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Copresented Dimension of Modules

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Abstract. In this paper, a new homological dimension of modules, copresented dimension, is defined. We study some basic properties of this homological dimension. Some ring extensions are considered, too. For instance, we prove that if $S \geq R$ is a finite normalizing extension and S_R is a projective module, then for each right S-module M_S, the copresented dimension of M_S does not exceed the copresented dimension of $Hom_R(S, M)$.

Keywords: Coherent ring, Copresented dimension, Projective module.

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1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. First we recall some known notions and facts needed in the sequel. Let R be a ring, n a non-negative integer and M an R-module. Then

(1) M is said to be *finitely cogenerated* [1] if for every family $\{V_k\}_J$ of submodules of M with $\bigcap_J V_k = 0$, there is a finite subset $I \subset J$ with $\bigcap_I V_k = 0$.

(2) M is said to be *n-copresented* [14] if there is an exact sequence of R-modules $0 \to M \to E^0 \to E^1 \to \cdots \to E^n$, where each E^i is a finitely cogenerated injective module.

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(3) R is called right *co-coherent* [17] if every finitely cogenerated factor module of a finitely cogenerated injective R-module is finitely copresented.
(4) R is called n-cocoherent [14] in case every n-copresented R-module is $(n+1)$-copresented. It is easy to see that R is cocoherent if and only if it is 1-cocoherent. Recall that a ring R is called right *conoethrian* [4] if every factor module of a finitely cogenerated R-module is finitely cogenerated. By [4, Proposition 17], a ring R is co-noethrian if and only if it is 0-cocoherent.
(5) M is said to be *n-presented* [5] if there is an exact sequence of R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is a finitely generated free module.
(6) R is called *coherent* [18] in case every 0-presented R-module is 1-presented.
(7) A ring extension $R \subseteq R'$ with characteristic $p > 0$ is called a *purely inseparable extension* [10] if for every element $r^{'} \in R^{'}$, there exists a non-negative integer n such that ${r^{'}}^{p^n} \in R$.
(8) For any commutative ring R of prime characteristic $p > 0$, assume that $F_R : R \to R^{(e)}$ is the e-th iterated Frobenius map in which $R^{(e)} \cong R$. Then, the *perfect closure* [9] of R, denoted by R^{∞}, is defined as the limit of the following direct system:

$$R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots$$

(9) M is called (n,d)-injective [18] if $\operatorname{Ext}_R^{d+1}(N, M) = 0$ for any n-presented right R-module N. It is clear that M is $(0,0)$-injective if and only if M is injective.
(10) Assume that $S \geq R$ is a unitary ring extension. Then, the ring S is called right *R-projective* [6] in case, for any right S-module M_S with an S-module N_S, $N_R \mid M_R$ implies $N_S \mid M_S$, where $N \mid M$ means that N is a direct summand of M.
(11) The ring extension $S \geq R$ is called a *finite normalizing extension* [8] in case there is a finite subset $\{s_1, \cdots, s_n\} \subseteq S$ such that $S = \sum_{i=1}^{i=n} s_i R$ and $s_i R = R s_i$ for $i = 1, \cdots, n$.
(12) A finite normalizing extension $S \geq R$ is called an *almost excellent extension* [12] in case ${}_R S$ is flat, S_R is projective, and the ring S is right R-projective.

In this paper, we introduce the dual concepts of *presented dimensions* of R-modules. We also, introduce the *copresented dimension* of any R-module M:

FEd(M) = inf$\{m \mid$ there exists an injective resolution $0 \to M \to E^0 \to \cdots \to E^m \to \cdots \to E^{m+i} \to \cdots$, such that E^{m+i} are finitely cogenerated for

$i = 0, 1, 2, \cdots\}$. If $K = \ker(E^m \to E^{m+1})$, then K has an infinite finite copresentation. It is clear that any copresented dimension is finitely copresented dimension (see [16]). Also, the copresented dimension of ring R is defined to be:

$$\mathrm{FED}(R) = \sup\{\mathrm{FEd}(M) \mid M \text{ is a finitely cogenerated module}\}.$$

Then, some basic properties of the copresented dimensions of modules are studied. For example, it is shown that if $\mathrm{FEd}(M) < \infty$, then $\mathrm{id(M)} \leq n$ if and only if $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for every strongly copresented R-module N. Also, it is proved that $\mathrm{FED}(R \oplus S) = \sup\{\mathrm{FED}(R), \mathrm{FED}(S)\}$, for any two rings R and S. Also, some characterizations of the copresented dimensions of modules on Ring Extensions are determined. For instance, let $S \geq R$ be a finite normalizing extension with S_R projective as an R-module, then for any right R-module M_R, we have $\mathrm{FEd}(\mathrm{Hom}_R(S, M))_S \leq \mathrm{FEd}(M_R)$. Finally, we give a sufficient condition under which $\mathrm{FED}(S) \leq \mathrm{FED}(R)$ and or $\mathrm{FED}(R) < \mathrm{FED}(S) + \max\{k, d\}$, where $k = \mathrm{id}(S_R)$ and $d = \sup\{\mathrm{FEd}(M_R) \mid M \in \mathrm{Mod} - S \text{ and } \mathrm{FEd}(M_S) = 0\}$.

2. Main Results

We start this section with the following definition which is the dual of the presented dimension of a module.

Definition 2.1. For any R-module M, we define the copresented dimension of M to be $\mathrm{FEd}(M) = \inf\{m \mid$ there exists an injective resolution $0 \to M \to E^0 \to \cdots \to E^m \to \cdots \to E^{m+i} \to \cdots$, so that E^{m+i} are finitely cogenerated for $i = 0, 1, 2, \cdots\}$. In particular, a module M is called strongly copresented module if $\mathrm{FEd}(M) = 0$.

Proposition 2.2. *For any R-module M,* $\mathrm{FEd}(M) \leq \mathrm{id}(M) + 1$.

Proof. It is a direct consequence of Definition 2.1. □

Example 2.3. Let $R = \mathbb{Z}$. Since $\mathrm{id}(\mathbb{Z}_{p^\infty}) = 0$, we have $\mathrm{FEd}(\mathbb{Z}_{p^\infty}) \leq 1$. On the other hand, $\mathbb{Z}_{p^\infty}$ is finitely cogenerated by [1, p.124]. So by Definition 2.1, $\mathrm{FEd}(\mathbb{Z}_{p^\infty}) = 0$.

Now, we study the behavior of the copresented dimension on the exact sequences. Before this we need the following lemma.

Lemma 2.4. *Let $0 \to A \xrightarrow{f'} B \xrightarrow{f} C \to 0$ be a short exact sequence of R-modules. Then:*

(1) *If $0 \to A \to A^0 \to A^1 \to \cdots$ and $0 \to C \to C^0 \to C^1 \to \cdots$ are injective resolutions of A and C, respectively. Then the exact sequence*

$$0 \longrightarrow B \longrightarrow A^0 \oplus C^0 \longrightarrow A^1 \oplus C^1 \longrightarrow \cdots$$

is an injective resolution of B.

(2) *If* $0 \to B \to B^0 \to B^1 \to \cdots$ *and* $0 \to C \to C^0 \to C^1 \to \cdots$ *are injective resolutions of* B *and* C*, respectively. Then the exact sequence*

$$0 \longrightarrow A \longrightarrow B^0 \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \cdots$$

is an injective resolution of A*, where* $D^i = C^i \oplus B^{i+1}$ *for any* $i \geq 0$.

(3) *If* $0 \to B \to B^0 \to B^1 \to \cdots$ *and* $0 \to A \to A^0 \to A^1 \to \cdots$ *are injective resolutions of* B *and* A*, respectively. Then the exact sequence*

$$0 \longrightarrow C \longrightarrow F^0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

is an injective resolution of C*, where* $F^0 = B^0 \oplus A^1$ *and* $E^i = A^0 \oplus B^{i+1} \oplus A^{i+2}$ *for any* $i \geq 0$.

Proof. (1) The proof is similar to that of [3, Theorem 2.4].

(2) Let $0 \to B \to B^0 \to B^1 \to \cdots$ be an injective resolution of B. Then, the exact sequences

$0 \to K \to B^1 \to B^2 \to \cdots$ and $0 \to B \to B^0 \to K \to 0$ exist, where $K = \frac{B^0}{B}$. Now, we consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B^0 & \longrightarrow & D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By (1), there is an exact sequence

$$0 \longrightarrow D \longrightarrow D^0 \longrightarrow D^1 \longrightarrow D^2 \longrightarrow \cdots$$

of injective R-modules D^i such that $D^i = C^i \oplus B^{i+1}$ for any $i \geq 0$.

Combining this sequence with the exact sequence $0 \to A \to B^0 \to D \to 0$, we get the exact sequence

$$0 \longrightarrow A \longrightarrow B^0 \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \cdots,$$

where B^0 and D^i are injective for any $i \geq 0$.

(3) Let $0 \to A \to A^0 \to A^1 \to \cdots$ be an injective resolution of A. Then, the exact sequences

$0 \to K \to A^1 \to A^2 \to \cdots$ and $0 \to A \to A^0 \to K \to 0$ exist, where $K = \frac{A^0}{A}$. Now, we consider the following commutative diagram:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A^0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & C \arrow[r] & 0 \\ & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

By (1), there is an exact sequence

$$0 \longrightarrow F \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$$

of injective R-modules F^i such that $F^i = B^i \oplus A^{i+1}$ for any $i \geq 0$.

It is clear that $F = A^0 \oplus C$. So, the exact sequence $0 \to C \to F \to A^0 \to 0$ exists. Let $K = \frac{F^0}{F}$, then we obtain the following commutative diagram:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & C \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C \arrow[r] & F^0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\ & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

Therefore by (1), the sequence

$$0 \longrightarrow E \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$$

is an injective resolution of E, where $E^i = A^0 \oplus F^{i+1} = A^0 \oplus B^{i+1} \oplus A^{i+2}$ for any $i \geq 0$. Combining this sequence with the exact sequence $0 \to C \to F^0 \to E \to 0$, we get the exact sequence

$$0 \longrightarrow C \longrightarrow F^0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots,$$

where F^0 and E^i are injective for any $i \geq 0$. □

Theorem 2.5. *Let* $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ *be an exact sequence of* R*-modules. Then* $\mathrm{FEd}(B) \leq \max\{\mathrm{FEd}(A), \mathrm{FEd}(C)\}, \mathrm{FEd}(C) \leq \max\{\mathrm{FEd}(B), \mathrm{FEd}(A) + 1\}, \mathrm{FEd}(A) \leq \max\{\mathrm{FEd}(B), \mathrm{FEd}(C) - 1\}$.

Proof. Assume that $\mathbf{E}'$ is an injective resolution of A and $\mathbf{E}''$ is an injective resolution of C. Thus by Lemma 2.5(1), there exists an injective resolution $\mathbf{E}$ of B such that

$$0 \to \mathbf{E}'^{\mathbf{A}} \to \mathbf{E}^{\mathbf{B}} = \mathbf{E}'^{\mathbf{A}} \oplus \mathbf{E}''^{\mathbf{C}} \to \mathbf{E}''^{\mathbf{C}} \to 0$$

is an exact sequence of complexes. Hence for every $m \geq \max\{\mathrm{FEd}(A), \mathrm{FEd}(C)\}$, E^m is finitely cogenerated. So, we deduce that $\mathrm{FEd}(B) \leq \max\{\mathrm{FEd}(A), \mathrm{FEd}(C)\}$.

Assume that $\mathbf{E}^{''}$ is an injective resolution of C and $\mathbf{E}$ is an injective resolution of B. Thus by Lemma 2.5(2), the exact sequence

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \cdots \longrightarrow D^d \longrightarrow \cdots$$

is an injective resolution of A. So for every $d \geq \max\{\mathrm{FEd}(B), \mathrm{FEd}(C)-1\}$, D^d is finitely cogenerated. Thus, we have that $\mathrm{FEd}(A) \leq \max\{\mathrm{FEd}(B), \mathrm{FEd}(C) - 1\}$. Also, it is prove that $\mathrm{FEd}(C) \leq \max\{\mathrm{FEd}(B), \mathrm{FEd}(A)+1\}$. □

The proof of the following Corollary is similar to the proof of [19, Corollary 2.7].

Corollary 2.6. *If* $\mathrm{FEd}(M_1), \mathrm{FEd}(M_2), \cdots \mathrm{FEd}(M_d)$ *are finite, then:*

$$\mathrm{FEd}(\oplus M_i) = \max\{\mathrm{FEd}(M_i) \mid i = 1, \cdots, d\}.$$

Proof. For the case $m = 2$, the exact sequences

$$0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$$

and

$$0 \to M_2 \to M_2 \oplus M_1 \to M_1 \to 0$$

exist. Thus by Theorem 2.5, we deduce that

$$\mathrm{FEd}(M_2) \leq \max\{\mathrm{FEd}(M_1 \oplus M_2), \mathrm{FEd}(M_1) - 1\},$$
$$\mathrm{FEd}(M_1) \leq \max\{\mathrm{FEd}(M_1 \oplus M_2), \mathrm{FEd}(M_2) - 1\}$$

and

$$\mathrm{FEd}(M_1 \oplus M_2) \leq \max\{FEd(M_1), FEd(M_2)\}.$$

Assume that $\mathrm{FEd}(M_1) < \mathrm{FEd}(M_2)$. Then $\mathrm{FEd}(M_1) \leq \mathrm{FEd}(M_2) - 1$, and we have:

$$\mathrm{FEd}(M_2) \leq \max\{\mathrm{FEd}(M_1 \oplus M_2), \mathrm{FEd}(M_2) - 2\} = \mathrm{FEd}(M_1 \oplus M_2).$$

Also, similarly $\mathrm{FEd}(M_1) \leq \mathrm{FEd}(M_1 \oplus M_2)$. So, we conclude that $\mathrm{FEd}(M_1 \oplus M_2) = \max\{\mathrm{FEd}(M_1), \mathrm{FEd}(M_2)\}$. □

Proposition 2.7. *Let n be a non-negative integer. Then the following statements are equivalent:*

(1) $\mathrm{id}(M) \leq n$ *for every strongly copresented R-module M;*
(2) $\mathrm{Ext}_R^{n+1}(N, M) = 0$ *for every strongly copresented R-module N.*

Proof. (1) ⇒ (2) This is obvious.

(2) ⇒ (1) We use the induction on n. Let $n = 0$. Since $\mathrm{Ext}_R^1(N, M) = 0$ for any strongly copresented R-module N, by using the exact sequence $0 \to M \to E^0 \to L^0 \to 0$ where E^0 is finitely cogenerated and L^0 is strongly copresented, we deduce that $\mathrm{Ext}_R^1(L^0, M) = 0$. Therefore by [7, Theorem 7.31], the exact sequence ebove is split. So, M is injective and hence $\mathrm{id}(M) \leq 0$. Assume that

$n > 0$. By [7, Corollary 6.42], we have that $\mathrm{Ext}_R^{n+1}(N, M) \cong \mathrm{Ext}_R^n(N, L^0) = 0$. Thus by induction hypothesis, $\mathrm{id}(L^0) \leq n - 1$. Therefore from the exact sequence ebove, we deduce that $\mathrm{id}(M) \leq n$. $\square$

Proposition 2.8. *Let* $\mathrm{FEd}(M) \leq 1$. *Then the following statements are equivalent:*

(1) $\mathrm{id}(M) \leq n$;
(2) $\mathrm{Ext}_R^{n+1}(N, M) = 0$ *for every strongly copresented* R*-module* N.

Proof. Since $\mathrm{FEd}(M) \leq 1$, the exact sequence $0 \to M \to E^0 \to L^0 \to 0$ exists, where E^0 is injective and L^0 is strongly copresented. Thus, $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for any strongly copresented R-module N if and only if $\mathrm{Ext}_R^n(N, L^0) = 0$ if and only if $\mathrm{id}(L^0) \leq n - 1$ (by Proposition 2.7) if and only if $\mathrm{id}(M) \leq n$. $\square$

Theorem 2.9. *Let* $\mathrm{FEd}(M) < \infty$. *Then the following statements are equivalent:*

(1) $\mathrm{id}(M) \leq n$;
(2) $\mathrm{Ext}_R^{n+1}(N, M) = 0$ *for every strongly copresented* R*-module* N.

Proof. (1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) If $\mathrm{FEd}(M) = m$, then the exact sequence

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \overset{d^{m-1}}{\to} E^m \overset{d^m}{\to} \cdots \to E^{m+j} \to \cdots$$

exists, where E^i is finitely cogenerated for any $i \geq m$. By Proposition 2.2, $n + 1 \geq m$. Let $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for every strongly copresented R-module N. Thus by [7, Corollary 6.42], we have

$$\mathrm{Ext}_R^{n+1}(N, M) \cong \mathrm{Ext}_R^{n-m+1}(N, \mathrm{coker} d^{m-1}) = 0.$$

Since $\mathrm{coker} d^{m-1}$ is strongly copresented, Proposition 2.8 impleis that

$$\mathrm{id}(\mathrm{coker} d^{m-1}) \leq n - m$$

and so, we deduce that $\mathrm{id}(M) \leq n$. $\square$

Corollary 2.10. *Let* $\mathrm{D}(R) < \infty$. *Then:*

$$\mathrm{D}(R) = \sup\{\mathrm{pd}(N) \mid N \text{ is strongly copresented}\}.$$

Proof. Assume that $\mathrm{D}(R) \leq m$. Thus, $\mathrm{pd}(N^{'}) \leq m$ for any R-module $N^{'}$. So, for any strongly copresented R-module N, $\mathrm{pd}(N) \leq m$. Conversely, let $\mathrm{pd}(N) \leq m$ for every strongly copresented R-module N. Thus $\mathrm{Ext}_R^{m+1}(N, M) = 0$ for every strongly presented R-module M. Since $\mathrm{D}(R) < \infty$, $\mathrm{FEd}(M) < \infty$ by Proposition 2.2. Therefore by Theorem 2.9, $\mathrm{id}(M) \leq m$ and hence by [19, corollary 3.7], $\mathrm{D}(R) \leq m$. $\square$

Definition 2.11. For any ring R, we define the copresented dimension of R to be $\mathrm{FED}(R) = \sup\{\mathrm{FEd}(M) \mid M \text{ is a finitely cogenerated module}\}$.

Example 2.12. Let $R = k[x^3, x^3y, xy^3, y^3]$, where k is a field with characteristic $p = 3$. By Definition 2.11 and Proposition 2.2, $\mathrm{FED}(R^\infty) \leq \mathrm{D}(R^\infty) + 1$, where R^∞ is perfect closure of R. On the other hand, $k[x, y]$ is purely inseparable over R. Also, by [9, Proposition 3.3], $(k[x, y])^\infty$ is coherent. Therefore by [10, Remark 1.4], R^∞ is coherent. Since R is reduced, [2, Proposition 5.5] implies that $\mathrm{FED}(R^\infty) \leq \dim(R) + 1$ and so, $\mathrm{FED}(R^\infty) \leq 3$.

Proposition 2.13. *The following statements are equivalent:*

(1) $\mathrm{FED}(R) = 0$;
(2) *Every finitely cogenerated module has an infinite finite copresented;*
(3) *Every finitely cogenerated module is finitely copresented;*
(4) *R is co-noetherian.*

Proof. The implication $(1) \Longrightarrow (2) \Longrightarrow (3)$ follow immediately from Definiton 2.11.

$(3) \Longrightarrow (4) \Longrightarrow (1)$ are trivial. □

Corollary 2.14. *If* $\mathrm{FED}(R) \leq 0$*, then R is n-cocoherent.*

Proof. Since every n-copresented module M is finitely cogenerated, Proposition 2.13 implies that M is $(n+1)$-copresented. □

Next, we study the copresented dimension of the direct sum of rings. But before this we need the following lemma.

Lemma 2.15. *Let $f : R \to S$ be a ring epimorphism. If M_S is a right S-module (hence a right R-module) and N_R is a right R-module, then the following statements hold:*

(1) *$M \otimes_R S \cong M_S$.*
(2) *If f is flat and N_R is a finitely cogenerated right R-module, then $N \otimes_R S$ is a finitely cogenerated right S-module.*
(3) *If f is flat , then M_S is a finitely cogenerated right S-module if and only if M_R is a finitely cogenerated right R-module.*
(4) *If f is projective, then M_S is an injective right S-module if and only if M_R is an injective right R-module.*

Proof. (1) This is clear.

(2) For any family of submodules $\{N_i \otimes_R 1_S | i \in I\}$ in $N \otimes_R S$, if $\bigcap(N_i \otimes_R 1_S) = 0$, then we need to show that $\bigcap_{i\in F}(N_i \otimes_R 1_S) = 0$ for some finite subset F of I. Since f is flat, we have that $\bigcap_{i\in I} N_i \otimes_R 1_S = 0$. So, $\bigcap_{i\in I} N_i = 0$ and hence by hypotises $\bigcap_{i\in F} N_i = 0$ for some finite subset F of I. Therefore, $\bigcap_{i\in F}(N_i \otimes_R 1_S) = \bigcap_{i\in F} N_i \otimes_R 1_S = 0$.

(3) $(\Rightarrow)$: Let $\psi : M \to \prod_{i\in I} R$ is a monomorphism, then we claim that $\pi : M \to \prod_{i\in F} R$ is a monomorphism for some finite subset F of I. We have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\psi} & \prod_{i\in I} R \\ \downarrow\cong & & \downarrow g \\ M & \xrightarrow{h} & \prod_{i\in I} S, \end{array}$$

where since g is epimorphism and ψ is monomorphism, h is monomorphism. So by hypothesis, $\alpha : M \to \prod_{i\in F} S$ is a monomorphism for some finite subset F of I. Therefore the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & \prod_{i\in F} R \\ \downarrow\cong & & \downarrow \beta \\ M & \xrightarrow{\alpha} & \prod_{i\in F} S, \end{array}$$

where β is epimorphism and α is monomorphism, implies that γ is monomorphism.

$(\Leftarrow)$: This follows from (1) and (2)

(4) By [5, Lemma 3.3], M_S is an (n, d)-injective right S-module if and only if M_R is an (n, d)-injective right R-module. If $n = 0, d = 0$, Then (4) is hold. □

Theorem 2.16. *Assume thet R and S are two rings. Then:*

$$\mathrm{FED}(R \oplus S) = \sup\{\mathrm{FED}(R), \mathrm{FED}(S)\}.$$

Proof. We first show that $\mathrm{FED}(R \oplus S) \leq \sup\{\mathrm{FED}(R), \mathrm{FED}(S)\}$. Consider $\mathrm{FED}(R) = n, \mathrm{FED}(S) = m$ and $n \geq m$. Also, let M be a finitely cogenerated right $(R\oplus S)$-module. Then M has a unique decomposition $M = A\oplus B$, where A, B are right modules of rings R and S, respectively. By [15, Lemma 1.1], A and B are finitely cogenerated right $(R \oplus S)$-module. So by Lemma 2.15, A is finitely cogenerated right R-module and B is finitely cogenerated right S-module. Therefore $\mathrm{FEd}(A) \leq n$ and $\mathrm{FEd}(B) \leq m$, and hence there is an exact sequences

$$0 \to A \to E_a^0 \to E_a^1 \to \cdots \to E_a^{n-1} \to E_a^n \to \cdots ,$$
$$0 \to B \to E_b^0 \to E_b^1 \to \cdots \to E_b^{m-1} \to E_b^m \to \cdots$$

of injective right R-modules E_a^i and injective right S-modules E_b^i such that E_a^i, E_b^i are finitely cogenerated for any $i \geq n$ and $i \geq m$, respectively. So, we deduce that the exact sequence

$$0 \to A \oplus B \to E_a^0 \oplus E_b^0 \to E_a^1 \oplus E_b^1 \to \cdots \to E_a^{n-1} \oplus E_b^{m-1} \to E_a^n \oplus E_b^m \to \cdots$$

exists, where by Lemma 2.15, every $E_a^i \oplus E_b^i$ is injective right $(R \oplus S)$-module and also, every $E_a^i \oplus E_b^i$ is finitely cogenerated for any $i \geq n$. Therefore, we have $\mathrm{FED}(R \oplus S) \leq \sup\{\mathrm{FED}(R), \mathrm{FED}(S)\}$.

Conversely, Assume that $\mathrm{FED}(R \oplus S) = d$. If M is a finitely cogenerated right R-module. Then by Lemma 2.15, M is a finitely cogenerated right $(R\oplus S)$-module and hence $\mathrm{FED}(M_{(R\oplus S)}) \leq d$. Thus, the exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{d-1} \to E^d \to \cdots$ of injective right $(R \oplus S)$-modules

E^i exists, where every E^i is finitely cogenerated for any $i \geq d$. Let $E^i = C^i \oplus D^i$, where C^i is a R-module and D^i is a S-module. On the other hand, M is a right R-module, so we have the exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^{d-1} \to C^d \to \cdots$ of R-modules. But, every C^i is injective right $(R \oplus S)$-module and also every C^i is finitely cogenerated right $(R \oplus S)$-module for $i \geq d$. So by [15, Lemma 1.1] and Lemma 2.15, C^i is an injective right R-module and it is finitely cogenerated R-module for $i \geq d$. Therefore $\mathrm{FEd}(M) \leq d$ and hence $\mathrm{FED}(R) \leq d$. Similarly, $\mathrm{FED}(S) \leq d$ and implies that $\sup\{\mathrm{FED}(R), \mathrm{FED}(S)\} \leq \mathrm{FED}(R \oplus S)$. □

Proposition 2.17. *Let $S \geq R$ be a finite normalizing extension with S_R projective as an R-module. Then for any right R-module M_R,* $\mathrm{FEd}(\mathrm{Hom}_R(S, M))_S \leq \mathrm{FEd}(M_R)$.

Proof. Asume that $\mathrm{FEd}(M_R) = n$. Then there axists an exact sequence of injective R-modules

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots ,$$

where each E^i is finitely cogenerated for any $i \geq n$. Since S is projective, there is an exact sequence

$$0 \to \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, E^0) \to \cdots \to \mathrm{Hom}_R(S, E^n) \to \cdots$$

of injective S-modules $\mathrm{Hom}_R(S, E^i)$, where by [13, Propositon 8.3], $\mathrm{Hom}_R(S, E^i)$ is finitely cogenerated for any $i \geq n$. Thus $\mathrm{FEd}(\mathrm{Hom}_R(S, M))_S \leq n$ and hence, we have $\mathrm{FEd}(\mathrm{Hom}_R(S, M))_S \leq \mathrm{FEd}(M_R)$. □

Proposition 2.18. *Let $S \geq R$ be a finite normalizing extension, S_R be Projective, and S be R-projective. Then for each right S-module M_S,* $\mathrm{FEd}(M_S) \leq \mathrm{FEd}(\mathrm{Hom}_R(S, M))$.

Proof. By [12, Lemma 1.1], M_S is isomorphic to a direct summand of $\mathrm{Hom}_R(S, M)$. So, from Corollary 2.6, we deduce that $\mathrm{FEd}(M_S) \leq \mathrm{FEd}(\mathrm{Hom}_R(S, M))$. □

Proposition 2.19. *Let $S \geq R$ be an almost excellent extension. Then for each right S-module M_S,* $\mathrm{FEd}(M_R) \leq \mathrm{FEd}(M_S)$.

Proof. Asume that $\mathrm{FEd}(M_S) = n$. So, there axists an exact sequence of injective S-modules

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots ,$$

where each E^i is finitely cogenerated for any $i \geq n$. Thus by [18, Proposition 5.1], every E^i is an injective R-module and also, it is a finitely cogenerated R-module for $i \geq n$ by [14, Theorem 5]. Therefore, it follows that $\mathrm{FEd}(M_R) \leq \mathrm{FEd}(M_S)$. □

Corollary 2.20. *Let $S \geq R$ be an almost excellent extension. Then for each right S-module M_S,* $\mathrm{FEd}(M_R) = \mathrm{FEd}(M_S) = \mathrm{FEd}(\mathrm{Hom}_R(S, M))$.

Theorem 2.21. *Asume that $S \geq R$ is a finite normalizing extension and S_R is Projective. Then:*

(1) *If S is R-projective and* $\mathrm{FED}(S) < \infty$*, then* $\mathrm{FED}(S) \leq \mathrm{FED}(R)$.
(2) *If* $\mathrm{FED}(R) < \infty$*, then* $\mathrm{FED}(R) < \mathrm{FED}(S) + \max\{k, d\}$*, where* $k = id(S_R)$ *and* $d = \sup\{\mathrm{FEd}(M_R) \mid M \in \mathrm{Mod} - S$ *and* $\mathrm{FEd}(M_S) = 0\}$.

Proof. (1) Asume that $\mathrm{FED}(S) = n$ and $\mathrm{FEd}(M_S) = n$ for a finitely cogenerated S-module M. Since S_R is projective, by hypothesis and [12, Lemma 1.1], M_S is isomorphic to a direct summand of $\mathrm{Hom}_R(S, M)$ and hence we have:

$$0 \to K \to \mathrm{Hom}_R(S, M)) \to M_S \to 0.$$

By [14, Lemma 4], $\mathrm{Hom}_R(S, M))$ is finitely cogenerated S-module, since M_R is a finitely cogenerated R-module. So, $\mathrm{FEd}(\mathrm{Hom}_R(S, M)_S) \leq n$. On the other hand, by Theorem 2.5,

$$\mathrm{FEd}(K) \leq \max\{n, n-1\},$$

$$n = \mathrm{FEd}(M_S) \leq \max\{\mathrm{FEd}(\mathrm{Hom}_R(S, M)_S), \mathrm{FEd}(K_S) - 1\} \leq \mathrm{FED}(S) = n.$$

Therefore $\mathrm{FEd}(\mathrm{Hom}_R(S, M)_S) = n$. Thus, Proposition 2.17 implies that

$$\mathrm{FEd}(\mathrm{Hom}_R(S, M)_S) \leq \mathrm{FEd}(M_R)$$

and hence $\mathrm{FED}(S) \leq \mathrm{FED}(R)$.

(2) Asume that $\mathrm{FED}(R) = n$ and $\mathrm{FEd}(M_R) = n$ for a finitely cogenerated R-module M. Since S_R is projective, by [12, Lemma 1.1], M_R is isomorphic to a direct summand of $\mathrm{Hom}_R(S, M)$ which induces the following short exact sequence of R-modules:

$$0 \to K \to \mathrm{Hom}_R(S_R, M)) \to M_R \to 0.$$

It is clear that $\mathrm{Hom}_R(S_R, M))$ is a finitely cogenerated R-module. Thus Theorem 2.5 implies that

$$n = \mathrm{FEd}(M_R) \leq \max\{\mathrm{FEd}(\mathrm{Hom}_R(S_R, M)), \mathrm{FEd}(K_R) - 1\} \leq \mathrm{FED}(R) = n,$$

and hence $\mathrm{FEd}(\mathrm{Hom}_R(S_R, M)) = n$.

If $\mathrm{FEd}(\mathrm{Hom}_R(S, M))_S = m \leq \mathrm{FED}(S)$, then there is an injective resolution

$$0 \longrightarrow \mathrm{Hom}_R(S, M) \xrightarrow{f_0} E^0 \xrightarrow{f_1} E^1 \longrightarrow \cdots \longrightarrow E^{m-1} \xrightarrow{f_m} E^m \xrightarrow{f_{m+1}} \cdots$$

of $\mathrm{Hom}_R(S, M)$, where every E^i is a finitely cogenerated S-module for any $i \geq m$. Let $D^i = \mathrm{coker}(f_i)$ for every $i \geq 0$. Thus, the following short exact sequences

$$0 \longrightarrow \mathrm{Hom}_R(S, M) \longrightarrow E^0 \to D^0 \longrightarrow 0,$$

$$\cdots$$

$$0 \longrightarrow D^{m-2} \longrightarrow E^{m-1} \longrightarrow D^{m-1} \longrightarrow 0,$$

$$0 \longrightarrow D^{m-1} \longrightarrow E^m \longrightarrow D^m \longrightarrow 0$$

exists, where $\mathrm{FEd}(D^{m-1}) = 0$. But by hypothesis and Proposition 2.2, we have:

$$\mathrm{FEd}(D^i)_R \leq \mathrm{id}(D^i)_R + 1 \leq \mathrm{id}(S_R) + 1 = k+1 \quad , \quad \mathrm{FEd}(D^{m-1})_R \leq d.$$

Therefore by Theorem 2.5, we deduce that:

$\mathrm{FEd}(D^{m-2})_R \leq \max\{\mathrm{FEd}(E^{m-1})_R, \mathrm{FEd}(D^{m-1})_R + 1\} < \max\{k+1, d+1\} = 1 + \max\{k, d\}$,

$$\mathrm{FEd}(D^{m-3})_R \leq \max\{\mathrm{FEd}(E^{m-2})_R, \mathrm{FEd}(D^{m-2})_R + 1\} < 2 + \max\{k, d\},$$

$$\vdots$$

$$\mathrm{FEd}(D^0)_R \leq \max\{\mathrm{FEd}(E^1)_R, \mathrm{FEd}(D^1)_R + 1\} < m - 1 + \max\{k, d\},$$

$n = \mathrm{FEd}(\mathrm{Hom}_R(S, M))_R \leq \max\{\mathrm{FEd}(E^0)_R, \mathrm{FEd}(D^0)_R + 1\} < m + \max\{k, d\}$.

Thus $\mathrm{FED}(R) < m + \max\{k, d\} \leq \mathrm{FED}(S) + \max\{k, d\}$ and so, the proof is complete. □

Corollary 2.22. *Let $S \geq R$ be an almost excellent extension. Then* $\mathrm{FED}(R) < \mathrm{FED}(S) + \mathrm{id}(S)_R$.

Proof. By Proposition 2.19 and Theorem 2.21, this is clear. □

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