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# Groups whose Bipartite Divisor Graph for Character Degrees Has Five Vertices

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Abstract. Let $G$ be a finite group and $\mathrm{cd}^*(G)$ be the set of nonlinear irreducible character degrees of $G$. Suppose that $\rho(G)$ denotes the set of primes dividing some element of $\mathrm{cd}^*(G)$. The bipartite divisor graph for the set of character degrees which is denoted by $B(G)$, is a bipartite graph whose vertices are the disjoint union of $\rho(G)$ and $\mathrm{cd}^*(G)$, and a vertex $p \in \rho(G)$ is connected to a vertex $a \in \mathrm{cd}^*(G)$ if and only if $p|a$. In this paper, we investigate the structure of a group $G$ whose graph $B(G)$ has five vertices. Especially we show that all these groups are solvable.

**Keywords:** Bipartite divisor graph, Character degree, Solvable group.

**2000 Mathematics subject classification:** Primary:20C15; Secondary:20D60.

## 1. Introduction

Throughout this paper, $G$ is a finite group. We write $\mathrm{cd}(G)$ to denote the set of irreducible character degrees of the group $G$, and we use $\mathrm{cd}^*(G)$ for the set $\mathrm{cd}(G) \setminus \{1\}$. Suppose that $\rho(G)$ is the set of primes dividing some element of $\mathrm{cd}^*(G)$. Exploring the interplay between the structure of a finite group $G$ and the set $\mathrm{cd}(G)$ is a favorite research field in group theory. One of the questions that was studied extensively is the graphs attached to the set $\mathrm{cd}(G)$. A comprehensive survey on this topic can be found in [6]. The prime graph $\Delta(G)$ and the common divisor graph $\Gamma(G)$ are two important graphs associated to $\mathrm{cd}(G)$. The prime graph $\Delta(G)$ is the graph with vertex set $\rho(G)$

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and there is an edge between two vertices $p$ and $q$ if there exist some $n \in \mathrm{cd}(G)$ which is divisible by $pq$. The common-divisor graph $\Gamma(G)$ is the graph with vertex set $\mathrm{cd}^*(G)$ and two vertices $m$ and $n$ are connected if $\gcd(m,n) > 1$. In this paper we focus on bipartite divisor graph for $\mathrm{cd}^*(G)$. The bipartite divisor graph $B(G)$ is a bipartite graph whose vertices are the disjoint union of $\rho(G)$ and $\mathrm{cd}^*(G)$ and a vertex $p \in \rho(G)$ is connected to a vertex $a \in \mathrm{cd}^*(G)$ if and only if $p|a$. Groups whose $\Delta(G)$ or $\Gamma(G)$ has few vertices has been studied by many authors. For example, the prime graphs with four or fewer vertices are considered in papers [4, 8, 9]. In this note, we do an analogous work for bipartite divisor graph. The notion of $B(G)$ is introduced in [11] and groups whose $B(G)$ is a path or cycle are discussed in [2]. In this paper, we consider groups whose bipartite divisor graph has five vertices and obtain some group theoretical properties of these groups. We also provide examples of each possible graph.

## 2. Preliminaries

The following theorems will be used throughout the paper.

**Theorem 2.1** (corollary 12.34 of [5]). *Let $G$ be solvable. Then $G$ has a normal abelian Sylow $p$-subgroup iff every element of $\mathrm{cd}(G)$ is relatively prime to $p$.*

**Theorem 2.2** (corollary 12.2 of [5]). *Suppose $p|\chi(1)$ for every nonlinear $\chi \in \mathrm{Irr}(G)$, where $p$ is a prime. Then $G$ has a normal $p$-complement.*

**Theorem 2.3** (Theorem 4.5 of [11]). *Let $G$ be a group whose $B(G)$ is a complete bipartite graph. Then one of the following cases occurs:*

(a) *$G = AH$, where $A$ is an abelian normal Hall subgroup of $G$ and $H$ is abelian, i.e. $G$ is metabelian.*
(b) *$G = AH$, where $A$ is an abelian normal Hall subgroup of $G$ and $H$ is a non-abelian $p$-group for some prime $p$. In particular, $\rho(G) = \{p\}$.*

*Remark* 2.4. Theorem 2.3 implies that the subgroup $H$ is a $\rho(G)$-subgroup and $A$ is a $\rho(G)'$-subgroup of $G$.

The following theorem from [12], helps us to obtain some examples of groups with a given set of character degrees.

**Theorem 2.5** (Theorem 4.1 of [12]). *Let $1 < m_1 < \cdots < m_r$ be integers such that $m_i$ divides $m_{i+1}$ for all $i = 1, 2, \cdots, r-1$. Then there exists a group $G$ with $\mathrm{cd}(G) = \{1, m_1, \cdots, m_r\}$.*

We also use the library of the small groups in GAP [1] for many examples and the $k$th group of order $n$ in this library is recognizable by command SmallGroup$(n,k)$ which is the symbol we use for this group.

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## 3. Groups whose $B(G)$ has five vertices

By definition of $\rho(G)$ and $\mathrm{cd}^*(G)$, each $p \in \rho(G)$ divides some element in $\mathrm{cd}^*(G)$ and every element $n \in \mathrm{cd}^*(G)$ is divisible by a prime number which lies in $\rho(G)$. Therefore, the graph $B(G)$ has no isolated vertex. Let $G$ be a group whose graph $B(G)$ has five vertices. Using the fact that vertices of $B(G)$ are disjoint union of $\rho(G)$ and $\mathrm{cd}^*(G)$, one of the following cases occur:

(i) $|\rho(G)| = 1$ and $\mathrm{cd}^*(G) = 4$.
(ii) $|\rho(G)| = 4$ and $\mathrm{cd}^*(G) = 1$.
(iii) $|\rho(G)| = 2$ and $\mathrm{cd}^*(G) = 3$.
(iv) $|\rho(G)| = 3$ and $\mathrm{cd}^*(G) = 2$.

We investigate each case separately and determine the structure of groups of each possible case.

**Theorem 3.1.** *Let $G$ be a group whose $B(G)$ has five vertices. If $|\rho(G)| = 1$ then $G = AP$, where $P$ is a Sylow $p$-subgroup for some prime $p$ and $A$ is a normal abelian $p$-complement.*

*Proof.* By hypothesis, $|\rho(G)| = 1$, hence $\mathrm{cd}(G) = \{1, p^a, p^b, p^c, p^d\}$ where $p$ is a prime number. Since $p$ divides the degree of every character, by Theorem 2.2, we conclude that $G$ has a normal $p$-complement. Therefore $G = AP$ where $P$ is a Sylow $p$-subgroup of $G$ and $A$ is a normal $p$-complement. In addition, every prime divisor of $|A|$ is coprime to $p$ and by corollary 12.34 of [5], $A$ is abelian and we are done. □

**Theorem 3.2.** *Let $G$ be a group whose $B(G)$ has five vertices. If $|\rho(G)| = 4$ then $G'$ is abelian, $G' \cap Z(G) = 1$ and $G/Z(G)$ is a Frobenius group with cyclic complement.*

*Proof.* Since $B(G)$ has five vertices and $|\rho(G)| = 4$, therefore $\mathrm{cd}(G) = \{1, m\}$ where $m$ is divisible by exactly four prime numbers. By corollary 12.6 of [5], $G'$ is abelian. Assume that $G$ is nilpotent. Since $|\mathrm{cd}(G)| = 2$, $G$ is nonabelian. Let $P$ be a nonabelian Sylow $p$-subgroup of $G$. If $G$ has another nonabelian Sylow $q$-subgroup for some prime $q \neq p$, then $|\mathrm{cd}(G)| > 2$ which is a contradiction. Hence $P$ is the only nonabelian Sylow subgroup of $G$ which implies that every element of $\mathrm{cd}(G)$ must be a prime power which is a contradiction. Therefore $G$ is not nilpotent. Now by using Theorem (C) of [3], we obtain the results. □

Example 3.3. Groups that satisfy hypothesis of Theorems 3.1 and 3.2 exist. For example, let $P$ be a $p$-group of order $p^3$ with $\mathrm{cd}(P) = \{1, p\}$ and let $G_1 = P \times P \times P \times P$, then

$$\mathrm{cd}(G_1) = \{1, p, p^2, p^3, p^4\}.$$

Therefore $G$ is a group that satisfy hypothesis of Theorem 3.1. Furthermore, if we replace $P$ by the direct product of an abelian group $A$ and the group $P$,

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then we have an example of Theorem 3.1 with non-trivial subgroup $A$.
For a group which satisfy Theorem 3.2, suppose that $F = GF(211)$ is a finite field with 211 elements and let $H$ be the multiplicative group of $F$ with 210 elements. Then $H$ act Frobeniously on $F$ and so the corresponding semidirect product $G_2 = FH$ is a Frobenius group with abelian kernel and complement. It is easy to check that $\mathrm{cd}(G_2) = \{1, 210\}$ and the results of Theorem 3.2 hold.

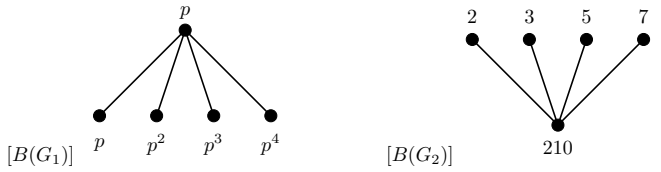


FIGURE 1. Graphs of example 3.3.

**Theorem 3.4.** *Let $G$ be a group whose $B(G)$ has five vertices. If $|\rho(G)| = 2$ then $G$ is solvable and one of the following cases occurs:*

(i) *$G = HN$ where $H$ is a Sylow $p$-subgroup of $G$ or a Hall $\{p, q\}$-subgroup and $N$ is a normal complement.*
(ii) *$G$ is one of the families stated in* [7].

*Proof.* Since $B(G)$ has no isolated vertex, it is easy to check that $B(G)$ is one of the graphs in Figure 2.

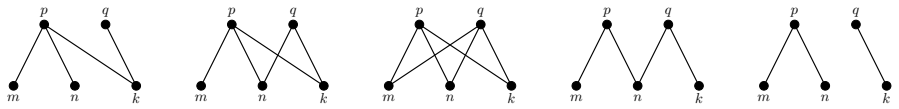


FIGURE 2. Possible graphs with $|\rho(G)| = 2$.

First, suppose that $B(G)$ is as graph (a) or (b) in Figure 2 where $p$ and $q$ are prime numbers and $\mathrm{cd}(G) = \{1, m, n, k\}$. Since $p$ divides every nonlinear character degree of $G$, by Theorem 2.2, $G$ has a normal $p$-complement, therefore $G = PN$ where $P$ is a Sylow $p$-subgroup of $G$ and $N$ is a normal $p$-complement and case (i) occurs. We claim that in both cases $G$ is solvable. Suppose that $G$ is not solvable. Note that in both graphs (a) and (b) of Figure 2 there is a prime which divides every nonlinear character degree. Since $|\mathrm{cd}(G)| = 4$, using Theorem A and B of [10] we have $\mathrm{cd}(G) = \{1, r-1, r, r+1\}$ for some prime power $r$ or $\mathrm{cd}(G) = \{1, 9, 10, 16\}$. In both cases there is no prime which divides every nonlinear character degree, thus graphs (a) and (b) can not occur

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for these groups. Therefore $G$ is a solvable group. If $B(G)$ is as graph (c) in Figure 2, then $B(G)$ is a complete bipartite graph and by Theorem 2.3 case (i) holds. Furthermore, by corollary 4.2 of [11], $G$ is solvable. If $B(G)$ is as graph (d) in Figure 2, then $B(G)$ is a path of length four and by proposition 2 of [2], $G$ is a solvable group. Since for every prime $r \neq p, q$, $r$ divides no character degree, using Theorem 2.1, we see that case (i) occurs. Now suppose that $B(G)$ is as graph (e) in Figure 2. Therefore $B(G)$ is a disconnected graph and has two connected components. We prove that $G$ is solvable. Assume that $G$ is not solvable. Again by Theorem A and B of [10], we must have $\mathrm{cd}(G) = \{1, r-1, r, r+1\}$ for some prime power $r$. Since $\mathrm{cd}(G) = \{1, p^a, p^b, q^c\}$ it follows that $p$ divides two consecutive numbers which is impossible. Hence $G$ is solvable and by Theorem 2.1 of [11], $G$ belongs to a family of groups stated in [7] and case (ii) holds. □

The following example shows that all graphs in Figure 2 occur as $B(G)$ for some group $G$.

Example 3.5. Let $G_1 = \mathrm{SmallGroup}(108, 17)$, then $\mathrm{cd}(G_1) = \{1, 2, 4, 6\}$ and $B(G_1)$ is the same as graph (a) in Figure 2.
By Theorem 2.5, there is a group $G_2$ with $\mathrm{cd}(G_2) = \{1, 2, 6, 12\}$ and $B(G_2)$ is as graph (b) in Figure 2.
Suppose that $G_3 = \mathrm{SmallGroup}(108, 17)$, then $\mathrm{cd}(G_3) = \{1, 6, 12, 18\}$ and $B(G_3)$ is the graph (c) in Figure 2.
Assume that $G_4 = \mathrm{SmallGroup}(72, 15)$, then we have $\mathrm{cd}(G_4) = \{1, 2, 3, 6\}$ and $B(G_4)$ is the graph (d) in Figure 2.
Put $G_5 = \mathrm{SmallGroup}(48, 28)$, then $\mathrm{cd}(G_5) = \{1, 2, 3, 4\}$ and $B(G_5)$ is the graph (e) in Figure 2.

**Theorem 3.6.** *Let $G$ be a group whose $B(G)$ has five vertices. If $|\rho(G)| = 3$ then $G$ is solvable and one of the following cases holds:*

(i) *$G = HN$ where $H$ is a Sylow $p$-subgroup or a Hall $\{p, q\}$-subgroup or a Hall abelian $\{p, q, r\}$-subgroup of $G$ and $N$ is its normal complement.*
(ii) *$G = QN$ where $Q$ is an abelian Sylow $q$-subgroup of $G$ and $N$ is its normal complement.*
(iii) *$G$ is one of the families stated in* [7].

*Proof.* It's easy to verify that $B(G)$ is one of the graphs in Figure 3. Since $|\mathrm{cd}(G)| = 3$ Theorem 12.15 of [5] shows that in all cases $G$ is a solvable group. Now we investigate each graph separately.

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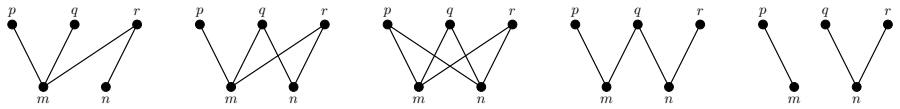


FIGURE 3. Possible graphs with $|\rho(G)| = 3$.

First suppose that $B(G)$ is as graph (a) in Figure 3. Then $\mathrm{cd}(G) = \{1, p^a q^b r^c, r^d\}$ and $r$ divides every nonlinear character degree.Thus Theorem 2.2 implies that $G$ has a normal $r$-complement. Therefore $G = HN$ where $H$ is a Sylow $r$-subgroup of $G$ and $N$ is a normal $r$-complement and case (i) of theorem holds. Now assume that $B(G)$ is the graph (b) in Figure 3. In this case, we have $\mathrm{cd}(G) = \{1, p^a q^b r^c, q^d r^h\}$ and every nonlinear character degree is divisible by both $q$ and $r$. Again Theorem 2.1 applies and $G = HN$ which $H$ is a Hall $\{r, q\}$-subgroup and $N$ is its normal complement. Hence case (i) occurs. If $B(G)$ is the graph (c) in Figure 3, then $B(G)$ is a complete graph. Applying Theorem 2.3, we have $G = HN$ where $H$ is a Hall abelian $\rho(G)$-subgroup and $N$ is an abelian normal complement of $H$. Therefore again case (i) of theorem holds. Now suppose that $B(G)$ is the graph (d) in Figure 3. Then $\mathrm{cd}(G) = \{1, p^a q^b, q^c r^d\}$. Since $q$ divides every nonlinear character degree, $G$ has a normal $q$-complement. Therefore, $G = QN$ where $Q$ is a Sylow $q$-subgroup and $N$ is its normal complement. Since $Q \cong G/N$ and $\mathrm{cd}(G)$ contains no powers of $q$, therefore $Q$ is abelian. Hence case (ii) of theorem holds. Finally, suppose that $B(G)$ is the graph (e) in Figure 3. Then $B(G)$ is disconnected and has two connected components. Since $G$ is solvable, Theorem 2.1 of [11] implies that $G$ belongs to a family of groups stated in [7] and case (iii) holds. □

EXAMPLE 3.7. We show that all graphs in Figure 3 really occur as $B(G)$ for some group $G$.
By Theorem 2.5 there exists a group $G_1$ with $\mathrm{cd}(G_1) = \{1, 5, 30\}$ and the graph $B(G_1)$ is the graph (a) in Figure 3.
Since $15|30|60$, Theorem 2.5 implies that there is a group $G_2$ with $\mathrm{cd}(G_2) = \{1, 15, 30\}$ and there is a group $G_3$ with $\mathrm{cd}(G) = \{1, 30, 60\}$. Thus the graphs $B(G_2)$ and $B(G_3)$ are the graphs (b) and (c) in Figure 3, respectively.
Suppose that $G_4 = \mathrm{SmallGroup}(960, 5748)$, then $\mathrm{cd}(G_4) = \{1, 12, 15\}$ and $B(G_4)$ is the graph (d) in Figure 3.
Let $G_5 = \mathrm{SmallGroup}(480, 1188)$, then $\mathrm{cd}(G_5) = \{1, 2, 15\}$ and the graph $B(G_5)$ is the graph (e) in Figure 3.

Suppose that $G$ is a group which satisfy hypothesis of Theorem 3.6, then $\mathrm{cd}(G) = \{1, m, n\}$. If we apply Theorems of [12], then we can obtain more information about the structure of $G$, depending on the relation between $\pi(m)$

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and $\pi(n)$, where $\pi(l)$ denotes the prime divisors of $l$. For details see [12].

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