

## Copresented Dimension of Modules

M. Amini\*, F. Hassani

Department of Mathematics, Payame Noor University, Tehran, Iran.

E-mail: mostafa.amini@pnu.ac.ir

E-mail: Hassani@pnu.ac.ir

**ABSTRACT.** In this paper, a new homological dimension of modules, copresented dimension, is defined. We study some basic properties of this homological dimension. Some ring extensions are considered, too. For instance, we prove that if  $S \geq R$  is a finite normalizing extension and  $S_R$  is a projective module, then for each right  $S$ -module  $M_S$ , the copresented dimension of  $M_S$  does not exceed the copresented dimension of  $\text{Hom}_R(S, M)$ .

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### 1. INTRODUCTION

Throughout this paper,  $R$  is an associative ring with identity and all modules are unitary. First we recall some known notions and facts needed in the sequel. Let  $R$  be a ring,  $n$  a non-negative integer and  $M$  an  $R$ -module. Then

- (1)  $M$  is said to be *finitely cogenerated* [1] if for every family  $\{V_k\}_J$  of submodules of  $M$  with  $\bigcap_J V_k = 0$ , there is a finite subset  $I \subset J$  with  $\bigcap_I V_k = 0$ .
- (2)  $M$  is said to be  *$n$ -copresented* [14] if there is an exact sequence of  $R$ -modules  $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^n$ , where each  $E^i$  is a finitely cogenerated injective module.

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\*Corresponding Author

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- (3)  $R$  is called *right co-coherent* [17] if every finitely cogenerated factor module of a finitely cogenerated injective  $R$ -module is finitely copresented.
- (4)  $R$  is called *n-cocoherent* [14] in case every  $n$ -copresented  $R$ -module is  $(n + 1)$ -copresented. It is easy to see that  $R$  is cocoherent if and only if it is 1-cocoherent. Recall that a ring  $R$  is called *right conoethrian* [4] if every factor module of a finitely cogenerated  $R$ -module is finitely cogenerated. By [4, Proposition 17], a ring  $R$  is co-noethrian if and only if it is 0-cocoherent.
- (5)  $M$  is said to be *n-presented* [5] if there is an exact sequence of  $R$ -modules  $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$ , where each  $F_i$  is a finitely generated free module.
- (6)  $R$  is called *coherent* [18] in case every 0-presented  $R$ -module is 1-presented.
- (7) A ring extension  $R \subseteq R'$  with characteristic  $p > 0$  is called a *purely inseparable extension* [10] if for every element  $r' \in R'$ , there exists a non-negative integer  $n$  such that  $r'^{p^n} \in R$ .
- (8) For any commutative ring  $R$  of prime characteristic  $p > 0$ , assume that  $F_R : R \rightarrow R^{(e)}$  is the  $e$ -th iterated Frobenius map in which  $R^{(e)} \cong R$ . Then, the *perfect closure* [9] of  $R$ , denoted by  $R^\infty$ , is defined as the limit of the following direct system:

$$R \xrightarrow{F_R} R \xrightarrow{F_R} R \xrightarrow{F_R} \cdots$$

- (9)  $M$  is called  $(n, d)$ -injective [18] if  $\text{Ext}_R^{d+1}(N, M) = 0$  for any  $n$ -presented right  $R$ -module  $N$ . It is clear that  $M$  is  $(0, 0)$ -injective if and only if  $M$  is injective.
- (10) Assume that  $S \geq R$  is a unitary ring extension. Then, the ring  $S$  is called *right R-projective* [6] in case, for any right  $S$ -module  $M_S$  with an  $S$ -module  $N_S$ ,  $N_R \mid M_R$  implies  $N_S \mid M_S$ , where  $N \mid M$  means that  $N$  is a direct summand of  $M$ .
- (11) The ring extension  $S \geq R$  is called a *finite normalizing extension* [8] in case there is a finite subset  $\{s_1, \dots, s_n\} \subseteq S$  such that  $S = \sum_{i=1}^n s_i R$  and  $s_i R = R s_i$  for  $i = 1, \dots, n$ .
- (12) A finite normalizing extension  $S \geq R$  is called an *almost excellent extension* [12] in case  ${}_R S$  is flat,  $S_R$  is projective, and the ring  $S$  is right  $R$ -projective.

In this paper, we introduce the dual concepts of *presented dimensions* of  $R$ -modules. We also, introduce the *copresented dimension* of any  $R$ -module  $M$ :

$\text{FE}d(M) = \inf\{m \mid \text{there exists an injective resolution } 0 \rightarrow M \rightarrow E^0 \rightarrow \cdots \rightarrow E^m \rightarrow \cdots \rightarrow E^{m+i} \rightarrow \cdots, \text{ such that } E^{m+i} \text{ are finitely cogenerated for}$

$i = 0, 1, 2, \dots\}$ . If  $K = \ker(E^m \rightarrow E^{m+1})$ , then  $K$  has an infinite finite copresentation. It is clear that any copresented dimension is finitely copresented dimension (see [16]). Also, the copresented dimension of ring  $R$  is defined to be:

$$\text{FED}(R) = \sup\{\text{FEd}(M) \mid M \text{ is a finitely cogenerated module}\}.$$

Then, some basic properties of the copresented dimensions of modules are studied. For example, it is shown that if  $\text{FEd}(M) < \infty$ , then  $\text{id}(M) \leq n$  if and only if  $\text{Ext}_R^{n+1}(N, M) = 0$  for every strongly copresented  $R$ -module  $N$ . Also, it is proved that  $\text{FED}(R \oplus S) = \sup\{\text{FED}(R), \text{FED}(S)\}$ , for any two rings  $R$  and  $S$ . Also, some characterizations of the copresented dimensions of modules on Ring Extensions are determined. For instance, let  $S \geq R$  be a finite normalizing extension with  $S_R$  projective as an  $R$ -module, then for any right  $R$ -module  $M_R$ , we have  $\text{FED}(\text{Hom}_R(S, M))_S \leq \text{FEd}(M_R)$ . Finally, we give a sufficient condition under which  $\text{FED}(S) \leq \text{FED}(R)$  and or  $\text{FED}(R) < \text{FED}(S) + \max\{k, d\}$ , where  $k = \text{id}(S_R)$  and  $d = \sup\{\text{FEd}(M_R) \mid M \in \text{Mod} - S \text{ and } \text{FEd}(M_S) = 0\}$ .

## 2. MAIN RESULTS

We start this section with the following definition which is the dual of the presented dimension of a module.

**Definition 2.1.** For any  $R$ -module  $M$ , we define the copresented dimension of  $M$  to be  $\text{FEd}(M) = \inf\{m \mid \text{there exists an injective resolution } 0 \rightarrow M \rightarrow E^0 \rightarrow \dots \rightarrow E^m \rightarrow \dots \rightarrow E^{m+i} \rightarrow \dots, \text{ so that } E^{m+i} \text{ are finitely cogenerated for } i = 0, 1, 2, \dots\}$ . In particular, a module  $M$  is called strongly copresented module if  $\text{FEd}(M) = 0$ .

**Proposition 2.2.** For any  $R$ -module  $M$ ,  $\text{FEd}(M) \leq \text{id}(M) + 1$ .

*Proof.* It is a direct consequence of Definition 2.1. □

**EXAMPLE 2.3.** Let  $R = \mathbb{Z}$ . Since  $\text{id}(\mathbb{Z}_{p^\infty}) = 0$ , we have  $\text{FEd}(\mathbb{Z}_{p^\infty}) \leq 1$ . On the other hand,  $\mathbb{Z}_{p^\infty}$  is finitely cogenerated by [1, p.124]. So by Definition 2.1,  $\text{FEd}(\mathbb{Z}_{p^\infty}) = 0$ .

Now, we study the behavior of the copresented dimension on the exact sequences. Before this we need the following lemma.

**Lemma 2.4.** Let  $0 \rightarrow A \xrightarrow{f'} B \xrightarrow{f} C \rightarrow 0$  be a short exact sequence of  $R$ -modules. Then:

- (1) If  $0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \dots$  and  $0 \rightarrow C \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$  are injective resolutions of  $A$  and  $C$ , respectively. Then the exact sequence

$$0 \longrightarrow B \longrightarrow A^0 \oplus C^0 \longrightarrow A^1 \oplus C^1 \longrightarrow \dots$$

is an injective resolution of  $B$ .

- (2) If  $0 \rightarrow B \rightarrow B^0 \rightarrow B^1 \rightarrow \dots$  and  $0 \rightarrow C \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$  are injective resolutions of  $B$  and  $C$ , respectively. Then the exact sequence

$$0 \rightarrow A \rightarrow B^0 \rightarrow D^0 \rightarrow D^1 \rightarrow \dots$$

is an injective resolution of  $A$ , where  $D^i = C^i \oplus B^{i+1}$  for any  $i \geq 0$ .

- (3) If  $0 \rightarrow B \rightarrow B^0 \rightarrow B^1 \rightarrow \dots$  and  $0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \dots$  are injective resolutions of  $B$  and  $A$ , respectively. Then the exact sequence

$$0 \rightarrow C \rightarrow F^0 \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$$

is an injective resolution of  $C$ , where  $F^0 = B^0 \oplus A^1$  and  $E^i = A^0 \oplus B^{i+1} \oplus A^{i+2}$  for any  $i \geq 0$ .

*Proof.* (1) The proof is similar to that of [3, Theorem 2.4].

(2) Let  $0 \rightarrow B \rightarrow B^0 \rightarrow B^1 \rightarrow \dots$  be an injective resolution of  $B$ . Then, the exact sequences

$0 \rightarrow K \rightarrow B^1 \rightarrow B^2 \rightarrow \dots$  and  $0 \rightarrow B \rightarrow B^0 \rightarrow K \rightarrow 0$  exist, where  $K = \frac{B^0}{B}$ . Now, we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & A & \rightarrow & B^0 & \rightarrow & D \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & K & = & K \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

By (1), there is an exact sequence

$$0 \rightarrow D \rightarrow D^0 \rightarrow D^1 \rightarrow D^2 \rightarrow \dots$$

of injective  $R$ -modules  $D^i$  such that  $D^i = C^i \oplus B^{i+1}$  for any  $i \geq 0$ .

Combining this sequence with the exact sequence  $0 \rightarrow A \rightarrow B^0 \rightarrow D \rightarrow 0$ , we get the exact sequence

$$0 \rightarrow A \rightarrow B^0 \rightarrow D^0 \rightarrow D^1 \rightarrow \dots,$$

where  $B^0$  and  $D^i$  are injective for any  $i \geq 0$ .

(3) Let  $0 \rightarrow A \rightarrow A^0 \rightarrow A^1 \rightarrow \dots$  be an injective resolution of  $A$ . Then, the exact sequences

$0 \rightarrow K \rightarrow A^1 \rightarrow A^2 \rightarrow \dots$  and  $0 \rightarrow A \rightarrow A^0 \rightarrow K \rightarrow 0$  exist, where  $K = \frac{A^0}{A}$ . Now, we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & A^0 & \longrightarrow & F & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & K & = & K & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

By (1), there is an exact sequence

$$0 \longrightarrow F \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \dots$$

of injective  $R$ -modules  $F^i$  such that  $F^i = B^i \oplus A^{i+1}$  for any  $i \geq 0$ .

It is clear that  $F = A^0 \oplus C$ . So, the exact sequence  $0 \rightarrow C \rightarrow F \rightarrow A^0 \rightarrow 0$  exists. Let  $K = \frac{F^0}{F}$ , then we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & C & \longrightarrow & F & \longrightarrow & A^0 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & C & \longrightarrow & F^0 & \longrightarrow & E \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & K & = & K \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Therefore by (1), the sequence

$$0 \longrightarrow E \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \dots$$

is an injective resolution of  $E$ , where  $E^i = A^0 \oplus F^{i+1} = A^0 \oplus B^{i+1} \oplus A^{i+2}$  for any  $i \geq 0$ . Combining this sequence with the exact sequence  $0 \rightarrow C \rightarrow F^0 \rightarrow E \rightarrow 0$ , we get the exact sequence

$$0 \longrightarrow C \longrightarrow F^0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \dots,$$

where  $F^0$  and  $E^i$  are injective for any  $i \geq 0$ . □

**Theorem 2.5.** Let  $0 \rightarrow A \xrightarrow{g} B \xrightarrow{f} C \rightarrow 0$  be an exact sequence of  $R$ -modules. Then  $\text{FEd}(B) \leq \max\{\text{FEd}(A), \text{FEd}(C)\}$ ,  $\text{FEd}(C) \leq \max\{\text{FEd}(B), \text{FEd}(A) + 1\}$ ,  $\text{FEd}(A) \leq \max\{\text{FEd}(B), \text{FEd}(C) - 1\}$ .

*Proof.* Assume that  $\mathbf{E}'$  is an injective resolution of  $A$  and  $\mathbf{E}''$  is an injective resolution of  $C$ . Thus by Lemma 2.5(1), there exists an injective resolution  $\mathbf{E}$  of  $B$  such that

$$0 \rightarrow \mathbf{E}'^A \rightarrow \mathbf{E}^B = \mathbf{E}'^A \oplus \mathbf{E}''^C \rightarrow \mathbf{E}''^C \rightarrow 0$$

is an exact sequence of complexes. Hence for every  $m \geq \max\{\text{FEd}(A), \text{FEd}(C)\}$ ,  $E^m$  is finitely cogenerated. So, we deduce that  $\text{FEd}(B) \leq \max\{\text{FEd}(A), \text{FEd}(C)\}$ .

Assume that  $\mathbf{E}''$  is an injective resolution of  $C$  and  $\mathbf{E}$  is an injective resolution of  $B$ . Thus by Lemma 2.5(2), the exact sequence

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \cdots \longrightarrow D^d \longrightarrow \cdots$$

is an injective resolution of  $A$ . So for every  $d \geq \max\{\text{FEd}(B), \text{FEd}(C) - 1\}$ ,  $D^d$  is finitely cogenerated. Thus, we have that  $\text{FEd}(A) \leq \max\{\text{FEd}(B), \text{FEd}(C) - 1\}$ . Also, it is prove that  $\text{FEd}(C) \leq \max\{\text{FEd}(B), \text{FEd}(A) + 1\}$ .  $\square$

The proof of the following Corollary is similar to the proof of [19, Corollary 2.7].

**Corollary 2.6.** *If  $\text{FEd}(M_1), \text{FEd}(M_2), \dots, \text{FEd}(M_d)$  are finite, then:*

$$\text{FEd}(\oplus M_i) = \max\{\text{FEd}(M_i) \mid i = 1, \dots, d\}.$$

*Proof.* For the case  $m = 2$ , the exact sequences

$$0 \rightarrow M_1 \rightarrow M_1 \oplus M_2 \rightarrow M_2 \rightarrow 0$$

and

$$0 \rightarrow M_2 \rightarrow M_2 \oplus M_1 \rightarrow M_1 \rightarrow 0$$

exist. Thus by Theorem 2.5, we deduce that

$$\text{FEd}(M_2) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_1) - 1\},$$

$$\text{FEd}(M_1) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_2) - 1\}$$

and

$$\text{FEd}(M_1 \oplus M_2) \leq \max\{\text{FEd}(M_1), \text{FEd}(M_2)\}.$$

Assume that  $\text{FEd}(M_1) < \text{FEd}(M_2)$ . Then  $\text{FEd}(M_1) \leq \text{FEd}(M_2) - 1$ , and we have:

$$\text{FEd}(M_2) \leq \max\{\text{FEd}(M_1 \oplus M_2), \text{FEd}(M_2) - 2\} = \text{FEd}(M_1 \oplus M_2).$$

Also, similarly  $\text{FEd}(M_1) \leq \text{FEd}(M_1 \oplus M_2)$ . So, we conclude that  $\text{FEd}(M_1 \oplus M_2) = \max\{\text{FEd}(M_1), \text{FEd}(M_2)\}$ .  $\square$

**Proposition 2.7.** *Let  $n$  be a non-negative integer. Then the following statements are equivalent:*

- (1)  $\text{id}(M) \leq n$  for every strongly copresented  $R$ -module  $M$ ;
- (2)  $\text{Ext}_R^{n+1}(N, M) = 0$  for every strongly copresented  $R$ -module  $N$ .

*Proof.* (1)  $\Rightarrow$  (2) This is obvious.

(2)  $\Rightarrow$  (1) We use the induction on  $n$ . Let  $n = 0$ . Since  $\text{Ext}_R^1(N, M) = 0$  for any strongly copresented  $R$ -module  $N$ , by using the exact sequence  $0 \rightarrow M \rightarrow E^0 \rightarrow L^0 \rightarrow 0$  where  $E^0$  is finitely cogenerated and  $L^0$  is strongly copresented, we deduce that  $\text{Ext}_R^1(L^0, M) = 0$ . Therefore by [7, Theorem 7.31], the exact sequence above is split. So,  $M$  is injective and hence  $\text{id}(M) \leq 0$ . Assume that

$n > 0$ . By [7, Corollary 6.42], we have that  $\text{Ext}_R^{n+1}(N, M) \cong \text{Ext}_R^n(N, L^0) = 0$ . Thus by induction hypothesis,  $\text{id}(L^0) \leq n - 1$ . Therefore from the exact sequence above, we deduce that  $\text{id}(M) \leq n$ .  $\square$

**Proposition 2.8.** *Let  $\text{FEd}(M) \leq 1$ . Then the following statements are equivalent:*

- (1)  $\text{id}(M) \leq n$ ;
- (2)  $\text{Ext}_R^{n+1}(N, M) = 0$  for every strongly copresented  $R$ -module  $N$ .

*Proof.* Since  $\text{FEd}(M) \leq 1$ , the exact sequence  $0 \rightarrow M \rightarrow E^0 \rightarrow L^0 \rightarrow 0$  exists, where  $E^0$  is injective and  $L^0$  is strongly copresented. Thus,  $\text{Ext}_R^{n+1}(N, M) = 0$  for any strongly copresented  $R$ -module  $N$  if and only if  $\text{Ext}_R^n(N, L^0) = 0$  if and only if  $\text{id}(L^0) \leq n - 1$  (by Proposition 2.7) if and only if  $\text{id}(M) \leq n$ .  $\square$

**Theorem 2.9.** *Let  $\text{FEd}(M) < \infty$ . Then the following statements are equivalent:*

- (1)  $\text{id}(M) \leq n$ ;
- (2)  $\text{Ext}_R^{n+1}(N, M) = 0$  for every strongly copresented  $R$ -module  $N$ .

*Proof.* (1)  $\Rightarrow$  (2) It is clear.

(2)  $\Rightarrow$  (1) If  $\text{FEd}(M) = m$ , then the exact sequence

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{m-1} \xrightarrow{d^{m-1}} E^m \xrightarrow{d^m} \dots \rightarrow E^{m+j} \rightarrow \dots$$

exists, where  $E^i$  is finitely cogenerated for any  $i \geq m$ . By Proposition 2.2,  $n + 1 \geq m$ . Let  $\text{Ext}_R^{n+1}(N, M) = 0$  for every strongly copresented  $R$ -module  $N$ . Thus by [7, Corollary 6.42], we have

$$\text{Ext}_R^{n+1}(N, M) \cong \text{Ext}_R^{n-m+1}(N, \text{coker } d^{m-1}) = 0.$$

Since  $\text{coker } d^{m-1}$  is strongly copresented, Proposition 2.8 implies that

$$\text{id}(\text{coker } d^{m-1}) \leq n - m$$

and so, we deduce that  $\text{id}(M) \leq n$ .  $\square$

**Corollary 2.10.** *Let  $D(R) < \infty$ . Then:*

$$D(R) = \sup\{\text{pd}(N) \mid N \text{ is strongly copresented}\}.$$

*Proof.* Assume that  $D(R) \leq m$ . Thus,  $\text{pd}(N') \leq m$  for any  $R$ -module  $N'$ . So, for any strongly copresented  $R$ -module  $N$ ,  $\text{pd}(N) \leq m$ . Conversely, let  $\text{pd}(N) \leq m$  for every strongly copresented  $R$ -module  $N$ . Thus  $\text{Ext}_R^{m+1}(N, M) = 0$  for every strongly presented  $R$ -module  $M$ . Since  $D(R) < \infty$ ,  $\text{FEd}(M) < \infty$  by Proposition 2.2. Therefore by Theorem 2.9,  $\text{id}(M) \leq m$  and hence by [19, corollary 3.7],  $D(R) \leq m$ .  $\square$

**Definition 2.11.** For any ring  $R$ , we define the copresented dimension of  $R$  to be  $\text{FED}(R) = \sup\{\text{FEd}(M) \mid M \text{ is a finitely cogenerated module}\}$ .

EXAMPLE 2.12. Let  $R = k[x^3, x^3y, xy^3, y^3]$ , where  $k$  is a field with characteristic  $p = 3$ . By Definition 2.11 and Proposition 2.2,  $\text{FED}(R^\infty) \leq D(R^\infty) + 1$ , where  $R^\infty$  is perfect closure of  $R$ . On the other hand,  $k[x, y]$  is purely inseparable over  $R$ . Also, by [9, Proposition 3.3],  $(k[x, y])^\infty$  is coherent. Therefore by [10, Remark 1.4],  $R^\infty$  is coherent. Since  $R$  is reduced, [2, Proposition 5.5] implies that  $\text{FED}(R^\infty) \leq \dim(R) + 1$  and so,  $\text{FED}(R^\infty) \leq 3$ .

**Proposition 2.13.** *The following statements are equivalent:*

- (1)  $\text{FED}(R) = 0$ ;
- (2) *Every finitely cogenerated module has an infinite finite copresented;*
- (3) *Every finitely cogenerated module is finitely copresented;*
- (4)  *$R$  is co-noetherian.*

*Proof.* The implication  $(1) \implies (2) \implies (3)$  follow immediately from Definition 2.11.

$(3) \implies (4) \implies (1)$  are trivial.  $\square$

**Corollary 2.14.** *If  $\text{FED}(R) \leq 0$ , then  $R$  is  $n$ -cocoherent.*

*Proof.* Since every  $n$ -copresented module  $M$  is finitely cogenerated, Proposition 2.13 implies that  $M$  is  $(n + 1)$ -copresented.  $\square$

Next, we study the copresented dimension of the direct sum of rings. But before this we need the following lemma.

**Lemma 2.15.** *Let  $f : R \rightarrow S$  be a ring epimorphism. If  $M_S$  is a right  $S$ -module (hence a right  $R$ -module) and  $N_R$  is a right  $R$ -module, then the following statements hold:*

- (1)  $M \otimes_R S \cong M_S$ .
- (2) *If  $f$  is flat and  $N_R$  is a finitely cogenerated right  $R$ -module, then  $N \otimes_R S$  is a finitely cogenerated right  $S$ -module.*
- (3) *If  $f$  is flat, then  $M_S$  is a finitely cogenerated right  $S$ -module if and only if  $M_R$  is a finitely cogenerated right  $R$ -module.*
- (4) *If  $f$  is projective, then  $M_S$  is an injective right  $S$ -module if and only if  $M_R$  is an injective right  $R$ -module.*

*Proof.* (1) This is clear.

(2) For any family of submodules  $\{N_i \otimes_R 1_S | i \in I\}$  in  $N \otimes_R S$ , if  $\bigcap (N_i \otimes_R 1_S) = 0$ , then we need to show that  $\bigcap_{i \in F} (N_i \otimes_R 1_S) = 0$  for some finite subset  $F$  of  $I$ . Since  $f$  is flat, we have that  $\bigcap_{i \in I} N_i \otimes_R 1_S = 0$ . So,  $\bigcap_{i \in I} N_i = 0$  and hence by hypothesis  $\bigcap_{i \in F} N_i = 0$  for some finite subset  $F$  of  $I$ . Therefore,  $\bigcap_{i \in F} (N_i \otimes_R 1_S) = \bigcap_{i \in F} N_i \otimes_R 1_S = 0$ .

(3)  $(\implies)$ : Let  $\psi : M \rightarrow \prod_{i \in I} R$  is a monomorphism, then we claim that  $\pi : M \rightarrow \prod_{i \in F} R$  is a monomorphism for some finite subset  $F$  of  $I$ . We have the following commutative diagram:



$$\begin{array}{ccc} M & \xrightarrow{\psi} & \prod_{i \in I} R \\ \downarrow \cong & & \downarrow g \\ M & \xrightarrow{h} & \prod_{i \in I} S, \end{array}$$

where since  $g$  is epimorphism and  $\psi$  is monomorphism,  $h$  is monomorphism. So by hypothesis,  $\alpha : M \rightarrow \prod_{i \in F} S$  is a monomorphism for some finite subset  $F$  of  $I$ . Therefore the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & \prod_{i \in F} R \\ \downarrow \cong & & \downarrow \beta \\ M & \xrightarrow{\alpha} & \prod_{i \in F} S, \end{array}$$

where  $\beta$  is epimorphism and  $\alpha$  is monomorphism, implies that  $\gamma$  is monomorphism.

( $\Leftarrow$ ) : This follows from (1) and (2)

(4) By [5, Lemma 3.3],  $M_S$  is an  $(n, d)$ -injective right  $S$ -module if and only if  $M_R$  is an  $(n, d)$ -injective right  $R$ -module. If  $n = 0, d = 0$ , Then (4) is hold.  $\square$

**Theorem 2.16.** Assume that  $R$  and  $S$  are two rings. Then:

$$\text{FED}(R \oplus S) = \sup\{\text{FED}(R), \text{FED}(S)\}.$$

*Proof.* We first show that  $\text{FED}(R \oplus S) \leq \sup\{\text{FED}(R), \text{FED}(S)\}$ . Consider  $\text{FED}(R) = n, \text{FED}(S) = m$  and  $n \geq m$ . Also, let  $M$  be a finitely cogenerated right  $(R \oplus S)$ -module. Then  $M$  has a unique decomposition  $M = A \oplus B$ , where  $A, B$  are right modules of rings  $R$  and  $S$ , respectively. By [15, Lemma 1.1],  $A$  and  $B$  are finitely cogenerated right  $(R \oplus S)$ -module. So by Lemma 2.15,  $A$  is finitely cogenerated right  $R$ -module and  $B$  is finitely cogenerated right  $S$ -module. Therefore  $\text{FED}(A) \leq n$  and  $\text{FED}(B) \leq m$ , and hence there is an exact sequences

$$\begin{aligned} 0 \rightarrow A \rightarrow E_a^0 \rightarrow E_a^1 \rightarrow \cdots \rightarrow E_a^{n-1} \rightarrow E_a^n \rightarrow \cdots, \\ 0 \rightarrow B \rightarrow E_b^0 \rightarrow E_b^1 \rightarrow \cdots \rightarrow E_b^{m-1} \rightarrow E_b^m \rightarrow \cdots \end{aligned}$$

of injective right  $R$ -modules  $E_a^i$  and injective right  $S$ -modules  $E_b^i$  such that  $E_a^i, E_b^i$  are finitely cogenerated for any  $i \geq n$  and  $i \geq m$ , respectively. So, we deduce that the exact sequence

$$0 \rightarrow A \oplus B \rightarrow E_a^0 \oplus E_b^0 \rightarrow E_a^1 \oplus E_b^1 \rightarrow \cdots \rightarrow E_a^{n-1} \oplus E_b^{m-1} \rightarrow E_a^n \oplus E_b^m \rightarrow \cdots$$

exists, where by Lemma 2.15, every  $E_a^i \oplus E_b^i$  is injective right  $(R \oplus S)$ -module and also, every  $E_a^i \oplus E_b^i$  is finitely cogenerated for any  $i \geq n$ . Therefore, we have  $\text{FED}(R \oplus S) \leq \sup\{\text{FED}(R), \text{FED}(S)\}$ .

Conversely, Assume that  $\text{FED}(R \oplus S) = d$ . If  $M$  is a finitely cogenerated right  $R$ -module. Then by Lemma 2.15,  $M$  is a finitely cogenerated right  $(R \oplus S)$ -module and hence  $\text{FED}(M_{(R \oplus S)}) \leq d$ . Thus, the exact sequence  $0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{d-1} \rightarrow E^d \rightarrow \cdots$  of injective right  $(R \oplus S)$ -modules

$E^i$  exists, where every  $E^i$  is finitely cogenerated for any  $i \geq d$ . Let  $E^i = C^i \oplus D^i$ , where  $C^i$  is a  $R$ -module and  $D^i$  is a  $S$ -module. On the other hand,  $M$  is a right  $R$ -module, so we have the exact sequence  $0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots \rightarrow C^{d-1} \rightarrow C^d \rightarrow \dots$  of  $R$ -modules. But, every  $C^i$  is injective right  $(R \oplus S)$ -module and also every  $C^i$  is finitely cogenerated right  $(R \oplus S)$ -module for  $i \geq d$ . So by [15, Lemma 1.1] and Lemma 2.15,  $C^i$  is an injective right  $R$ -module and it is finitely cogenerated  $R$ -module for  $i \geq d$ . Therefore  $\text{FEd}(M) \leq d$  and hence  $\text{FEd}(R) \leq d$ . Similarly,  $\text{FEd}(S) \leq d$  and implies that  $\sup\{\text{FEd}(R), \text{FEd}(S)\} \leq \text{FEd}(R \oplus S)$ .  $\square$

**Proposition 2.17.** *Let  $S \geq R$  be a finite normalizing extension with  $S_R$  projective as an  $R$ -module. Then for any right  $R$ -module  $M_R$ ,  $\text{FEd}(\text{Hom}_R(S, M))_S \leq \text{FEd}(M_R)$ .*

*Proof.* Assume that  $\text{FEd}(M_R) = n$ . Then there exists an exact sequence of injective  $R$ -modules

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{n-1} \rightarrow E^n \rightarrow \dots,$$

where each  $E^i$  is finitely cogenerated for any  $i \geq n$ . Since  $S$  is projective, there is an exact sequence

$$0 \rightarrow \text{Hom}_R(S, M) \rightarrow \text{Hom}_R(S, E^0) \rightarrow \dots \rightarrow \text{Hom}_R(S, E^n) \rightarrow \dots$$

of injective  $S$ -modules  $\text{Hom}_R(S, E^i)$ , where by [13, Proposition 8.3],  $\text{Hom}_R(S, E^i)$  is finitely cogenerated for any  $i \geq n$ . Thus  $\text{FEd}(\text{Hom}_R(S, M))_S \leq n$  and hence, we have  $\text{FEd}(\text{Hom}_R(S, M))_S \leq \text{FEd}(M_R)$ .  $\square$

**Proposition 2.18.** *Let  $S \geq R$  be a finite normalizing extension,  $S_R$  be Projective, and  $S$  be  $R$ -projective. Then for each right  $S$ -module  $M_S$ ,  $\text{FEd}(M_S) \leq \text{FEd}(\text{Hom}_R(S, M))$ .*

*Proof.* By [12, Lemma 1.1],  $M_S$  is isomorphic to a direct summand of  $\text{Hom}_R(S, M)$ . So, from Corollary 2.6, we deduce that  $\text{FEd}(M_S) \leq \text{FEd}(\text{Hom}_R(S, M))$ .  $\square$

**Proposition 2.19.** *Let  $S \geq R$  be an almost excellent extension. Then for each right  $S$ -module  $M_S$ ,  $\text{FEd}(M_R) \leq \text{FEd}(M_S)$ .*

*Proof.* Assume that  $\text{FEd}(M_S) = n$ . So, there exists an exact sequence of injective  $S$ -modules

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{n-1} \rightarrow E^n \rightarrow \dots,$$

where each  $E^i$  is finitely cogenerated for any  $i \geq n$ . Thus by [18, Proposition 5.1], every  $E^i$  is an injective  $R$ -module and also, it is a finitely cogenerated  $R$ -module for  $i \geq n$  by [14, Theorem 5]. Therefore, it follows that  $\text{FEd}(M_R) \leq \text{FEd}(M_S)$ .  $\square$

**Corollary 2.20.** *Let  $S \geq R$  be an almost excellent extension. Then for each right  $S$ -module  $M_S$ ,  $\text{FEd}(M_R) = \text{FEd}(M_S) = \text{FEd}(\text{Hom}_R(S, M))$ .*

**Theorem 2.21.** *Asume that  $S \geq R$  is a finite normalizing extension and  $S_R$  is Projective. Then:*

- (1) *If  $S$  is  $R$ -projective and  $\text{FED}(S) < \infty$ , then  $\text{FED}(S) \leq \text{FED}(R)$ .*
- (2) *If  $\text{FED}(R) < \infty$ , then  $\text{FED}(R) < \text{FED}(S) + \max\{k, d\}$ , where  $k = \text{id}(S_R)$  and  $d = \sup\{\text{FE}d(M_R) \mid M \in \text{Mod} - S \text{ and } \text{FE}d(M_S) = 0\}$ .*

*Proof.* (1) Asume that  $\text{FED}(S) = n$  and  $\text{FE}d(M_S) = n$  for a finitely cogenerated  $S$ -module  $M$ . Since  $S_R$  is projective, by hypothesis and [12, Lemma 1.1],  $M_S$  is isomorphic to a direct summand of  $\text{Hom}_R(S, M)$  and hence we have:

$$0 \rightarrow K \rightarrow \text{Hom}_R(S, M) \rightarrow M_S \rightarrow 0.$$

By [14, Lemma 4],  $\text{Hom}_R(S, M)$  is finitely cogenerated  $S$ -module, since  $M_R$  is a finitely cogenerated  $R$ -module. So,  $\text{FE}d(\text{Hom}_R(S, M)_S) \leq n$ . On the other hand, by Theorem 2.5,

$$\text{FE}d(K) \leq \max\{n, n-1\},$$

$$n = \text{FE}d(M_S) \leq \max\{\text{FE}d(\text{Hom}_R(S, M)_S), \text{FE}d(K_S) - 1\} \leq \text{FED}(S) = n.$$

Therefore  $\text{FE}d(\text{Hom}_R(S, M)_S) = n$ . Thus, Proposition 2.17 implies that

$$\text{FE}d(\text{Hom}_R(S, M)_S) \leq \text{FE}d(M_R)$$

and hence  $\text{FED}(S) \leq \text{FED}(R)$ .

(2) Asume that  $\text{FED}(R) = n$  and  $\text{FE}d(M_R) = n$  for a finitely cogenerated  $R$ -module  $M$ . Since  $S_R$  is projective, by [12, Lemma 1.1],  $M_R$  is isomorphic to a direct summand of  $\text{Hom}_R(S, M)$  which induces the following short exact sequence of  $R$ -modules:

$$0 \rightarrow K \rightarrow \text{Hom}_R(S_R, M) \rightarrow M_R \rightarrow 0.$$

It is clear that  $\text{Hom}_R(S_R, M)$  is a finitely cogenerated  $R$ -module. Thus Theorem 2.5 implies that

$$n = \text{FE}d(M_R) \leq \max\{\text{FE}d(\text{Hom}_R(S_R, M)), \text{FE}d(K_R) - 1\} \leq \text{FED}(R) = n,$$

and hence  $\text{FE}d(\text{Hom}_R(S_R, M)) = n$ .

If  $\text{FE}d(\text{Hom}_R(S, M))_S = m \leq \text{FED}(S)$ , then there is an injective resolution

$$0 \longrightarrow \text{Hom}_R(S, M) \xrightarrow{f_0} E^0 \xrightarrow{f_1} E^1 \longrightarrow \dots \longrightarrow E^{m-1} \xrightarrow{f_m} E^m \xrightarrow{f_{m+1}} \dots$$

of  $\text{Hom}_R(S, M)$ , where every  $E^i$  is a finitely cogenerated  $S$ -module for any  $i \geq m$ . Let  $D^i = \text{coker}(f_i)$  for every  $i \geq 0$ . Thus, the following short exact sequences

$$0 \longrightarrow \text{Hom}_R(S, M) \longrightarrow E^0 \rightarrow D^0 \longrightarrow 0,$$

...

$$0 \longrightarrow D^{m-2} \longrightarrow E^{m-1} \longrightarrow D^{m-1} \longrightarrow 0,$$

$$0 \longrightarrow D^{m-1} \longrightarrow E^m \longrightarrow D^m \longrightarrow 0$$

exists, where  $\text{FEd}(D^{m-1}) = 0$ . But by hypothesis and Proposition 2.2, we have:

$$\text{FEd}(D^i)_R \leq \text{id}(D^i)_R + 1 \leq \text{id}(S_R) + 1 = k + 1, \quad \text{FEd}(D^{m-1})_R \leq d.$$

Therefore by Theorem 2.5, we deduce that:

$$\text{FEd}(D^{m-2})_R \leq \max\{\text{FEd}(E^{m-1})_R, \text{FEd}(D^{m-1})_R + 1\} < \max\{k + 1, d + 1\} = 1 + \max\{k, d\},$$

$$\text{FEd}(D^{m-3})_R \leq \max\{\text{FEd}(E^{m-2})_R, \text{FEd}(D^{m-2})_R + 1\} < 2 + \max\{k, d\},$$

$$\vdots$$

$$\text{FEd}(D^0)_R \leq \max\{\text{FEd}(E^1)_R, \text{FEd}(D^1)_R + 1\} < m - 1 + \max\{k, d\},$$

$$n = \text{FEd}(\text{Hom}_R(S, M))_R \leq \max\{\text{FEd}(E^0)_R, \text{FEd}(D^0)_R + 1\} < m + \max\{k, d\}.$$

Thus  $\text{FED}(R) < m + \max\{k, d\} \leq \text{FED}(S) + \max\{k, d\}$  and so, the proof is complete.  $\square$

**Corollary 2.22.** *Let  $S \geq R$  be an almost excellent extension. Then  $\text{FED}(R) < \text{FED}(S) + \text{id}(S)_R$ .*

*Proof.* By Proposition 2.19 and Theorem 2.21, this is clear.  $\square$

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