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Generalized Frames for $B(\mathcal{H}, \mathcal{K})$

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Abstract. Frames play significant role in various areas of science and engineering. Motivated by the work of Chander Shekhar, S. K. Kaushik and Abas Askarizadeh, Mohammad Ali Dehghan, we introduce the concepts of K-frames for $B(\mathcal{H}, \mathcal{K})$ and we establish some result. Also, we consider the relationships between K-Frames and K-Operator Frames for $B(\mathcal{H})$.

Keywords: Frame, K-operator frame, C^*-algebra, Hilbert C^*-modules.

2000 Mathematics subject classification: 42C15, 47A05.

1. Introduction and preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [7] in 1952 to study some deep problems in nonharmonic Fourier series, after the fundamental paper [5] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [8].

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Traditionally, frames have been used in signal processing, image processing, data compression and sampling theory. A discreet frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements.

In this paper, we introduce a new notion of K-frames for $B(\mathcal{H},\mathcal{K})$ and we consider the relationships between K-frames and K-operator Frames for $B(\mathcal{H})$, the set of all bounded operators on a Hilbert space $\mathcal{H}$.

Let I be a finite or countable index subset of $\mathbb{N}$. In this section we briefly recall the definitions and basic properties of C^*-algebra, Hilbert $\mathcal{A}$-modules, Frames, K-perator Frames for $B(\mathcal{H})$ and K-g-frames. For information about frames in Hilbert spaces we refer to [3]. Our reference for C^*-algebras is [6, 4]. For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. [4]. If $\mathcal{A}$ is a Banach algebra, an involution is a map $a \to a^*$ of $\mathcal{A}$ into itself such that for all a and b in $\mathcal{A}$ and all scalars α the following conditions hold:

(1) $(a^*)^* = a$.
(2) $(ab)^* = b^*a^*$.
(3) $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$.

Definition 1.2. [4]. A $\mathcal{C}^*$-algebra $\mathcal{A}$ is a Banach algebra with involution such that :

$$\|a^*a\| = \|a\|^2$$

for every a in $\mathcal{A}$.

Example 1.3. $\mathcal{B} = B(\mathcal{H})$ the algebra of bounded operators on a Hilbert space, is a $\mathcal{C}^*$-algebra, where for each operator A, A^* is the adjoint of A.

Definition 1.4. [9]. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(1) $\langle x, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.
(2) $\langle ax + y, z\rangle = a\langle x, y\rangle + \langle y, z\rangle$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
(3) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

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EXAMPLE 1.5. Let $\mathcal{H}$ be a Hilbert space, then $B(\mathcal{H})$ is a Hilbert C^*-module with the inner product $\langle T,S\rangle = TS^*, \forall T,S \in B(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, A map $T:\mathcal{H}\to\mathcal{K}$ is said to be adjointable if there exists a map $T^*:\mathcal{K}\to\mathcal{H}$ such that $\langle Tx,y\rangle_{\mathcal{A}} = \langle x,T^*y\rangle_{\mathcal{A}}$ for all $x\in\mathcal{H}$ and $y\in\mathcal{K}$.

We also reserve the notation $End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H})$ is abbreviated to $End^*_{\mathcal{A}}(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and let $B(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. Then $B(\mathcal{H},\mathcal{K})$ is a Hilbert $B(\mathcal{K})$-module with the inner product $\langle T,S\rangle = TS^*, \forall T,S\in B(\mathcal{H},\mathcal{K})$.

Definition 1.6. [2] A sequence $\{T_i \in B(\mathcal{H},\mathcal{K}) : i\in I\}$ is said to be a frame for $B(\mathcal{H},\mathcal{K})$ if there exist $0<A,B<\infty$ sach that

$$A\langle T,T\rangle \le \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \le B\langle T,T\rangle, \forall T\in B(\mathcal{H},\mathcal{K}), \tag{1.1}$$

where the series converges in the strong operator topology.

Definition 1.7. [1] Let $K\in B(\mathcal{H})$. A sequence $\{\Lambda_i \in B(\mathcal{H},\mathcal{K}_i) : i\in I\}$ is called a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$, if there exist constants $A,B>0$ such that

$$A\|K^*x\|^2 \le \sum_{i\in I}\|\Lambda_i x\|^2 \le B\|x\|^2, \forall x\in\mathcal{H}. \tag{1.2}$$

The constants A and B are called lower and upper bounds for the K-g-frame, respectively. A K-g-frame $\{\Lambda_i\}_{i\in I}$ is said to be tight if there exists a constant $A>0$ such that

$$A\|K^*x\|^2 = \sum_{i\in I}\|\Lambda_i x\|^2, \forall x\in\mathcal{H}. \tag{1.3}$$

It is called Parseval K-g-frame if $A=1$ in 1.3.

Definition 1.8. [11] Let $K\in B(\mathcal{H})$. A family of bounded linear operators $\{T_i\}_{i\in I}$ on a Hilbert space $\mathcal{H}$ is said to be a K-operator frame for $B(\mathcal{H})$, if there exist positive constants $A,B>0$ such that

$$A\|K^*x\|^2 \le \sum_{i\in I}\|T_i x\|^2 \le B\|x\|^2, \forall x\in\mathcal{H}. \tag{1.4}$$

where A and B are called lower and upper bounds for the K-operator frame, respectively. A K-operator frame $\{T_i\}_{i\in I}$ is said to be tight if there exists a constant$A>0$ such that

$$A\|K^*x\|^2 = \sum_{i\in I}\|T_i x\|^2, \forall x\in\mathcal{H}. \tag{1.5}$$

It is called Parseval K-operator frame if $A=1$ in 1.5.

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2. K-FRAME FOR $B(\mathcal{H},\mathcal{K})$

Now we are ready to define the K-frame for $B(\mathcal{H},\mathcal{K})$.

Definition 2.1. Let $K \in B(\mathcal{H})$. A sequence $\{T_i \in B(\mathcal{H},\mathcal{K}) : i \in I\}$ is said to be a K-frame for $B(\mathcal{H},\mathcal{K})$ if there exist $0 < A, B < \infty$ sach that

$$A\langle TK, TK\rangle \leq \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \leq B\langle T, T\rangle, \forall T \in B(\mathcal{H},\mathcal{K}), \tag{2.1}$$

where the series converges in the strong operator topology.

Similar to Remark 1 in [10] we have

Remark 2.2. (1) Every frame for $B(\mathcal{H},\mathcal{K})$ is a K-frame, for any $K \in B(\mathcal{H})$: $K \neq 0$.
(2) If $K \in B(\mathcal{H})$ is a surjective operator, then every K-frame for $B(\mathcal{H},\mathcal{K})$ is a frame for $B(\mathcal{H},\mathcal{K})$.

The frame operator $S : B(\mathcal{H},\mathcal{K}) \to B(\mathcal{H},\mathcal{K})$ for the K-frame is given by

$$ST = \sum_{i\in I}\langle T, T_i\rangle T_i = \sum_{i\in I} TT_i^*T_i.$$

Remark 2.3. The frame operator is positive and adjointable, but not invertible in general.

Theorem 2.4. *Let $K \in B(\mathcal{H})$ and $\{T_i \in B(\mathcal{H},\mathcal{K}) : i \in I\}$ be a frame for $B(\mathcal{H},\mathcal{K})$. Then $\{T_iK^*\}_{i\in I}$ is a K-frame for $B(\mathcal{H},\mathcal{K})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a frame for $B(\mathcal{H},\mathcal{K})$.

Then there exists two positive constants A and B, such that

$$A\langle T, T\rangle \leq \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \leq B\langle T, T\rangle, \forall T \in B(\mathcal{H},\mathcal{K}). \tag{2.2}$$

Replacing T by TK in (2.2) we obtain

$$A\langle TK, TK\rangle \leq \sum_{i\in I}\langle TK, T_i\rangle\langle T_i, TK\rangle \leq B\langle TK, TK\rangle, \forall T \in B(\mathcal{H},\mathcal{K}).$$

But

$$\begin{aligned}\sum_{i\in I}\langle TK, T_i\rangle\langle T_i, TK\rangle &= \sum_{i\in I} TKT_i^*T_i(TK)^* \\ &= \sum_{i\in I} TKT_i^*T_iK^*T^* \\ &= \sum_{i\in I} T(T_iK^*)^*T_iK^*T^* \\ &= \sum_{i\in I}\langle T, T_iK^*\rangle\langle T_iK^*, T\rangle.\end{aligned}$$

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So

$$A\langle TK,TK\rangle \leq \sum_{i\in I}\langle T,T_iK^*\rangle\langle T_iK^*,T\rangle \leq B\langle TK,TK\rangle, \forall T\in B(\mathcal{H},\mathcal{K}).$$

On the other hand, for all $x\in\mathcal{K}$, we have

$$\begin{aligned} B\langle\langle TK,TK\rangle x,x\rangle &= B\langle TK(TK)^*x,x\rangle \\ &= B\langle TKK^*T^*x,x\rangle \\ &= B\langle K^*T^*x,K^*T^*x\rangle \\ &= B\|K^*T^*x\|^2 \\ &\leq B\|K\|^2\|T^*x\|^2 \\ &= B\|K\|^2\langle T^*x,T^*x\rangle \\ &= B\|K\|^2\langle TT^*x,x\rangle \\ &= B\|K\|^2\langle\langle T,T\rangle x,x\rangle. \end{aligned}$$

Hence

$$B\langle TK,TK\rangle \leq B\|K\|^2\langle T,T\rangle.$$

(Where $\|K\|$ is the operator norm)

From the above we have

$$A\langle TK,TK\rangle \leq \sum_{i\in I}\langle T,T_iK^*\rangle\langle T_iK^*,T\rangle \leq B\|K\|^2\langle T,T\rangle, \forall T\in B(\mathcal{H},\mathcal{K}).$$

Then $\{T_iK^*\}_{i\in I}$ is a K-frame for $B(\mathcal{H},\mathcal{K})$. □

Theorem 2.5. *Let $K_1,K_2\in B(\mathcal{H})$ and $\{T_i\in B(\mathcal{H},\mathcal{K}):i\in I\}$ be a K_1-frame for $B(\mathcal{H},\mathcal{K})$. Then $\{T_iK_2^*\}_{i\in I}$ is a K_2K_1-frame for $B(\mathcal{H},\mathcal{K})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a K_1-frame for $B(\mathcal{H},\mathcal{K})$.

Then there exists two positive constants A and B, such that

$$A\langle TK_1,TK_1\rangle \leq \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \leq B\langle T,T\rangle, \forall T\in B(\mathcal{H},\mathcal{K}). \tag{2.3}$$

Replacing T by TK_2 in (2.3) we obtain

$$A\langle TK_2K_1,TK_2K_1\rangle \leq \sum_{i\in I}\langle TK_2,T_i\rangle\langle T_i,TK_2\rangle \leq B\langle TK_2,TK_2\rangle, \forall T\in B(\mathcal{H},\mathcal{K}).$$

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But

$$\begin{aligned}\sum_{i\in I}\langle TK_2,T_i\rangle\langle T_i,TK_2\rangle &= \sum_{i\in I} TK_2T_i^*T_i(TK_2)^*\\ &= \sum_{i\in I} TK_2T_i^*T_iK_2^*T^*\\ &= \sum_{i\in I} T(T_iK_2^*)^*T_iK_2^*T^*\\ &= \sum_{i\in I}\langle T,T_iK_2^*\rangle\langle T_iK_2^*,T\rangle.\end{aligned}$$

So

$$A\langle TK_2K_1,TK_2K_1\rangle \leq \sum_{i\in I}\langle T,T_iK_2^*\rangle\langle T_iK_2^*,T\rangle \leq B\langle TK_2,TK_2\rangle, \forall T\in B(\mathcal{H},\mathcal{K}).$$

On the other hand, for all $x\in\mathcal{K}$, we have

$$\begin{aligned}B\langle\langle TK_2,TK_2\rangle x,x\rangle &= B\langle TK_2(TK_2)^*x,x\rangle\\ &= B\langle TK_2K_2^*T^*x,x\rangle\\ &= B\langle K_2^*T^*x,K_2^*T^*x\rangle\\ &= B\|K_2^*T^*x\|^2\\ &\leq B\|K_2\|^2\|T^*x\|^2\\ &= B\|K_2\|^2\langle T^*x,T^*x\rangle\\ &= B\|K_2\|^2\langle TT^*x,x\rangle\\ &= B\|K_2\|^2\langle\langle T,T\rangle x,x\rangle.\end{aligned}$$

Hence

$$B\langle TK_2,TK_2\rangle \leq B\|K_2\|^2\langle T,T\rangle.$$

(Where $\|K_2\|$ is the operator norm)

From the above we have

$$A\langle TK_2K_1,TK_2K_1\rangle \leq \sum_{i\in I}\langle T,T_iK_2^*\rangle\langle T_iK_2^*,T\rangle \leq B\|K_2\|^2\langle T,T\rangle, \forall T\in B(\mathcal{H},\mathcal{K}).$$

Then $\{T_iK_2^*\}_{i\in I}$ is a K_2K_1-frame for $B(\mathcal{H},\mathcal{K})$. □

Corollary 2.6. *Let $K\in B(\mathcal{H})$ and $\{T_i\in B(\mathcal{H},\mathcal{K}) : i\in I\}$ be a K-frame for $B(\mathcal{H},\mathcal{K})$. Then $\{T_i(K^*)^N\}_{i\in I}$ is a K^{N+1}-frame for $B(\mathcal{H},\mathcal{K})$.*

Proof. It follows from the previous theorem. □

For a sequence of Hilbert spaces $\{\mathcal{K}_i\}_{i\in I}$, define the space $l^2(\{\mathcal{K}_i\}_{i\in I})$ by

$$l^2(\{\mathcal{K}_i\}_{i\in I}) = \left\{\{x_i\}_{i\in I} : x_i\in\mathcal{K}_i, i\in I, \sum_{i\in I}\|x_i\|^2<\infty\right\},$$

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with the inner product

$$\Big\langle \{x_i\}_{i\in I}, \{y_i\}_{i\in I} \Big\rangle = \sum_{i\in I} \langle x_i, y_i\rangle,$$

it is a Hilbert space.

Proposition 2.7. *Let $K \in B(\mathcal{H})$ and $\mathcal{K} = l^2(\{\mathcal{K}_i\}_{i\in I})$. The sequence $\{\Lambda_i \in B(\mathcal{H},\mathcal{K}_i) : i \in I\}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$ if and only if the sequence $\{\tilde{\Lambda}_i \in B(\mathcal{H},\mathcal{K}) : i \in I\}$ is a K-g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$, with $\tilde{\Lambda}_i x = (...,0,0,\Lambda_i x,0,0,...), \forall x \in \mathcal{H}$.*

The follwing theorem show that the definition 1.7 is equivalent with our definition 2.1.

Theorem 2.8. *A sequence $\{T_i \in B(\mathcal{H},\mathcal{K}) : i \in I\}$ is a K-frame for $B(\mathcal{H},\mathcal{K})$ if and only if it is a K-g-frame for $\mathcal{H}$ whit respect to $\mathcal{K}$.*

Proof. Let $\{T_i\}_{i\in I}$ be a K-g-frame for $\mathcal{H}$ whit respect to $\mathcal{K}$.

Then there exists two positive constants A and B, such that

$$A\|K^*x\|^2 \leq \sum_{i\in I} \|T_i x\|^2 \leq B\|x\|^2, \forall x \in \mathcal{H},$$

i.e

$$A\langle K^*x, K^*x\rangle \leq \sum_{i\in I} \langle T_i^*T_i x, x\rangle \leq B\langle x,x\rangle, \forall x \in \mathcal{H}.$$

So

$$AKK^* \leq \sum_{i\in I} T_i^*T_i \leq BI_{\mathcal{H}}.$$

Hence

$$ATKK^*T^* \leq \sum_{i\in I} TT_i^*T_iT^* \leq BTT^*, \forall T \in B(\mathcal{H},\mathcal{K}).$$

Thus

$$A\langle TK, TK\rangle \leq \sum_{i\in I} \langle T, T_i\rangle\langle T_i, T\rangle \leq B\langle T,T\rangle, \forall T \in B(\mathcal{H},\mathcal{K}),$$

i.e $\{T_i\}_{i\in I}$ is a K-frame for $B(\mathcal{H},\mathcal{K})$.

Conversely, assume that $\{T_i\}_{i\in I}$ be a K-frame for $B(\mathcal{H},\mathcal{K})$.

Then there exists two positive constants A and B, such that

$$A\langle TK, TK\rangle \leq \sum_{i\in I} \langle T, T_i\rangle\langle T_i, T\rangle \leq B\langle T,T\rangle, \forall T \in B(\mathcal{H},\mathcal{K}),$$

i.e

$$ATKK^*T^* \leq \sum_{i\in I} TT_i^*T_iT^* \leq BTT^*, \forall T \in B(\mathcal{H},\mathcal{K}).$$

So

$$A\langle TKK^*T^*x, x\rangle \leq \sum_{i\in I} \langle TT_i^*T_iT^*x, x\rangle \leq B\langle TT^*x, x\rangle, \forall T \in B(\mathcal{H},\mathcal{K}), \forall x \in \mathcal{K}.$$

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Let $y \in \mathcal{H}$ and $T \in B(\mathcal{H}, \mathcal{K})$ such that $T^*x = y$, then

$$A\langle K^*y, K^*y\rangle \leq \sum_{i\in I}\langle T_i^*T_iy, y\rangle \leq B\langle y, y\rangle, \forall y \in \mathcal{H},$$

i.e

$$A\|K^*y\|^2 \leq \sum_{i\in I}\|T_iy\|^2 \leq B\|y\|^2, \forall y \in \mathcal{H},$$

thus $\{T_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$ whit respect to $\mathcal{K}$. □

Corollary 2.9. *A sequence $\{T_i \in B(\mathcal{H}) : i \in I\}$ is a tight K-frame for $B(\mathcal{H}, \mathcal{K})$ if and only if it is a tight K-g-frame for $\mathcal{H}$ with respect to $\mathcal{K}$.*

3. The relationships between K-frames and K-operator Frames for $B(\mathcal{H})$

Now we suppose that $\mathcal{H} = \mathcal{K}$, and we will define the K-frame for $B(\mathcal{H})$.

Definition 3.1. Let $K \in B(\mathcal{H})$. A sequence $\{T_i \in B(\mathcal{H}) : i \in I\}$ is said to be a K-frame for $B(\mathcal{H})$ if there exist $0 < A, B < \infty$ sach that

$$A\langle TK, TK\rangle \leq \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \leq B\langle T, T\rangle, \forall T \in B(\mathcal{H}), \tag{3.1}$$

where the series converges in the strong operator topology.

In the following we will show the equivalence between K-frames and K-operator Frames for $B(\mathcal{H})$.

Theorem 3.2. *A sequence $\{T_i \in B(\mathcal{H}) : i \in I\}$ is a K-frame for $B(\mathcal{H})$ if and only if it is a K-operator frame for $B(\mathcal{H})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a K-operator frame for $B(\mathcal{H})$.

Then there exists two positive constants A and B, such that

$$A\|K^*x\|^2 \leq \sum_{i\in I}\|T_ix\|^2 \leq B\|x\|^2, \forall x \in \mathcal{H},$$

i.e

$$A\langle K^*x, K^*x\rangle \leq \sum_{i\in I}\langle T_i^*T_ix, x\rangle \leq B\langle x, x\rangle, \forall x \in \mathcal{H}.$$

So

$$AKK^* \leq \sum_{i\in I}T_i^*T_i \leq BI_{\mathcal{H}}.$$

Hence

$$ATKK^*T^* \leq \sum_{i\in I}TT_i^*T_iT^* \leq BTT^*, \forall T \in B(\mathcal{H}).$$

Thus

$$A\langle TK, TK\rangle \leq \sum_{i\in I}\langle T, T_i\rangle\langle T_i, T\rangle \leq B\langle T, T\rangle, \forall T \in B(\mathcal{H}),$$

i.e $\{T_i\}_{i\in I}$ is a K-frame for $B(\mathcal{H})$.

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Conversely, assume that $\{T_i\}_{i\in I}$ be a K-frame for $B(\mathcal{H})$.

Then there exists two positive constants A and B, such that

$$A\langle TK,TK\rangle \leq \sum_{i\in I}\langle T,T_i\rangle\langle T_i,T\rangle \leq B\langle T,T\rangle, \forall T\in B(\mathcal{H}),$$

i.e

$$ATKK^*T^* \leq \sum_{i\in I} TT_i^*T_iT^* \leq BTT^*, \forall T\in B(\mathcal{H}).$$

So

$$A\langle TKK^*T^*x,x\rangle \leq \sum_{i\in I}\langle TT_i^*T_iT^*x,x\rangle \leq B\langle TT^*x,x\rangle, \forall T\in B(\mathcal{H}), \forall x\in\mathcal{H}.$$

Let $y\in\mathcal{H}$ and $T\in B(\mathcal{H})$ such that $T^*x=y$, then

$$A\langle K^*y,K^*y\rangle \leq \sum_{i\in I}\langle T_i^*T_iy,y\rangle \leq B\langle y,y\rangle, \forall y\in\mathcal{H},$$

i.e

$$A\|K^*y\|^2 \leq \sum_{i\in I}\|T_iy\|^2 \leq B\|y\|^2, \forall y\in\mathcal{H},$$

thus $\{T_i\}_{i\in I}$ is a K-operator frame for $B(\mathcal{H})$. □

Corollary 3.3. *A sequence $\{T_i\in B(\mathcal{H}) : i\in I\}$ is a tight K-frame for $B(\mathcal{H})$ if and only if it is a tight K-operator frame for $B(\mathcal{H})$.*

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