Iranian Journal of Mathematical Sciences and Informatics
Vol. 16, No. 2 (2021), pp 11-23
DOI: 10.29252/ijmsi.16.2.11

Fixed Point in Semi-linear Uniform Spaces and Convex Metric Spaces

Amani Rawshdeh, Abdalla Tallafha *

Department of Mathematics, The University of Jordan, Amman-Jordan

E-mail: a_rawashdeh85@yahoo.com
E-mail:a.tallafha@ju.edu.jo

Abstract.Tallafha, A. and Alhihi S. in [15], asked the following question. If f is a contraction from a complete semi-linear uniform space (X,Γ) to it self, is f has a unique fixed point. In this paper, we shall answer this question negatively and we shall show that convex metric space and M-space are equivalent except uniqueness. Also, we shall characterize convex metric spaces and use this characterization to give some application using semi-linear uniform spaces

Keywords: Uniform spaces, Semi-linear uniform spaces, Contractions.
2000 Mathematics subject classification: 54E35, 41A65. .

1. Introduction

Uniform spaces are topological spaces with additional structure that is used to define uniform properties such as completeness, uniform continuity and uniform convergence. The notion of uniformity has been investigated by several mathematician as Weil [16],[17], and [18] L.W.Cohen [4], and [5] Graves [7]. The theory of uniform spaces was given by Burbaki in [3], Also Wiel´s in his booklet [16] define uniformly continuous mapping.

Tallafha, A. and Khalil, R. in 2009 defined a new type of uniform space namely, semi-linear uniform space [11]. They studied some cases of best approximation in such spaces, besides they defined a set valued map ρ, called metric type, on semi-linear uniform spaces that enables one to study analytical concepts on semi-linear uniform spaces.

*Corresponding Author

Received 01 October 2017; Accepted 26 July 2018

In [12], [13] and [14] Tallafha, A. answered the question "Is there a semi-linear uniform space which is not metrizable?". Also he defined another set valued map called δ on $X \times X$, which is used with ρ to give more properties of semi-linear uniform spaces. Finally he studied the relation between ρ and δ and he showed that $\rho(x,y) = \rho(s,t)$ if and only if $\delta(x,y) = \delta(s,t)$. Also he defined Lipschitz condition, and contraction mapping on semi-linear uniform spaces, which enables one to study fixed point for such functions.

Since Lipschitz condition, and contractions are usually discussed in metric and normed spaces, and never been studied in other weaker spaces. We believe that the structure of semi-linear uniform spaces is very rich and all the known results on fixed point theory can be generalized.

In [1], Alhihi,S. Gave more properties of semi-linear uniform space. Also Tallafha, A. and Alhihi, S. [15] gave another properties of semi-linear uniform spaces and asked the following question. **If f is a contraction from a complete semi-linear uniform space (X, Γ) to it self, is f has a unique fixed point.** Finally S. Alhihi and M. AlFayaad in [2] gave some topological properties of semi-linear uniform spaces.

Let X be a none empty set and D_X be a collection of all relations on X such that each element V of D_X is reflexive and symmetric. D_X is called the family of all entourages of the diagonal.

Let Γ be a sub collection of D_X such that,

(i) If $V_1$and V_2 are in Γ, then $V_1 \cap V_2 \in \Gamma$

(ii) For every $V \in \Gamma$, there exists $U \in \Gamma$ such that, $U \circ U \subset V$.

(iii) $\bigcap_{V \in \Gamma} V = \Delta$

(vi) If $V \in \Gamma$ and $V \subseteq W \in D_X$, then $W \in \Gamma$.

Then the pair (X, Γ) is called a uniform space. We refer the reader to [6] and [8] for the basic structure of uniform spaces.

Let (X, Γ) be a uniform space, by a chain in $X \times X$ we mean a totally (or linearly) ordered collection of subsets of $X \times X$, ordered by set inclusion.

Definition 1.1. [11]. Let Γ be a subcollection of D_X such that

(i) For every $V \in \Gamma$, there exists $U \in \Gamma$ such that $U \circ U \subset V$.

(ii) $\bigcap_{V \in \Gamma} V = \Delta$ (vi) $\bigcup_{V \in \Gamma} V = X \times X$. (v) Γ is a chain.

Then the pair (X, Γ) is called a semi-linear uniform space.

The following is an example of a semi-linear uniform space which is metrizable.

EXAMPLE 1.2. Let $V_t = \{(x, y) : y - t < x < y + t, -\infty < y < \infty\}$. Then $(\mathbb{R}, \Gamma)$ with $\Gamma = \{V_t : 0 < t < \infty\}$ is a semi-linear uniform space.

The following example is a semi-linear uniform space which is not metrizable.

EXAMPLE 1.3. Let (X,Γ) be a semi-linear uniform where $\Gamma = \{V_\epsilon, \epsilon > 0\}, V_\epsilon = \{(x,y) : x^2 + y^2 < \epsilon\} \cup \{\Delta\}$.

Definition 1.4. [11]. Let (X,Γ) be a semi-linear uniform space. For $(x,y) \in X \times X$, let $\Gamma_{(x,y)} = \{V \in \Gamma : (x,y) \in V\}$. Then, the set valued map ρ on $X \times X$, is defined by $\rho(x,y) = \bigcap\{V : V \in \Gamma_{(x,y)}\}$.

Clearly for all $(x,y) \in X \times X$, we have $\rho(x,y) = \rho(y,x)$, and $\Delta \subseteq \rho(x,y)$.

Definition 1.5. [12]. Let (X,Γ) be a semi-linear uniform space. For $(x,y) \in X \times X$, let $\Gamma^c_{(x,y)} = \{V \in \Gamma : (x,y) \notin V\}$. Then the set valued map δ on $X \times X$, is defined by

$$\delta(x,y) = \left\{ \begin{array}{cc} \bigcup\{V : V \in \Gamma^c_{(x,y)}\} & x \neq y \\ \phi & x = y \end{array} \right\}$$

In [12],Tallafha gave some important properties of semi-linear uniform spaces, using the set valued map ρ and δ, some of these properties are given in the following proposition.

Proposition 1.6. [12]*Let (X,Γ) be a semi-linear uniform space. Then,*
i) If $V \in \Gamma^c_{(x,y)}$, then $V \subsetneqq \rho(x,y)$. ii) $\delta(x,y) \subseteq \rho(x,y)$ for all $(x,y) \in X \times X$.
iii) If $V \in \Gamma_{(x,y)}$, then $\delta(x,y) \subseteq V$. iv) If $(x,y) \in \rho(s,t)$, then $\rho(x,y) \subseteq \rho(s,t)$.
v) If $(x,y) \in \delta(s,t)$, then $\delta(x,y) \subseteq \delta(s,t)$.

Also in [1] Alhihi gave more properties of semi-linear uniform spaces as:

Theorem 1.7. *Let $A \in \Gamma$, for $n \in \mathbb{N}$, we have,*
(i) $n\left(\frac{1}{n}A\right) \subseteq A$ *(ii) If $B \in \Gamma$ satisfies $nB \subseteq A$, then $B \subseteq \frac{1}{n}A$.*
(iii) $\frac{1}{n+1}A \subseteq \frac{1}{n}A$ *(iv)* $\frac{1}{n}A \subseteq A$

(v) $$\frac{1}{n}\ \delta(x,y) = \begin{cases} \bigcap\limits_{V \in \Gamma^c_{(x,y)}} \frac{1}{n}V & if\ x \neq y \\ \phi & if\ x = y \end{cases}$$
(vi) $$\frac{1}{n}\rho(x,y) = \frac{1}{n} \bigcap_{V \in \Gamma_{(x,y)}} V \subseteq \bigcap_{V \in \Gamma_{(x,y)}} \frac{1}{n}V$$

(vii) For $x,y \in X$ where (X,Γ) is a semi-linear uniform spaces, then $n\left(\frac{1}{n}\rho(x,y)\right) \subseteq \rho(x,y)$.

(viii) Let x, y *be any distinct points in semi-linear uniform spaces* (X,Γ). *Then,* $n\left(\frac{1}{n}\delta(x,y)\right) \subseteq \delta(x,y) \subseteq \frac{1}{n}(n\delta(x,y))$.

Best approximation is an important concept discussed in metric spaces with application in other sciences.

Definition 1.8. [6] Let (X,d) be a metric space, $E \subseteq X$. The set E is called proximinal if for any $x \in X$, there exists some $e_x \in E$ such that $d(x,E) = d(x,e_x)$ where $d(x,E) = \inf_{e \in E} d(x,e)$.

In [11], A.Tallafha and R.Khalil define Proximinality using semi-linear uniform spaces instead of metric spaces.

Definition 1.9. [11] Let (X,Γ) be a semi-linear uniform space, for $x \in X$ and $E \subseteq X$, define $\rho(x,E) = \bigcap_{e \in E} \rho(x,e)$.

Definition 1.10. [11] Let (X,Γ) be a semi-linear uniform space, $E \subseteq X$. The set E is called proximinal if for any $x \in X$, there exists some $e_x \in E$ such that $\rho(x,E) = \rho(x,e_x)$.

In [2], S. Alhihi and M. Fayyad showed that every semi-linear uniform space induced a Tychonoff space (X,T_Γ) where T_Γ induced by local base $B_x = \{B(x,U) : U \in \Gamma\}$ where $B(x,U) = \{y : (x,y) \in U\}$. For more topological properties of T_Γ we refer the reader to [2].

Also it is known that every metric space (X,d) induce a semi-linear space (X,Γ_d).

2. Semi-linear Uniform Space Induced by Metric Space

In this section we shall show that the structure of semi-linear uniform space is very rich structure and the classical definitions in metric spaces can be carried to semi-linear uniform spaces.

Definition 2.1. [14] **Let** (X,d) **be a metric space. Define** $V_\epsilon = \{(x,y) : d(x,y) < \epsilon\}$.**Then** (X,Γ) **where** $\Gamma = \{V_\epsilon : \epsilon > 0\}$ **is a semi-linear uniform space induced by** (X,d). **This semi-linear uniform space will be denoted by** (X,Γ_d).

So semi-linear uniform space is a space weaker than metric space and stronger than topological spaces.

Lemma 2.2. *Let* (X,Γ_d) *be a semi-linear uniform space induced by the metric space* (X,d). *Then,*

(1) $\rho(x,y) = \{(s,t) \in X \times X : d(s,t) \leq d(x,y)\}$
(2) $\delta(x,y) = \{(s,t) \in X \times X : d(s,t) < d(x,y)\}$

Proof. 1. We want to show $\bigcap\{V_\varepsilon, \varepsilon > 0 : V_\varepsilon \in \Gamma_{(x,y)}\} = \{(s,t) \in X \times X : d(s,t) \leq d(x,y)\}$. It is clear that $\{(s,t) : d(s,t) \leq d(x,y)\} \subseteq \bigcap\{V_\varepsilon, \varepsilon > 0 : V_\varepsilon \in \Gamma_{(x,y)}\}$. Conversely, suppose $(s,t) \in X \times X$ such that $(s,t) \in \bigcap\{V_\varepsilon, \varepsilon > 0 : V_\varepsilon \in \Gamma_{(x,y)}\}$, if $d(s,t) > d(x,y)$, then there exists r such that $d(s,t) > r > d(x,y)$ which means $V_r \in \Gamma_{(x,y)}$ and $(s,t) \notin V_r$, which contradicts the assumption.

2. If $x = y$ the result is true. So we assume $x \neq y$. We want to show that $\bigcup\{V_\varepsilon : V_\varepsilon \in \Gamma^c_{(x,y)}\} = \{(s,t) \in X \times X : d(s,t) < d(x,y)\}$, it is clear $\bigcup\{V_\varepsilon : V_\varepsilon \in \Gamma^c_{(x,y)}\} \subseteq \{(s,t) : d(s,t) < d(x,y)\}$.

Conversely, suppose $(s,t) \in X \times X$ such that $d(s,t) < d(x,y)$, if $(s,t) \notin \bigcup\{V : V \in \Gamma^c_{(x,y)}\}$, then there exists r such that $d(s,t) < r < d(x,y)$ which mean $V_r \in \Gamma^c_{(x,y)}$ and $(s,t) \in V_r$, which is a contradiction. □

In 1928, K. Menger defined convex metric space then Khalil R. define M-space. Now we shall characterize M- spaces and show that they are equivalent except uniqueness..

Definition 2.3. [10]Let (X,d) be a metric space. For $x \in X, r > 0$, let $B[x,r] = \{t : d(x,t) \leq r\}$ .A metric space (X,d) is convex, if for all $x, y \in X$, $B[x,r_1] \cap B[y,r_2] \neq \phi$ whenever $r_1 + r_2 \geq d(x,y)$.

Definition 2.4. [9]. A metric space (X,d) is M-space, if for all $(x,y) \in X \times X$, and $\lambda = d(x,y)$, if $\alpha \in [0,\lambda]$, there exists a unique $z_\alpha \in X$ such $B[x,\alpha] \cap B[y,\lambda-\alpha] = \{z_\alpha\}$.

In Definition 2.3 if r_1 or $r_2 = 0$, then $B[x,r_1] \cap B[y,r_2] \neq \phi$. Therefore we have the following.

Proposition 2.5. *1- If a metric space (X,d) is convex and $r_1, r_2 \geq 0$, such that $d(x,y) \leq r_1 + r_2$ then $B[x,r_1] \cap B[y,r_2] \neq \phi$.*

2- If a metric space (X,d) is M-space, then for all $(x,y) \in X \times X$ and $\alpha \in [0, d(x,y)]$ there exists a unique $z_\alpha \in X$ such that $d(x,z_\alpha) = \alpha$ and $d(y,z_\alpha) = d(x,y) - \alpha$.

Proof. 1- Let (X,d) be a convex metric space and $r_1, r_2 \geq 0$, such that $d(x,y) \leq r_1 + r_2$.If $r_1 = 0$, then $x \in B[x,r_1] \cap B[y,r_2]$. If $r_2 = 0$, then $y \in B[x,r_1] \cap B[y,r_2]$.

2- Let (X,d) be M-space, $(x,y) \in X \times X$ and $\alpha \in [0, d(x,y)]$. Then there exists a unique $z_\alpha \in X$ such that $d(x,z_\alpha) \leq \alpha$ and $d(y,z_\alpha) \leq d(x,y) - \alpha$, if $d(x,z_\alpha) < \alpha$ or $d(y,z_\alpha) < d(x,y) - \alpha$, then $d(x,y) \leq d(x,z_\alpha) + d(y,z_\alpha) < d(x,y)$, therefore $d(x,z_\alpha) = \alpha$ and $d(y,z_\alpha) = d(x,y) - \alpha$. □

Proposition 2.6. *Convex and M-spaces are equivalent except uniqueness.*

Proof. Let (X,d) be a convex metric space and $x, y \in X$, let $\lambda = d(x,y)$. For $\alpha \in [0,\lambda]$, let $r_1 = \alpha$ and $r_2 = \lambda - \alpha$ then there exists $z \in X$ such that $d(x,z) \leq r_1$ and $d(z,y) \leq r_2$.Which implies $B(x,\alpha) \cap B(y,\lambda-\alpha) \neq \phi$.

Conversely, let (X,d) be M-metric space, for $x,y\in X$ and $r_1,r_2>0$ such that $d(x,y)\leq r_1+r_2$. If $d(x,y)\leq r_1$ then take $z=y$. Hence we may assume that $d(x,y)>r_1$. Therefore $B(x,r_1)\cap B(y,d(x,y)-r_1)\neq\phi$. Thus there exists $z\in X$ such that $d(x,z)\leq r_1$ and $d(z,y)\leq d(x,y)-r_1\leq r_2$. □

Now we shall characterize convex spaces.

Lemma 2.7. *A metric space (X,d) is convex if and only if for all $(x,y)\in X\times X$ and $\alpha\in[0,1]$, there exists $z\in X$ such that $d(x,z)=\alpha d(x,y)$ and $d(z,y)=(1-\alpha)\,d(x,y)$.*

Proof. Let (X,d) be a convex space and $0\leq\alpha\leq 1$. Then for all $x,y\in X$ we have $0\leq\alpha d(x,y)\leq d(x,y)$. Therefore, there exists $z\in X$ such that $\alpha d(x,y)=d(x,z)$ and $d\,(y,z)=d\,(x,y)-\alpha d(x,y)$.

Conversely, if $d(x,y)=0$ then $z=x$. So we may assume $d(x,y)>0$.For $0\leq\alpha\leq d(x,y)$, we have $0\leq\frac{\alpha}{d(x,y)}\leq 1$. Hence there exists $z\in X$ such that $d(x,z)=\frac{\alpha}{d(x,y)}d(x,y)$ and $d(z,y)=(1-\alpha)\,d(x,y)=\left(1-\frac{\alpha}{d(x,y)}\right)d(x,y)=d\,(x,y)-\alpha$. □

Clearly if $\alpha=\frac{1}{2}$ we obtain the mid point property. So we have the following corollary.

Corollary 2.8. *Every convex or M-space has the mid point property.*

Lemma 2.9. *Let (X,Γ_d) be a semi-linear uniform space induced by the metric space (X,d) which has the mid point property. Then for all $x\in X$ there exist $\{w_n\}\in X\backslash\{x\}$ such that $\{w_n\}$ converges to x provided X contain more than one point.*

Proof. Let $x\in X$ and $\varepsilon>0$, let $z\in X\backslash\{x\}$. By mid point property let w_1 be such that $d(x,w_1)=d(w_1,z)=\frac{1}{2}d\,(x,z)$. So we continuo this way to obtained $w_1,w_2,...w_n$ such that $d\,(x,w_n)=\frac{1}{2^n}d\,(x,z)<\varepsilon$. So $\{w_n\}$ converges to x. □

The following is an important property of semi-linear uniform space induced by a convex metric space (X,d).

Lemma 2.10. *Let (X,Γ_d) be a semi-linear uniform space induced by a convex metric space (X,d). Then $V_{\varepsilon_1}\circ V_{\varepsilon_2}=V_{\varepsilon_1+\varepsilon_2}$.*

Proof. Let (X,d) be a convex metric space. From the triangle inequality of metric space we have $V_{\varepsilon_1}\circ V_{\varepsilon_2}\subseteq V_{\varepsilon_1+\varepsilon_2}$. Let $(s,t)\in V_{\varepsilon_1+\varepsilon_2}$. Then $d(s,t)<\varepsilon_1+\varepsilon_2$. Let γ be such that $d(s,t)+\gamma=\varepsilon_1+\varepsilon_2$. Since $\varepsilon_1+\varepsilon_2-\gamma\geq 0$ then $\varepsilon_1-\frac{\gamma}{2}\geq 0$ or $\varepsilon_2-\frac{\gamma}{2}\geq 0$, say $\varepsilon_1-\frac{\gamma}{2}\geq 0$. If $d(s,t)\leq\varepsilon_1-\frac{\gamma}{2}$ we are done . If not, i.e., there exists y such that $d(s,y)=\varepsilon_1-\frac{\gamma}{2}<\varepsilon_1$ and $d(y,t)=d\,(s,t)-\left(\varepsilon_1-\frac{\gamma}{2}\right)<\varepsilon_2$. Hence $(s,t)\in V_{\varepsilon_1}\circ V_{\varepsilon_2}$. □

Theorem 2.11. *Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d). Then $n\rho(x, y) = \{(s, t) \in X \times X : d(s, t) \leq nd(x, y)\}$.*

Proof. Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d). Clearly $n\rho(x, y) \subseteq \{(s, t) \in X \times X : d(s, t) \leq nd(x, y)\}$. Now we want to use induction to show that $\{(s, t) \in X \times X : d(s, t) \leq nd(x, y)\} \subseteq n\rho(x, y)$....... $(*)$. By Lemma 2.2 $(*)$ is true for $n = 1$. Suppose $\{(s, t) \in X \times X : d(s, t) \leq kd(x, y)\} \subseteq k\rho(x, y)$. To show $\{(s, t) \in X \times X : d(s, t) \leq (k + 1) d(x, y)\} \subseteq (k + 1) \rho(x, y)$, let (s, t) be such that $d(s, t) \leq kd(x, y) + d(x, y)$. If $d(s, t) \leq kd(x, y)$,we are done, if not then $d(s, t) > kd(x, y)$ which implies that there exists $z \in X$, such that $d(s, z) \leq kd(x, y)$ and $d(z, t) \leq d(s, t) - kd(x, y) \leq d(x, y)$. Hence $(s, z) \in k\rho(x, y)$ and $(z, t) \in \rho(x, y)$ which implies $(s, t) \in (k + 1) \rho(x, y)$. □

Now we shall give the definitions of continuous function, uniformly continuous function, converges of sequences in semi-linear uniform spaces and complete semi-linear space.

Definition 2.12. [11]. Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ then,

1- f is continuous at $x_\circ$ if for all $U \in \Gamma_Y$, there exists $V \in \Gamma_X$, such that if $(x, x_\circ) \in V$, then $(f(x), f(x_\circ)) \in U$.

2- f is uniformly continuous if for all $U \in \Gamma_Y$, there exists $V \in \Gamma_X$, such that if $(x, y) \in V$, then $(f(x), f(y)) \in U$.

Definition 2.13. [11]. Let (X, Γ) be a semi-linear uniform space and (x_n) be a sequence in X. then,

1-. (x_n) converges to x in X and denoted by $x_n \to x$, if for every $V \in \Gamma$ there exists k such that $(x_n, x) \in V$ for every $n \geq k$.

2- (x_n) is called Cauchy if for every $V \in \Gamma$ there exists k such that $(x_n, x_m) \in V$ for every $n, m \geq k$.

Definition 2.14. [11]. Let (X, Γ) be a semi-linear uniform space. Then (X, Γ) is called complete, if every Cauchy sequence is convergent.

In definition 2.1 we mentioned that every metric space (X, d) induced a semi-linear uniform space (X, Γ_d), the natural questions arises what is the relation between the continuity (uniform continuity) of functions on metric space and continuity (uniform continuity) of functions on semi-linear uniform space induced by metric space. Also, what is the relation between converges in (X, d) and converges on (X, Γ_d). The following Theorems answer these questions.

Theorem 2.15. *Let $(X, \Gamma_X), (Y, \Gamma_Y)$ be two semi-linear uniform spaces induced by the metric spaces (X, d_X) and (Y, d_Y) respectively. Then $f : (X, d_X) \to (Y, d_Y)$ is continuous if and only if $f : (X, \Gamma_X) \longrightarrow (Y, \Gamma_Y)$ is continuous.*

Proof. Let $f : (X, d_X) \to (Y, d_Y)$ be continuous at $x_\circ \in X$. Let $U \in \Gamma_Y$. Then there exists $\epsilon > 0$ such that $U = U_\epsilon$, since $f : (X, d_X) \to (Y, d_Y)$ is continuous, then there exists $\delta > 0$ such that $d_X(x, x_\circ) < \delta$, implies $d_Y(f(x), f(x_\circ)) < \epsilon$. So there exists $V \in \Gamma_X$, $V = V_\delta$ such that if $(x, x_\circ) \in V_\delta$, then $(f(x), f(x_\circ)) \in U_\epsilon$. Hence $f : (X, \Gamma_X) \longrightarrow (Y, \Gamma_Y)$ is continuous at $x_\circ$.

Conversely, let $f : (X, \Gamma_X) \longrightarrow (Y, \Gamma_Y)$ be continuous at $x_\circ$ and $\epsilon > 0$. Then $U_\epsilon \in \Gamma_Y$. So there exists $V_\delta \in \Gamma_X$ such that, if $(x,x_\circ) \in V_\delta$, then $(f(x), f(x_\circ)) \in U_\epsilon$. Thus there exists $\delta > 0$ such that if $d_X(x, x_\circ) < \delta$, then $d_Y(f(x), f(x_\circ)) < \epsilon$. Hence $f : (X, d_X) \to (Y, d_Y)$ is continuous at $x_\circ$. □

Theorem 2.16. *Let $(X, \Gamma_X), (Y, \Gamma_Y)$ be two semi-linear uniform spaces induced by the metric spaces (X, d_X) and (Y, d_Y) respectively. Then $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous if and only if $f : (X, \Gamma_X) \longrightarrow (Y, \Gamma_Y)$ is uniformly continuous.*

Proof. Let $f : (X, d_X) \to (Y, d_Y)$ be uniformly continuous. Let $U \in \Gamma_Y$. Then there exists $\epsilon > 0$ such that $U = U_\epsilon$. Since $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous, then there exists $\delta > 0$ such that $d_X(x, y) < \delta$, implies $d_Y(f(x), f(y)) < \epsilon$. So if $(x, y) \in V_\delta$, then $(f(x), f(x_\circ)) \in U_\epsilon$. Hence $f : (X, \Gamma_X) \longrightarrow (Y, \Gamma_Y)$ is uniformly continuous.

Conversely, let $f : (X, \Gamma_X) \longrightarrow (Y, \Gamma_Y)$ be uniformly continuous and $\epsilon > 0$. Then $U_\epsilon \in \Gamma_Y$, so there exists $V_\delta \in \Gamma_X$ such that, if $(x,y) \in V_\delta$, then $(f(x), f(y)) \in U_\epsilon$. Thus there exists $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$. Hence $f : (X, d_X) \to (Y, d_Y)$ is uniformly continuous. □

In the following Theorems we shall discuss the convergent of sequences besides the completeness of semi-linear uniform spaces.

Theorem 2.17. *Let (X, Γ) be a semi-linear uniform space induced by the metric spaces (X, d) and (x_n) be a sequence in X. Then (x_n) is converge in (X, d) if and only (x_n) is converges in (X, Γ).*

Proof. Let (x_n) be converges to x in (X, d) and $U \in \Gamma$. Then there exists $\epsilon > 0$ such that $U = U_\epsilon$. Since (x_n) is converges in (X, d), then there exists k such that $d(x_n, x) < \epsilon$ for every $n \geq k$, it follows $(x_n, x) \in U_\epsilon$ for every $n \geq k$. Which mean $x_n \to x$ in (X, Γ).

Conversely, let (x_n) be converge to x in (X, Γ) and $\epsilon > 0$. Then $U_\epsilon \in \Gamma$, so there exists k such that $(x_n, x) \in U_\epsilon$ for every $n \geq k$. Thus $d(x_n, x) < \epsilon$ for every $n \geq k$. Hence $(x_n) \to x$ in (X, d). □

Theorem 2.18. *Let (X, Γ) be a semi-linear uniform space induced by the metric spaces (X, d) and (x_n) be a sequence in X. Then (x_n) is Cauchy in (X, d) if and only (x_n) is Cauchy in (X, Γ).*

Proof. Let (x_n) be a Cauchy in (X, d). Let $U \in \Gamma$. Then there exists $\epsilon > 0$ such that $U = U_\epsilon$. Since (x_n) is Cauchy in (X, d), then there exists k such that $d(x_n, x_m) < \epsilon$ for every $n, m \geq k$, it follows $(x_n, x_m) \in U_\epsilon$ for every $n, m \geq k$. which mean x_n is Cauchy in (X, Γ).

Conversely, let (x_n) be Cauchy in (X, Γ) and $\epsilon > 0$. Then $U_\epsilon \in \Gamma$, so there exists k such that $(x_n, x_m) \in U_\epsilon$ for every $n, m \geq k$. Thus $d(x_n, x_m) < \epsilon$ for every $n, m \geq k$. Hence (x_n) Cauchy in (X, d). □

Corollary 2.19. *Let (X, Γ) be a semi-linear uniform space induced by the metric space (X, d).Then (X, d) is complete if and only if (X, Γ) is complete.*

Now we shall show that $E \subseteq X$ is proximinal in (X, d) if and only if it is proximinal in (X, Γ).

Theorem 2.20. *Let (X, Γ) be a semi-linear uniform space induced by the metric spaces (X, d). Then $E \subseteq X$ is proximinal in (X, d) if and only if $E \subseteq X$ is proximinal in (X, Γ).*

Proof. Let $E \subseteq X$ be proximinal in (X, d). Now there exists $e_x \in E$ such that $d(x, E) = d(x, e_x)$. We want to show $\rho(x, E) = \rho(x, e_x)$, since $\rho(x, E) = \bigcap_{e \in E} \rho(x, e)$, *so* we want to show $\rho(x, e_x) \subseteq \rho(x, E)$. If not, i.e., $\rho(x, e_x) \nsubseteq \rho(x, E)$, by Lemma 2.2 $\rho(x, e_x) = \{(s, t) \in X \times X : d(s, t) \leq d(x, e_x)\} \nsubseteq \rho(x, E)$. So there exists $(s, t) \in X \times X$ such that $d(s, t) \leq d(x, e_x)$ and $(s, t) \notin \rho(x, E) = \bigcap_{e \in E} \rho(x, e)$. Hence there exists $e^* \in E$ such that $(s, t) \notin \rho(x, e^*)$, which means $d(x, e^*) < d(s, t) \leq d(x, e_x)$, this contradicts $d(x, e_x) = \inf\{d(x, e) : e \in E\}$.

Conversely, let $E \subseteq X$ be proximinal in (X, Γ). Now there exists $e_x \in E$ such that $\rho(x, E) = \rho(x, e_x)$. We want to show that $d(x, E) = d(x, e_x)$. If $d(x, E) \neq d(x, e_x)$, then there exists $e^* \in E$ such that $d(x, e^*) < d(x, e_x)$. Since $\rho(x, e_x) = \rho(x, E) = \bigcap_{e \in E} \rho(x, e)$, it follows $(x, e_x) \in \rho(x, e^*)$. Which means $d(x, e_x) < d(x, e^*)$.This contradicts the assumption. □

3. **Main results**

In[15] Tallafha A and Alhihi S. defined contraction on semi-linear uniform spaces.

Definition 3.1. [14]Let $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ be a mapping. Then f is contraction if there exist $m, n \in N$ such that $m > n$ and
$m\ \rho(f(x), f(y)) \subseteq n\rho(x, y)\ \forall x, y \in X$.

This definition is an interested one, since tell now we need metric spaces or normed spaces to define contraction. So to define contraction using a weaker space is a promising idea by which most of the results of fixed point theorem in metric spaces can be discussed in a semi-linear uniform spaces.

The first question one can ask if (X, Γ) is induced by (X, d) and $f : (X, d) \longrightarrow (X, d)$ be a contraction, must $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ be contraction. What about the converse.

Now we shall answer our question positively and gave an example to show the converse need not be true.

Theorem 3.2. *Let (X, Γ) be a semi-linear uniform space induced by unbounded convex metric space (X, d). Then $f : (X, d) \longrightarrow (X, d)$ is a contraction if and only if $f : f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is a contraction.*

Proof. Let (X, Γ) be a semi-linear uniform space induced by unbounded convex metric space (X, d). If $f : (X, d) \longrightarrow (X, d)$ is a contraction, then there exists $0 < r < 1$ such that $d(f(x), f(y)) \leq rd(x, y)$ $\forall x, y \in X$, hence, there exists $m, n \in N$ such that $r \leq \frac{n}{m} < 1$. This implies $d(f(x), f(y)) \leq \frac{n}{m} d(x, y)$ for all $x, y \in X$, hence $md(f(x), f(y)) \leq nd(x, y)$. Thus if $d(s, t) \leq md(f(x), f(y)$, then $d(s, t) \leq nd(x, y)$. by Theorem 2.11 $m\rho(f(x), f(y) \subseteq n\rho(x, y)$. Which means $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is a contraction. Now suppose $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is a contraction. Then $\exists m, n \in \mathbb{N}$ such that $m > n$ and $m\rho(f(x), f(y)) \subseteq n\rho(x, y)$ $\forall x, y \in X$. To complete the prove we want to show $md(f(x), f(y)) \leq nd(x, y)$ $\forall x, y \in X$. that is, $md(f(x_0), f(y_0)) > nd(x_0, y_0)$ for some $x_0, y_0 \in X$. Let $(s, t) \in X \times X$ be such that $d(s, t) \geq md(f(x_0), f(y_0))$. Hence there exists $z \in X$ such that $d(s, z) = md(f(x_0), f(y_0))$. Therefore $(s, z) \in m\rho(f(x_0), f(y_0)) \subseteq n\rho(x_0, y_0)$ which is a contradiction. Therefore $d(s, t) \leq md(f(x_0), f(y_0))$, for all $(s, t) \in X \times X$. Hence (X, d) is bounded. □

Example 3.3. Let $X = \mathbb{R}$ and d be the discrete metric space. Then $\Gamma = \{\Delta, X \times X\}$, so any function $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is a contraction, but the only contraction for $(X, dis) \longrightarrow (X, dis)$ is the constant function.

So, what is the additional conditions on a semi-linear uniform space induced by a metric space make f contraction on a metric space too. We will introduce some Theorems which illustrate what is the relationship between contraction on a metric space and contraction on semi-uniform space induce by a metric space using the new types of metric spaces we introduced in section 2.

Theorem 3.4. *If (X,d) is a metric space induce by a normed space $(X, \|.\|)$, then (X, d) is unbounded convex metric space.*

Proof. Let (X, d) be a metric space induced by a normed space $(X, \|.\|)$. For $x, y \in X$, $r_1, r_2 > 0$, and $d(x, y) \leq r_1 + r_2$, If $d(x, y) = \|x - y\| \leq r_1$or

$\|x-y\| \leq r_2$, then it is clear that $(X, \|.\|)$ is convex metric space. If not then $0 < 1 - \frac{r_1}{\|x-y\|} \leq \frac{r_2}{\|x-y\|} < 1$. Let α be such that $1 - \frac{r_1}{\|x-y\|} \leq \alpha \leq \frac{r_2}{\|x-y\|}$ and $z = \alpha x + (1-\alpha) y$, so $d(x,z) = \|x-z\| = (1-\alpha) d(x,y) \leq r_1$ and $d(z,y) = \|z-y\| = \alpha\|x-y\| \leq r_2$.

To show that (X,d) is unbounded, let $t \in [0,\infty)$ and $x \neq 0$, then $d(0,y) = t$ where $y = \frac{t}{\|x\|}x$. □

Remark 3.5. Theorem 3.4 can be used to check if a metric space induce by a normed space or not.

By Theorem3.2, and Theorem 3.4, we have the following Corollary.

Corollary 3.6. *Let (X, Γ) be a semi-linear uniform induced by a normed space $(X, \|.\|)$. Then $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is a contraction if and only if $f : (X, \|.\|) \longrightarrow (X, \|.\|)$ is a contraction.*

In [15] Tallafha A. asked the following question. If (X, Γ) is a complete semi-linear space and $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is a contraction. Does f has a unique fixed point. The following example answer this question negatively.

Example 3.7. Let $\Gamma = \{V_\epsilon, \epsilon > 0\}, V_\epsilon = \{(x,y) : x^2 + y^2 < \epsilon\} \cup \{\Delta\}$. Then (X, Γ) is a semi-linear uniform space. Now (X, Γ) is complete since if x_n is Cauchy, then $x_n \to 0$ or x_n has a constant tail, so x_n is convergent sequence. Let $f(x) = x\ sin(x)$. Then $f(x)$ is a contraction and has $\left\{\frac{\pi}{2} + 2n\pi : n = 0,1,2\right\} \cup \{0\}$ as fixed points.

In section 2, we showed that $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ satisfies the property P if and only if $f : (X,d) \longrightarrow (X,d)$ satisfies the same property P, where P is continuos, uniformly continuous, but this is not true for contraction see Example 3.3. We suggest two ways to solve this problem, by strength the definition of contraction or by strength the definition of semi-linear uniform space.

Definition 3.8. A semi-linear uniform space (X, Γ) is called a strong semi-linear uniform space if Γ satisfies the following additional condition for all $V \in \Gamma$,we have $\bigcup\limits_{n=1}^{\infty} nV = X \times X$.

Definition 3.9. Let (X, Γ) be a semi-linear uniform space. A mapping $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is called strong contraction if there exists $m, n \in N$ such that $m > n$ and $m\ \rho(f(x), f(y)) \subseteq n\rho(x,y)$ and $(m+1)\ \rho(f(x), f(y)) \nsubseteq n\rho(x,y)$.

Lemma 3.10. *Every metric space* (X,d) *with mid point property, induces a strong semi-linear uniform space* (X,Γ), *where* $\Gamma=\{V_\varepsilon:\varepsilon>0\}$, $V_\varepsilon=\{(x,y)\in X\times X: d(x,y)<\varepsilon\}$.

Proof. Let (X,Γ) be a semi-linear uniform space induced by a metric space (X,d) which has mid point property. To show that (X,Γ) is a strong semi-linear uniform space, let$V\in\Gamma$. Then there exists $\epsilon>0$ such that $V=V_\varepsilon$. Let $x,y\in X$, if $d(x,y)<\varepsilon$ the proof is done. If not, there exists $x_1\in X$ such that $\frac{1}{2}d(x,y)=d(x,x_1)=d(x_1,y)$, if $d(x,x_1)$ and $d(x_1,y)$ is less than ε we are done, if not we can continue this way to obtain $d(x,y)=d(x,x_n)+d(x_n,x_{n-1})+....+d(x_2,x_1)+d(x_1,y_m)+d(y_m,y_{m-1})+.....+d(y_1,y)$ and $\frac{1}{2^n}d(x,y)=d(x,x_n)=d(x_n,x_{n-1})=....=d(x_2,x_1)$ and $\frac{1}{2^m}d(x,y)=d(x_1,y_m)=d(y_m,y_{m-1})=.....=d(y_1,y)$. Thus for m,n satisfies $\text{Max}\{\frac{1}{2^n}d(x,y),\frac{1}{2^m}d(x,y)\}<\varepsilon$ we have $(x,y)\in(n+m)V_\varepsilon$. □

By Corollary 2.7. we have the following.

Corollary 3.11. *Every convex or M- metric space* (X,d) *induces a strong semi-linear uniform space* (X,Γ).

Remark 3.12. In Example 4.3. (X,dis) bounded not convex and (X,Γ) is not a strong semi-linear space. Also $x\sin x:(X,\Gamma)\longrightarrow(X,\Gamma)$ is not a strong contraction.

The previous discussion leads to the following open questions.

Question 1. Let (X,Γ) be a complete strong semi-linear uniform space . And $f:(X,\Gamma)\longrightarrow(X,\Gamma)$ be contraction. Does f has a unique fixed point?

Question 2. Let (X,Γ) be a complete semi-linear uniform space and $f:(X,\Gamma)\longrightarrow(X,\Gamma)$ be a strong contraction. Does f has a unique fixed point?

Question 3. Let (X,Γ) be a complete strong semi-linear uniform space and $f:(X,\Gamma)\longrightarrow(X,\Gamma)$ be a strong contraction. Does f has a unique fixed point?

Acknowledgments

The authors would like to thank the referee for her/his comments.

References

1. S. A. Alhihi, More properties of semi linear uniform spaces, *Journal of applied mathematics*, **6**, online, (2015).
2. S. A. Alhihi, M. A. Al-Fayyad, Topological properties of semi-linear uniform spaces, Glob., *J. Pure Appl. Math.*, **12(6)**, (2016), 5329-5341.
3. Bourbaki, Topologie Générale (General Topology); ISBN 0-387-19374-X, Paris, 1940.
4. L. W. Cohen, Uniformity properties in a topological space satisfying the first denumerability postulate, Duke Math. J., **3**, (1937), 610-615.
5. L. W. Cohen, On imbedding a space in a complete space, Duke Math. J., **5,** (1939), 174-183.

6. R. Engelking, Outline of General Topology, North-Holand, Amsterdam, 1968.
7. L. M. Graves, On the completing of a Housdroff space, Ann. Math., **38**, (1937), 61-64.
8. I. M. James, Topological and Uniform Spaces, Undergraduate Texts in Mathematics, Springer,Verlag, 1987.
9. R. Kalil, Best Approximation in Metric Spaces, Proceedings of the American mathematical society, **103(2)**,(1988), 579-586.
10. K. Menger, Untersuchungen ūber allegemeine Metrik, Math. Ann., **100**, (1928), 75-163.
11. A. Tallafha, and R. Khalil, Best Approximation in Uniformity type spaces, Eur. J. Pure Appl. Math., **2**, (2009), 231-238.
12. A. Tallafha, Some properties of semi-linear uniform spaces, Bol. Soc. Paran. Mat., **29(2)**, (2011), 9-14.
13. A. Tallafha, Open Problems in Semi-Linear Uniform Spaces, J. Applied Functional Analysis, **8(2)**, (2013), 223-228
14. A. Tallafha, Fixed point in Semi-Linear Uniform Spaces, Eur. J. Pure Appl. Math., **12(3)**, (2013), 334-340.
15. A. Tallafha, S. Alhihi, Fixed point in a non-metrizable spaces, Accepted for publication as a part of the proceedings volume " Computational Analysis: Contributions from AMAT2015" in Springer- New York.
16. A. Weil, Les recouvrements des espaces topologiques: espaces complete, espaces bicompact, C. R. Acad. Paris, 202, (1936), 1002-1005.
17. A. Weil, Sur les espaces a structure uniforme et sur la topologie generale, Act. Sci. Ind. 551, Paris, 1937.
18. A. Weil, Sur les espaces à structure uniforme et sur la topologic générale, Paris, 1938.