$ of G. In particular, if G is a p-group and $\exp(\Phi(G)) = p$, then $\operatorname{Aut}^{\Phi}(G) \cong \operatorname{Hom}(G/G', \Phi(G))$.

In the following theorem, we will make use Theorem 3.7, which is the structural theorem for p-groups with cyclic Frattini subgroup.

Theorem 3.10. Let G be a finite non-abelian p-group with cyclic Frattini subgroup. Then $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$ if and only if G is one of the following types: $C_2^m \times D_8^{*(s+1)}$ or $C_2^m \times (D_8^{*s} * Q_8)$, where $s, m \geq 0$.

Proof. Let $\operatorname{Aut}^L(G) = \operatorname{Aut}^\Phi(G)$. Hence $\operatorname{Aut}^\Phi(G)$ is abelian, G is of class 2 and by Theorem 3.8, $\Phi(G) \leq Z(G)$. It follows that $\exp(G') = \exp(G/Z(G)) = p$ and so |G'| = p. Assume that $|\Phi(G) : G'| = p^a$. Then $\Phi(G) \cong C_{p^{a+1}}$ and we observe that $\exp(G/G') \leq p^{a+1} = |\Phi(G)|$. Together with Lemma 3.9, we have $|\operatorname{Aut}^\Phi(G)| = |\operatorname{Hom}(G,\Phi(G))| = |G|/p$. Next, we note that $G' \cap L(G) \neq 1$; otherwise, $G' \cap L(G) = 1$ and $G' \times L(G)$ would be a subgroup of $\Phi(G)$. Hence either G' = 1 or L(G) = 1, a contradiction. Whence $G' \leq L(G)$. Now we are able to show that $G' = L(G) \cong C_p$. To do this, first assume that $L(G) \neq \Phi(G)$. By similar argument that was applied for Theorem 3.6, we have $\exp(G/G') \leq \exp(L(G))$, which implies that $\exp(G/L(G)) \leq \exp(G/G') \leq \exp(L(G)) = |L(G)|$. If $L(G) = \Phi(G)$, then $\exp(G/L(G)) = \exp(G/\Phi(G)) \leq \exp(L(G)) = |L(G)|$. Thus $|\operatorname{Aut}^L(G)| = |G/L(G)| = |\operatorname{Aut}^\Phi(G)| = |G/G'|$, by [12, Proposition 1] and so $G' = L(G) \cong C_p$. Now, we will make use of the notation of Theorem 3.7.

Since cl(G) = 2, by Theorem 3.7 and [5, Theorems 5.4.3 and 5.4.4], G_0 is one of the groups $M_p(n, 1)$, where $n \ge 3$, if p = 2; D_8 or Q_8 .

We claim that $G' = G'_0$ and $\Phi(G) = \Phi(G_0)$. To see this, since $G'_0 \cap G'_i \neq 1$ for $1 \leq i \leq s$ and $|G'_i| = p$, we have $G'_i \leq G'_0$ and so $G' = G'_0$. Also $\Phi(G) = G'G^p = G'_0E^pG_0^pG_1^p \cdots G_s^p = G'_0G_0^p = \Phi(G_0)$. To continue the proof, we may consider two cases:

Case I. E = 1.

Let $G = G_0 * T$, where T be one of the groups $M_p(1,1,1)^{*s}$, while p > 2 or D_8^{*s} , where all $s \ge 0$. Note that if s = 0, then $G = G_0$ and $Z(G) = Z(G_0) = \Phi(G_0) = \Phi(G)$; otherwise, since $1 \ne G_0 \cap T = Z(T) \le Z(G_0)$, then $Z(G) = Z(G_0)$, because |Z(T)| = p, which implies that $\Phi(G) = \Phi(G_0) = Z(G_0) = Z(G)$. We claim that G is an extraspecial p-group. To see this, since $G' = L(G) \cong C_p$, by Theorem 3.2, $\operatorname{Aut}^{\Phi}(G) = \operatorname{Aut}^{L}(G) = \operatorname{Inn}(G)$. This shows that G is an extraspecial p-group, by Theorem 2.2. If F > 2, then by Corollary 3.5, F = L(G) = 1, which is impossible. Whence F = 1 if F = 1 is yields that F = 1 is F = 1. This yields that F = 1 is an F = 1 in F

In this case $G_0 > 1$ and $G = E \times (G_0 * T)$, where T be one of the groups lying in Case I.

We claim that $\operatorname{Aut}^{\Phi(G_0*T)}(G_0*T)=\operatorname{Aut}^{L(G_0*T)}(G_0*T)$. Choose a non-trivial element σ of $\operatorname{Aut}^{\Phi(G_0*T)}(G_0*T)$. Then the map $\overline{\sigma}$ defined by $(ef)^{\overline{\sigma}}=ef^{\sigma}$, for all $e\in E, f\in G_0*T$ denotes an automorphism of $\operatorname{Aut}^{\Phi}(G)=\operatorname{Aut}^{L}(G)$. Since $G'\cap L(G_0*T)\neq 1$, then $L(G)\leq L(G_0*T)$ and so σ is in $\operatorname{Aut}^{L(G_0*T)}(G_0*T)$. This shows that $\operatorname{Aut}^{\Phi(G_0*T)}(G_0*T)=\operatorname{Aut}^{L(G_0*T)}(G_0*T)$, as required. Next, by a similar argument as mentioned for the previous case, G_0 be one of the groups: D_8 or Q_8 . Therefore G has one of the following types: $C_2^m\times D_8^{*(s+1)}$ or $C_2^m\times (D_8^{*s}*Q_8)$, where $s\geq 0, m>0$.

Conversely, assume that G be of the groups in Theorem 3.10. Hence $G' = L(G) \cong C_2$. Now the proof is complete, since $|\operatorname{Aut}^L(G)| = |\operatorname{Aut}^\Phi(G)| = |G|/2$.

4. Classify all finite *p*-groups G of order $p^n (3 \le n \le 5)$, such that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$

Let G be a non-abelian group of order p^3 . Then by Corollary 3.5, $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if p = 2. In the following corollaries, we use Theorems 4.7 and 5.1 of [11] and classify all finite p-groups G of order $p^n (4 \le n \le 5)$, such that $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. First we recall the following concept, which was introduced by Hall in [6].

Definition 4.1. Two finite groups G and H are said to be isoclinic if there exist isomorphisms $\phi: G/Z(G) \to H/Z(H)$ and $\theta: G' \to H'$ such that, if $(x_1Z(G))^{\phi} = y_1Z(H)$ and $(x_2Z(G))^{\phi} = y_2Z(H)$, then $[x_1, x_2]^{\theta} = [y_1, y_2]$. Notice that isoclinism is an equivalence relation among finite groups and the equivalence classes are called isoclinism families.

Corollary 4.2. Let G be a non-abelian group of order p^4 . Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if p = 2 and G is one of the following types: $M_2(3,1)$ or $M_2(2,1,1)$.

Proof. Assume that $|G| = p^4$ and $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. We claim that $|Z(G)| = p^2$. Suppose for a contradiction, that |Z(G)| = p. We observe that $G' \leq Z(G) \cong C_p$, by Theorem 3.2 and so G is an extraspecial p-group, a contradiction since the order of G is not of the form p^{2n+1} , for some natural number n. Therefore $G/Z(G) \cong C_p^2$, and hence |G'| = p. We consider two cases:

Case I. p an odd prime. It is straightforward to see that the map $\sigma: G \to G$ by $x^{\sigma} = x^{1+p}$, is an automorphism of G. Hence for any element x of L(G), $x = x^{\sigma} = x^{1+p}$, and so $x^p = 1$. Thus $\exp(L(G)) = p$ and so $G' = L(G) \cong C_p$, by Theorem 3.2. If $G/L(G) \cong C_{p^3}$, then by [3, Theorem 2.2], G is cyclic, a contradiction. Next, we assume that $G/L(G) \cong C_{p^2} \times C_p$. Then G is an abelian group by [11, Theorem 5.1], which is impossible. Finally, if $G/L(G) \cong C_p^3$, then $L(G) = \Phi(G)$ and so $\operatorname{Aut}^{\Phi}(G) = \operatorname{Inn}(G)$. Therefore by Theorem 2.2, G is an extraspecial p-group, a contradiction.

Case II. p=2. Since |G'|=2, and G' be a characteristic subgroup of G, we have $G' \leq L(G) \leq Z(G)$. Thus |L(G)|=2 or 4. If |L(G)|=4, then L(G)=Z(G) and $G/L(G)\cong C_2^2$. Hence by [11, Theorems 5.1 and 4.7], $G\cong M_2(2,2)$, and $L(G)\cong C_2^2$, which is a contradiction by Theorem 3.2. Next we assume that |L(G)|=2. So G'=L(G) and |G/L(G)|=8. By a similar argument, G is isomorphic to one of the following groups: $M_2(3,1)$ or $M_2(2,1,1)$. The converse follows at once from Theorem 3.2.

Corollary 4.3. Let G be a non-abelian group of order p^5 . Then $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$ if and only if p = 2 and G is one of the following types: $M_2(3,2)$, $M_2(4,1)$, $M_2(2,2,1)$, D_8^{*2} or $D_8 * Q_8$.

Proof. Let G be a finite group such that $|G| = p^5$ and $\operatorname{Aut}^L(G) = \operatorname{Inn}(G)$. We consider two cases:

Case I. p > 2. These groups lying in the isoclinism families (5), (4) or (2) of [8, 4.5] and we show that $\operatorname{Aut}^L(G) \neq \operatorname{Inn}(G)$.

First, let G denote one of the groups in the isoclinism family (5). Hence |Z(G)| = p and $G' = Z(G) = \Phi(G) \cong C_p$, by Theorem 3.2. So G is an extraspecial p-group and by Corollary 3.5, |L(G)| = 1, a contradiction.

Next, let G be one of the groups in the isoclinism family (4). Then $G' \cong C_p^2$, which is a contradiction, since G' is cyclic.

Finally, let G denote one of the groups in the isoclinism family (2). Then $G/Z(G)\cong C_p^2$ and so d(G/L(G))>1. We observe that $G'=L(G)\cong C_p$ and $Z(G)=\Phi(G)$, by using Theorems 2.2, 3.2, [3, Theorem 2.2] and [11, Theorem 5.1]. So d(G)=2 and by [16], G is a minimal non-abelian p-group. If $G/L(G)\cong C_{p^3}\times C_p$, then G is an abelian group, by [11, Theorem 5.1], a contradiction. If $G/L(G)\cong C_{p^2}$, then by [16], $G\cong M_p(3,2)$ or $G\cong M_p(2,2,1)$. Thus L(G)=1, by [11, Theorem 4.7], a contradiction. Finally, assume that $G/L(G)\cong C_{p^2}\times C_p^2$ or $G/L(G)\cong C_p^4$. In this cases, $\operatorname{Aut}^L(G)\neq \operatorname{Inn}(G)$, by Theorem 2.5.

Case II. p=2. We can see that |L(G)|=2,4, by [3, Theorem 2.2] and [11, Theorem 5.1]. First, we assume that |L(G)|=4. Since G is a non-cyclic group, by [3, Theorem 2.2], d(G/L(G))>1. It follows that G/L(G) is one of the groups C_2^3 or $C_4\times C_2$. Now in the first case, $L(G)=\Phi(G)$ and so G is an extraspecial 2-group by Theorem 2.2. Hence $G'=L(G)\cong C_2$, a contradiction. Therefore $G/L(G)\cong C_4\times C_2$ and by [11, Theorems 5.1 and 4.7], G is one of the groups: $M_2(2,3)$ or $M_2(3,1,1)$, and $L(G)\cong C_2^2$, a contradiction by Theorem 3.2. Now we may suppose that |L(G)|=2. So $G'=L(G)\cong C_2$. We discuss the following cases.

If $G/L(G) \cong C_2^4$, then $L(G) = \Phi(G)$ and so $\operatorname{Aut}^\Phi(G) = \operatorname{Inn}(G)$. Therefore by Theorem 2.2, G is an extraspecial 2-group. Thus G is one of the groups D_8^{*2} or $D_8 * Q_8$, by [21]. Next, suppose that $G/L(G) \cong C_4 \times C_2^2$. Hence $G/L(G) = \langle \bar{a}, \bar{b}, \bar{c} \rangle$, where $\bar{a} = aL(G), \bar{b} = bL(G), \bar{c} = cL(G)$ and $o(\bar{a}) = 4$, $o(\bar{b}) = o(\bar{c}) = 2$. Therefore $G = \langle a, b, c, L(G) \rangle = \langle a, b, c \rangle$, by [11, Corollary 3.7]. Since $\langle a^2 \rangle \times G' \leq Z(G)$, we have either $Z(G) \cong C_4 \times C_2$ or C_2^2 . If $Z(G) \cong C_4 \times C_2$, then $\operatorname{Aut}^L(G) \neq \operatorname{Inn}(G)$, by Theorem 2.5. Therefore $Z(G) \cong C_2^2$. Now by using GAP [4], we find that there are no such groups. Next, if $G/L(G) \cong C_8 \times C_2$, then $G \cong M_2(4,1)$, by [11, Theorem 5.1]. Finally, suppose that $G/L(G) \cong C_4^2$. Then d(G) = 2, by [11, Corollary 3.7] and $G' = L(G) \cong C_2$. Hence by [16], G is a minimal non-abelian 2-group. Thus G is isomorphic to the group $M_2(3,2)$ or $M_2(2,2,1)$. The converse follows at once from Theorem 3.2.

Acknowledgments

The author is grateful to the referees for their valuable suggestions. The paper was revised according to these comments. This research was in part supported by a grant from Payame Noor University.

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[DOI: 10.52547/ijmsi.17.2.97]

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