Graded $r$-Ideals

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ABSTRACT. Let $G$ be a group with identity $e$ and $R$ be a commutative $G$-graded ring with nonzero unity 1. In this article, we introduce the concept of graded $r$-ideals. A proper graded ideal $P$ of a graded ring $R$ is said to be a graded $r$-ideal if whenever $a, b \in h(R)$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$, then $b \in P$. We study and investigate the behavior of graded $r$-ideals to introduce several results. We introduced several characterizations for graded $r$-ideals; we proved that $P$ is a graded $r$-ideal of $R$ if and only if $aP = aR \cap P$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$. Also, $P$ is a graded $r$-ideal of $R$ if and only if $P = (P : a)$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$. Moreover, $P$ is a graded $r$-ideal of $R$ if and only if whenever $A, B$ are graded ideals of $R$ such that $AB \subseteq P$ and $A \cap r(h(R)) \neq \phi$, then $B \subseteq P$. In this article, we introduce the concept of a $huz$-rings. A graded ring $R$ is said to be a $huz$-ring if every homogeneous element of $R$ is either a zero divisor or a unit. In fact, we proved that $R$ is a $huz$-ring if and only if every graded ideal of $R$ is a graded $r$-ideal. Moreover, assuming that $R$ is a graded domain, we proved that $\{0\}$ is the only graded $r$-ideal of $R$.

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1. Introduction

Let $G$ be a group with identity $e$. A ring $R$ is said to be a $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of $R_g$ are called homogeneous of degree $g$ and $R_e$ (the identity component of $R$) is a subring of $R$ and $1 \in R_e$. For $x \in R$, $x$ can be written uniquely as $\sum x_g$ where $x_g$ is the component of $x$ in $R_g$. Also we write $h(R) = \bigcup_{g \in G} R_g$ and $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. For more details, see [3].

Let $R$ be a $G$-graded ring and $P$ be an ideal of $R$. Then $P$ is called a $G$-graded ideal if $P = \bigoplus_{g \in G} (P \cap R_g)$, i.e., if $x \in P$ and $x = \sum_{g \in G} x_g$, then $x_g \in P$ for all $g \in G$. An ideal of a graded ring need not be graded; see the following example.

Example 1.1. Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then $R$ is $G$-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Now, $P = (1 + i)$ is an ideal of $R$ with $1 + i \in P$. If $P$ is a graded ideal, then $1 \in P$, so $1 = a(1 + i)$ for some $a \in R$, i.e., $1 = (x + iy)(1 + i)$ for some $x, y \in \mathbb{Z}$. Thus $1 = x - y$ and $0 = x + y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, $P$ is not graded ideal.

Throughout this article, $R$ will be a commutative ring with nonzero unity 1. For $a \in R$, we define $\text{Ann}(a) = \{r \in R : ra = 0\}$. An element $a \in R$ is said to be a regular element if $\text{Ann}(a) = \{0\}$, the set of all regular elements of $R$ is denoted by $r(R)$. If $A$ is a subset of $R$ and $P$ is an ideal of $R$, then we define $(P : A) = \{r \in R : rA \subseteq P\}$.

The notion of $r$-ideals was introduced and studied by Rostam Mohamadian in [2]. A proper ideal $P$ of $R$ is said to be an $r$-ideal (resp. $pr$-ideal) if whenever $a, b \in R$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbb{N}$).

In this article, we introduce the concept of graded $r$-ideals. A proper graded ideal $P$ of a graded ring $R$ is said to be a graded $r$-ideal (resp. graded $pr$-ideal) if whenever $a, b \in h(R)$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbb{N}$). We study and investigate the behavior of graded $r$-ideals to introduce several results.

We introduce several characterizations for graded $r$-ideals; we prove that $P$ is a graded $r$-ideal of $R$ if and only if $a P = a R \cap P$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$. Also, $P$ is a graded $r$-ideal of $R$ if and only if $P = (P : a)$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$. Moreover, $P$ is a graded $r$-ideal of $R$ if and only if whenever $A, B$ are graded ideals of $R$ such that $A B \subseteq P$ and $A \cap r(h(R)) \neq \phi$, then $B \subseteq P$. 

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A proper graded ideal of a graded ring $R$ is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in P$, then either $a \in P$ or $b \in P$ ([1]). We prove that the intersection of two graded $r$-ideals is a graded $r$-ideal. On the other hand, if the intersection of two non-comparable graded prime ideals is a graded $r$-ideal, then both ideals are graded $r$-ideals. Moreover, we prove that every graded maximal $r$-ideal is graded prime.

If $P$ is a graded $r$-ideal of $R$, we prove that $P_x$ is an $r$-ideal of $R_x$ and $(P : a)_{\mathbb{Z}}$ is a graded $r$-ideal of $R$ for all $a \in h(R) - P$. Also, we prove that if $R$ is $\mathbb{Z}$-graded, then $P$ is a graded $pr$-ideal of $R$ if and only if $\sqrt{P}$ is a graded $r$-ideal of $R$.

In this article, we introduce the concept of $huz$-rings. A graded ring $R$ is said to be a $huz$-ring if every homogeneous element of $R$ is either a zero divisor or a unit. In fact, we prove that $R$ is a $huz$-ring if and only if every graded ideal of $R$ is a graded $r$-ideal. Moreover, assuming that $R$ is a graded domain, we prove that $\{0\}$ is the only graded $r$-ideal of $R$.

2. Graded $r$-Ideals

In this section, we introduce and study the concept of graded $r$-ideals.

**Definition 2.1.** Let $R$ be a $G$-graded ring. A proper graded ideal $P$ of $R$ is said to be a graded $r$-ideal (resp. graded $pr$-ideal) if whenever $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbb{N}$).

Note that for a graded ideal $P$ of a $G$-graded ring $R$, $P_g = P \cap R_g$ for $g \in G$.

**Theorem 2.2.** Let $R$ be a $G$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded $r$-ideal if and only if $aP = aR \cap P$ for every $a \in h(R)$ with $Ann(a) = \{0\}$.

**Proof.** ($\Rightarrow$) Let $a \in h(R)$ such that $Ann(a) = \{0\}$. Then $aP \subseteq P$ and $aP \subseteq aR$, i.e., $aP \subseteq aR \cap P$. Let $x \in aR \cap P$. Then $x = az \in P$ for some $z \in R$. Since $R$ is $G$-graded, $z = \sum_{g \in G} z_g$ and then $x = \sum_{g \in G} az_g \in P$ and since $P$ is a graded ideal, $az_g \in P$ for all $g \in G$. Since $P$ is a graded $r$-ideal, $z_g \in P$ for all $g \in G$ and then $z = \sum_{g \in G} z_g \in P$ which implies that $x = az \in aP$. Hence, $aP = aR \cap P$. ($\Leftarrow$) Let $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Then $ab \in aR \cap P = aP$ and then $ab = ax$ for some $x \in P$ which implies that $a(b - x) = 0$. Since $Ann(a) = \{0\}$, $b - x = 0$, i.e., $b = x \in P$. Hence, $P$ is a graded $r$-ideal. \[\square\]

**Theorem 2.3.** Let $R$ be a $G$-graded ring and $P$ be a graded ideal of $R$. If $aP_g = aR_h \cap P_g$ for all $g, h \in G$ and for all $a \in h(R)$ with $Ann(a) = \{0\}$, then $P$ is a graded $r$-ideal of $R$. 
Proof. Let \(a, b \in h(R)\) such that \(ab \in P\) and \(\text{Ann}(a) = \{0\}\). Then there exist \(g, h \in G\) such that \(a \in R_g\) and \(b \in R_h\) and then \(ab \in R_gR_h \subseteq R_{gh} \cap P = P_{gh}\). Now, \(ab \in aR_h \cap P_{gh} = aP_{gh}\), i.e., \(ab = ay\) for some \(y \in P_{gh}\) and then \(a(b - y) = 0\). Since \(\text{Ann}(a) = \{0\}\), \(b = y \in P_{gh} \subseteq P\). Hence, \(P\) is a graded r-ideal of \(R\). □

Theorem 2.4. Let \(R\) be a \(G\)-graded ring and \(P\) be a graded ideal of \(R\). Then \(P\) is a graded r-ideal if and only if \(P = (P : a)\) for all \(a \in h(R)\) with \(\text{Ann}(a) = \{0\}\).

Proof. Suppose that \(P\) is a graded r-ideal of \(R\). Let \(a \in h(R)\) with \(\text{Ann}(a) = \{0\}\). Clearly, \(P \subseteq (P : a)\). Let \(y \in (P : a)\). Then \(ya \in P\). Since \(R\) is \(G\)-graded, \(y = \sum_{g \in G} y_g\) and then \(ya = \sum_{g \in G} y_g a \in P\) and since \(P\) is graded, \(y_g a \in P\) for all \(g \in G\). Since \(P\) is a graded r-ideal, \(y_g \in P\) for all \(g \in G\) and then \(y = \sum_{g \in G} y_g \in P\). Hence, \(P = (P : a)\). Conversely, let \(a, b \in h(R)\) such that \(ab \in P\) and \(\text{Ann}(a) = \{0\}\). Then \(b \in (P : a)\). Hence, \(P\) is a graded r-ideal of \(R\). □

Theorem 2.5. Let \(R\) be a \(G\)-graded ring and \(P\) be a graded ideal of \(R\). If \(P_g = (P :_{R_g} a)\) for all \(g, h \in G\) and for all \(a \in h(R)\) with \(\text{Ann}(a) = \{0\}\), then \(P\) is a graded r-ideal of \(R\).

Proof. Let \(a, b \in h(R)\) such that \(ab \in P\) and \(\text{Ann}(a) = \{0\}\). Then \(a \in R_g\) and \(b \in R_h\) for some \(g, h \in G\) and then \(ab \in R_gR_h \subseteq R_{gh} \cap P = P_{gh}\), i.e., \(b \in (P_{gh} :_{R_h} a) = P_{gh} \subseteq P\). Hence, \(P\) is a graded r-ideal of \(R\). □

Theorem 2.6. Let \(R\) be a \(G\)-graded ring and \(P\) be a graded ideal of \(R\). Then \(P\) is a graded r-ideal if and only if whenever \(A, B\) are graded ideals of \(R\) such that \(AB \subseteq P\) and \(A \cap r(h(R)) \neq \phi\), then \(B \subseteq P\).

Proof. Suppose that \(P\) is a graded r-ideal of \(R\). Let \(A, B\) be two graded ideals of \(R\) such that \(AB \subseteq P\) and \(A \cap r(h(R)) \neq \phi\). Since \(A \cap r(h(R)) \neq \phi\), there exists \(a \in A \cap r(h(R))\). Let \(g \in G\) and \(b \in B_g\). Then \(ab \in AB_g \subseteq AB \subseteq P\). Since \(P\) is a graded r-ideal, \(b \in P\). So, \(B_g \subseteq P\) for all \(g \in G\) which implies that \(B \subseteq P\). Conversely, let \(a, b \in h(R)\) such that \(ab \in P\) and \(\text{Ann}(a) = \{0\}\). Then \(A = \langle a \rangle\) and \(B = \langle b \rangle\) are graded ideals of \(R\) such that \(AB \subseteq P\) and \(a \in A \cap r(h(R))\). By assumption, \(B \subseteq P\) and then \(b \in P\). Hence, \(P\) is a graded r-ideal of \(R\). □

Theorem 2.7. If \(R\) is a \(G\)-graded domain, then \(\{0\}\) is a unique graded r-ideal of \(R\).

Proof. Let \(P\) be a nonzero proper graded ideal of \(R\). Then there exists \(0 \neq a = \sum_{g \in G} a_g \in P\) and then \(a_g \in P\) for all \(g \in G\) since \(P\) is graded. Since \(R\) is
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a domain, \( \text{Ann}(a_g) = \{0\} \) with \( 1.a_g \in P \). If \( P \) is a graded r-ideal, then \( 1 \in P \) which is a contradiction. Hence, \( \{0\} \) is the only graded r-ideal of \( R \). \( \square \)

**Lemma 2.8.** If \( R \) is a \( G \)-graded ring, then \( R_e \) contains all homogeneous idempotent elements of \( R \).

*Proof.* Let \( 0 \neq x \in h(R) \) be an idempotent. Then \( x \in R_g \) for some \( g \in G \) and then \( x = x^2 \in R_g \cap R_g^2 \). Since \( 0 \neq x \in R_g \cap R_g^2 \), \( g^2 = g(\in G) \) which implies that \( g = e \). Hence, \( x \in R_e \). \( \square \)

**Theorem 2.9.** Let \( R \) be a \( G \)-graded ring. Suppose that \( \{x_i : i \in \Gamma\} \) is a set of homogeneous idempotent elements in \( R_e \). Then \( P = \sum_{i \in \Gamma} R_e x_i \) is an r-ideal of \( R_e \).

*Proof.* Let \( a, b \in R_e \) such that \( ab \in P \) and \( \text{Ann}(a) = \{0\} \). Let \( z = \prod_{k=1}^{n} (1 - x_{i_k}) \)

where \( ab = \sum_{j=1}^{n} r_j x_j \) for some \( r_1 \ldots r_n \in R_e \). Then \( abz = 0 \). Since \( \text{Ann}(a) = \{0\} \), \( bz = 0 \). On the other hand, there exists \( r \in P \) such that \( z = 1 - r \) and then \( b(1 - r) = 0 \) which implies that \( b = br \in P \). Hence, \( P \) is an r-ideal of \( R_e \). \( \square \)

The next lemma is well known and clear; so we omit the proof.

**Lemma 2.10.** If \( P_1 \) and \( P_2 \) are graded ideals of a graded ring \( R \), then \( P_1 \cap P_2 \) is a graded ideal of \( R \).

**Theorem 2.11.** Let \( R \) be a \( G \)-graded ring. If \( P_1 \) and \( P_2 \) are graded r-ideals of \( R \), then \( P_1 \cap P_2 \) is a graded r-ideal of \( R \).

*Proof.* By Lemma 2.10, \( P_1 \cap P_2 \) is a graded ideal of \( R \). Let \( a, b \in h(R) \) such that \( ab \in P_1 \cap P_2 \) and \( \text{Ann}(a) = \{0\} \). Then \( ab \in P_1 \). Since \( P_1 \) is a graded r-ideal, \( b \in P_1 \). Similarly, \( b \in P_2 \) and hence \( b \in P_1 \cap P_2 \). Therefore, \( P_1 \cap P_2 \) is a graded r-ideal of \( R \). \( \square \)

**Theorem 2.12.** Let \( R \) be a \( G \)-graded ring and \( P_1, P_2 \) be graded prime ideals of \( R \) which are not comparable. If \( P_1 \cap P_2 \) is a graded r-ideal of \( R \), then \( P_1 \) and \( P_2 \) are graded r-ideals of \( R \).

*Proof.* Let \( a, b \in h(R) \) such that \( ab \in P_1 \) and \( \text{Ann}(a) = \{0\} \). Suppose that \( y \in P_2 - P_1 \). Then \( aby \in P_1 \cap P_2 \). Since \( P_1 \cap P_2 \) is graded r-ideal, \( by \in P_1 \cap P_2 \) and then \( by \in P_1 \). Since \( P_1 \) is graded prime and \( y \notin P_1 \), \( b \in P_1 \). Hence, \( P_1 \) is a graded r-ideal of \( R \). Similarly, \( P_2 \) is a graded r-ideal of \( R \). \( \square \)

If \( P \) is a graded ideal of a \( G \)-graded ring \( R \), then \( \sqrt{P} \) need not to be a graded ideal of \( R \); see ([4], Exercises 17 and 13 on pp. 127-128). We introduce the following.
Lemma 2.13. If $P$ is a graded ideal of a $\mathbb{Z}$-graded ring $R$, then $\sqrt{P}$ is a graded ideal of $R$.

Proof. Clearly, $\sqrt{P}$ is an ideal of $R$. Let $x \in \sqrt{P}$ and write $x = \sum_{i=1}^{t} x_i$ where $x_i \in R_{n_i}$ and $n_1 < n_2 < \ldots < n_t$. Then $x^k \in P$ for some positive integer $k$. Of course, $x^k = x_1^k + \text{(higher terms)}$ and as $P$ is graded, we should have that $x_1^k \in P$. Thus, $x_1 \in \sqrt{P}$ which implies that $x - x_1 \in \sqrt{P}$. Now, induct on the number of homogeneous components to conclude that $x_i \in \sqrt{P}$ for all $1 \leq i \leq t$. Hence, $\sqrt{P}$ is a graded ideal of $R$. \qed

Theorem 2.14. Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded $pr$-ideal of $R$ if and only if $\sqrt{P}$ is a graded $r$-ideal of $R$.

Proof. Suppose that $P$ is a graded $pr$-ideal of $R$. By Lemma 2.13, $\sqrt{P}$ is a graded ideal of $R$. Let $a, b \in h(R)$ such that $ab \in \sqrt{P}$ and $\text{Ann}(a) = \{0\}$. Then $a^nb^n = (ab)^n \in P$ for some $n \in \mathbb{N}$. Since $a, b \in h(R)$, there exist $g, h \in G$ such that $a \in R_g$ and $b \in R_h$ and then $a^g \in R_{hg}$ and $b^h \in R_{hb}$ which implies that $a^g, b^h \in h(R)$ such that $a^g b^h \in P$. Clearly, $\text{Ann}(a^n) = \{0\}$ and since $P$ is a graded $pr$-ideal, $b^m = (b^n)^m \in P$ for some $m \in \mathbb{N}$ which implies that $b \in \sqrt{P}$. Hence, $\sqrt{P}$ is a graded $r$-ideal of $R$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$. Then $ab \in \sqrt{P}$ and since $\sqrt{P}$ is a graded $r$-ideal, $b \in \sqrt{P}$ which implies that $b^n \in P$ for some $n \in \mathbb{N}$. Hence, $P$ is a graded $pr$-ideal of $R$. \qed

Using Theorem 2.14 and Theorem 2.2, we have the next corollary.

Corollary 2.15. Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded $pr$-ideal if and only if $a\sqrt{P} = aR \cap \sqrt{P}$ for every $a \in h(R)$ with $\text{Ann}(a) = \{0\}$.

Using Theorem 2.14 and Theorem 2.4, we have the next corollary.

Corollary 2.16. Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded $pr$-ideal if and only if $\sqrt{P} = (\sqrt{P} : a)$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$.

Theorem 2.17. If $P$ is a graded $r$-ideal of a $G$-graded ring $R$, then $(P : a)$ is a graded $r$-ideal of $R$ for all $a \in h(R) - P$.

Proof. Let $a \in h(R) - P$. Clearly, $(P : a)$ is an ideal of $R$. Let $x \in (P : a)$. Then $x \in R$ such that $xa \in P$. Since $R$ is graded, $x = \sum_{g \in G} x_g$ where $x_g \in R_g$. Since $a \in h(R)$, $a \in R_h$ for some $h \in G$ and then $x_g a \in R_g R_h \subseteq R_{gh}$, i.e., $x_g a \in h(R)$ for all $g \in G$. Now, $xa = \sum_{g \in G} x_g a \in P$. Since $P$ is a graded, $x_g a \in P$ for all $g \in G$, i.e., $x_g \in (P : a)$ for all $g \in G$. Hence, $(P : a)$ is a graded ideal of $R$. 

Let $b, c \in b(R)$ such that $bc \in (P : a)$ and $Ann(b) = \{0\}$. Then $bca \in P$. Since $P$ is a graded $r$-ideal, $ca \in P$ which implies that $c \in (P : a)$. Therefore, $(P : a)$ is a graded $r$-ideal of $R$. □

**Theorem 2.18.** Every graded maximal $r$-ideal of a graded ring $R$ is graded prime.

*Proof.* Let $P$ be a graded maximal $r$-ideal of $R$. Suppose that $a, b \in h(R)$ such that $ab \in P$ and $a \notin P$. Then by Theorem 2.17, $(P : a)$ is a graded $r$-ideal of $R$. Clearly, $P \subseteq (P : a)$ and $b \in (P : a)$. By maximality of $P$, $P = (P : a)$ and then $b \in P$. Hence, $P$ is a graded prime ideal of $R$. □

**Definition 2.19.** A graded ring $R$ is said to be an huz-ring if every homogeneous element of $R$ is either a zero divisor or a unit.

The next theorem gives an example on huz-rings.

**Theorem 2.20.** Every graded finite ring is an huz-ring.

*Proof.* Let $R$ be a $G$-graded finite ring. Assume that $a \in h(R)$. Then $a \in R_g$ for some $g \in G$. Define $\phi : R_{g-1} \to R_g$ by $\phi(x) = ax$. If $\phi$ is injective, then since $R$ is finite, $\phi$ is surjective and as $1 \in R_e$, $1 = ax$ for some $x \in R_{g-1}$ and then $a$ is a unit. Suppose that $\phi$ is not injective. Then there exist $x, y \in R_{g-1}$ with $x \neq y$ such that $ax = ay$. But then $a(x - y) = 0$ and $x - y \neq 0$, so $a$ is a zero divisor. □

If we drop the finite condition in Theorem 2.20, then the result is not true in general. See the following example.

**Example 2.21.** Let $G = \Z$. Then clearly, the semigroup ring $R[X; \Z]$ is a $\Z$-graded ring. If $R$ is a field, then $R[X; \Z]$ is a huz-ring; and if $R = \Z$, then $R[X; \Z]$ is not a huz-ring.

Finally, we prove that a graded ring $R$ is an huz-ring if and only if every proper graded ideal of $R$ is a graded $r$-ideal.

**Theorem 2.22.** A graded ring $R$ is a huz-ring if and only if every proper graded ideal of $R$ is a graded $r$-ideal.

*Proof.* Suppose that $R$ is a huz-ring. Let $P$ be a proper graded ideal of $R$. Assume that $a, b \in h(R)$ such that $ab \in P$ and $Ann(a) = \{0\}$. Since $Ann(a) = \{0\}$, $a$ is not zero divisor and since $R$ is huz, $a$ is a unit and then $b = a^{-1}(ab) \in P$. Hence, $P$ is a graded $r$-ideal of $R$. Conversely, let $a \in h(R)$ such that $a$ is not a zero divisor. Then $Ann(a) = \{0\}$. Suppose that $P = (a)$. If $P$ is proper, then $P$ is a graded $r$-ideal of $R$ by assumption. Let $b \in h(R)$. Then $ab \in P$ and then $b \in P$ since $P$ is a graded $r$-ideal. So, $h(R) \subseteq P$. Since $1 \in R_e \subseteq h(R)$, $1 \in P$ which is a contradiction. So, $P = R$, then $1 \in P$ and then $1 = xa$ for some $x \in R$ which implies that $a$ is a unit and hence $R$ is an huz-ring. □
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