Graded $r$-Ideals

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**Abstract.** Let $G$ be a group with identity $e$ and $R$ be a commutative $G$-graded ring with nonzero unity $1$. In this article, we introduce the concept of graded $r$-ideals. A proper graded ideal $P$ of a graded ring $R$ is said to be a graded $r$-ideal if whenever $a, b \in h(R)$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$, then $b \in P$. We study and investigate the behavior of graded $r$-ideals to introduce several results. We introduced several characterizations for graded $r$-ideals; we proved that $P$ is a graded $r$-ideal of $R$ if and only if $aP = aR \cap P$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$. Also, $P$ is a graded $r$-ideal of $R$ if and only if $P = (P : a)$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$.

Moreover, $P$ is a graded $r$-ideal of $R$ if and only if whenever $A, B$ are graded ideals of $R$ such that $AB \subseteq P$ and $A \cap (h(R)) \neq \phi$, then $B \subseteq P$.

In this article, we introduce the concept of a $huz$-rings. A graded ring $R$ is said to be a $huz$-ring if every homogeneous element of $R$ is either a zero divisor or a unit. In fact, we proved that $R$ is a $huz$-ring if and only if every graded ideal of $R$ is a graded $r$-ideal. Moreover, assuming that $R$ is a graded domain, we proved that $\{0\}$ is the only graded $r$-ideal of $R$.

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1. Introduction

Let $G$ be a group with identity $e$. A ring $R$ is said to be a $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of $R_g$ are called homogeneous of degree $g$ and $R_e$ (the identity component of $R$) is a subring of $R$ and $1 \in R_e$. For $x \in R$, $x$ can be written uniquely as $\sum_{g \in G} x_g$ where $x_g$ is the component of $x$ in $R_g$. Also we write $h(R) = \bigcup_{g \in G} R_g$ and $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. For more details, see [3].

Let $R$ be a $G$-graded ring and $P$ be an ideal of $R$. Then $P$ is called a $G$-graded ideal if $P = \bigoplus_{g \in G} (P \cap R_g)$, i.e., if $x \in P$ and $x = \sum_{g \in G} x_g$, then $x_g \in P$ for all $g \in G$. An ideal of a graded ring need not be graded; see the following example.

**Example 1.1.** Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then $R$ is $G$-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Now, $P = (1+i)$ is an ideal of $R$ with $1+i \in P$. If $P$ is a graded ideal, then $1 \in P$, so $1 = a(1+i)$ for some $a \in R$, i.e., $1 = (x+iy)(1+i)$ for some $x, y \in \mathbb{Z}$. Thus $1 = x-y$ and $0 = x+y$, i.e., $2x = 1$ and hence $x = \frac{1}{2}$ a contradiction. So, $P$ is not graded ideal.

Throughout this article, $R$ will be a commutative ring with nonzero unity 1. For $a \in R$, we define $\text{Ann}(a) = \{r \in R : ra = 0\}$. An element $a \in R$ is said to be a regular element if $\text{Ann}(a) = \{0\}$, the set of all regular elements of $R$ is denoted by $r(R)$. If $A$ is a subset of $R$ and $P$ is an ideal of $R$, then we define $(P : A) = \{r \in R : rA \subseteq P\}$.

The notion of $r$-ideals was introduced and studied by Rostam Mohamadian in [2]. A proper ideal $P$ of $R$ is said to be an $r$-ideal (resp. pr-ideal) if whenever $a, b \in R$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbb{N}$).

In this article, we introduce the concept of graded $r$-ideals. A proper graded ideal $P$ of a graded ring $R$ is said to be a graded $r$-ideal (resp. graded pr-ideal) if whenever $a, b \in h(R)$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$, then $b \in P$ (resp. $b^n \in P$ for some $n \in \mathbb{N}$). We study and investigate the behavior of graded $r$-ideals to introduce several results.

We introduce several characterizations for graded $r$-ideals; we prove that $P$ is a graded $r$-ideal of $R$ if and only if $aP = aR \cap P$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$. Also, $P$ is a graded $r$-ideal of $R$ if and only if $P = (P : a)$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$. Moreover, $P$ is a graded $r$-ideal of $R$ if and only if whenever $A, B$ are graded ideals of $R$ such that $AB \subseteq P$ and $A \cap r(h(R)) \neq \phi$, then $B \subseteq P$. 

A proper graded ideal of a graded ring \( R \) is said to be graded prime if whenever \( a, b \in h(R) \) such that \( ab \in P \), then either \( a \in P \) or \( b \in P \) ([1]). We prove that the intersection of two graded \( r \)-ideals is a graded \( r \)-ideal. On the other hand, if the intersection of two non-comparable graded prime ideals is a graded \( r \)-ideal, then both ideals are graded \( r \)-ideals. Moreover, we prove that every graded maximal \( r \)-ideal is graded prime.

If \( P \) is a graded \( r \)-ideal of \( R \), we prove that \( P_g \) is an \( r \)-ideal of \( R \) and \( (P : a) \) is a graded \( r \)-ideal of \( R \) for all \( a \in h(R) - P \). Also, we prove that if \( R \) is \( \mathbb{Z} \)-graded, then \( P \) is a graded \( pr \)-ideal of \( R \) if and only if \( \sqrt{P} \) is a graded \( r \)-ideal of \( R \).

In this article, we introduce the concept of \( huz \)-rings. A graded ring \( R \) is said to be a \( huz \)-ring if every homogeneous element of \( R \) is either a zero divisor or a unit. In fact, we prove that \( R \) is a \( huz \)-ring if and only if every graded ideal of \( R \) is a graded \( r \)-ideal. Moreover, assuming that \( R \) is a graded domain, we prove that \( \{0\} \) is the only graded \( r \)-ideal of \( R \).

2. Graded \( r \)-Ideals

In this section, we introduce and study the concept of graded \( r \)-ideals.

**Definition 2.1.** Let \( R \) be a \( G \)-graded ring. A proper graded ideal \( P \) of \( R \) is said to be a graded \( r \)-ideal (resp. graded \( pr \)-ideal) if whenever \( a, b \in h(R) \) such that \( ab \in P \) and \( \text{Ann}(a) = \{0\} \), then \( b \in P \) (resp. \( b^n \in P \) for some \( n \in \mathbb{N} \)).

Note that for a graded ideal \( P \) of a \( G \)-graded ring \( R \), \( P_g = P \cap R_g \) for \( g \in G \).

**Theorem 2.2.** Let \( R \) be a \( G \)-graded ring and \( P \) be a graded ideal of \( R \). Then \( P \) is a graded \( r \)-ideal if and only if \( aP = aR \cap P \) for every \( a \in h(R) \) with \( \text{Ann}(a) = \{0\} \).

*Proof.* \((\Rightarrow)\) Let \( a \in h(R) \) such that \( \text{Ann}(a) = \{0\} \). Then \( aP \subseteq P \) and \( aP \subseteq aR \), i.e., \( aP \subseteq aR \cap P \). Let \( x \in aR \cap P \). Then \( x = az \in P \) for some \( z \in R \). Since \( R \) is \( G \)-graded, \( z = \sum_{g \in G} z_g \) and then \( x = \sum_{g \in G} az_g \in P \) and since \( P \) is a graded ideal, \( az_g \in P \) for all \( g \in G \). Since \( P \) is a graded \( r \)-ideal, \( z_g \in P \) for all \( g \in G \) and then \( z = \sum_{g \in G} z_g \in P \) which implies that \( x = az \in aP \). Hence, \( aP = aR \cap P \). \((\Leftarrow)\) Let \( a, b \in h(R) \) such that \( ab \in P \) and \( \text{Ann}(a) = \{0\} \). Then \( ab \in aR \cap P = aP \) and then \( ab = ax \) for some \( x \in P \) which implies that \( a(b - x) = 0 \). Since \( \text{Ann}(a) = \{0\} \), \( b - x = 0 \), i.e., \( b = x \in P \). Hence, \( P \) is a graded \( r \)-ideal. \( \square \)

**Theorem 2.3.** Let \( R \) be a \( G \)-graded ring and \( P \) be a graded ideal of \( R \). If \( aP_g = aR_h \cap P_g \) for all \( g, h \in G \) and for all \( a \in h(R) \) with \( \text{Ann}(a) = \{0\} \), then \( P \) is a graded \( r \)-ideal of \( R \).
Proof. Let \( a, b \in h(R) \) such that \( ab \in P \) and \( \text{Ann}(a) = \{0\} \). Then there exist \( g, h \in G \) such that \( a \in R_g \) and \( b \in R_h \) and then \( ab \in R_g R_h \cap P \subseteq R_{gh} \cap P = P_{gh} \). Now, \( ab = aR_h \cap P_{gh} = aP_{gh} \), i.e., \( ab = ay \) for some \( y \in P_{gh} \) and then \( a(b - y) = 0 \). Since \( \text{Ann}(a) = \{0\} \), \( b = y \in P_{gh} \subseteq P \). Hence, \( P \) is a graded \( r \)-ideal of \( R \).

\[ \square \]

**Theorem 2.4.** Let \( R \) be a \( G \)-graded ring and \( P \) be a graded ideal of \( R \). Then \( P \) is a graded \( r \)-ideal if and only if \( P = (P : a) \) for all \( a \in h(R) \) with \( \text{Ann}(a) = \{0\} \).

**Proof.** Suppose that \( P \) is a graded \( r \)-ideal of \( R \). Let \( a \in h(R) \) with \( \text{Ann}(a) = \{0\} \). Clearly, \( P \subseteq (P : a) \). Let \( y \in (P : a) \). Then \( ya \in P \). Since \( R \) is \( G \)-graded, \( y = \sum_{g \in G} y_g a \) and then \( ya = \sum_{g \in G} y_g a \in P \) and since \( P \) is graded, \( y_g a \in P \) for all \( g \in G \). Since \( P \) is a graded \( r \)-ideal, \( y_g \in P \) for all \( g \in G \) and then \( y = \sum_{g \in G} y_g \in P \). Hence, \( P = (P : a) \). Conversely, let \( a, b \in h(R) \) such that \( ab \in P \) and \( \text{Ann}(a) = \{0\} \). Then \( b \in (P : a) = P \). Hence, \( P \) is a graded \( r \)-ideal of \( R \).

\[ \square \]

**Theorem 2.5.** Let \( R \) be a \( G \)-graded ring and \( P \) be a graded ideal of \( R \). If \( P_g = (P_g : R_a) \) for all \( g, h \in G \) and for all \( a \in h(R) \) with \( \text{Ann}(a) = \{0\} \), then \( P \) is a graded \( r \)-ideal of \( R \).

**Proof.** Let \( a, b \in h(R) \) such that \( ab \in P \) and \( \text{Ann}(a) = \{0\} \). Then \( a \in R_g \) and \( b \in R_h \) for some \( g, h \in G \) and then \( ab \in R_g R_h \cap P \subseteq R_{gh} \cap P = P_{gh} \), i.e., \( b \in (P_g : R_a) = P_{gh} \subseteq P \). Hence, \( P \) is a graded \( r \)-ideal of \( R \).

\[ \square \]

**Theorem 2.6.** Let \( R \) be a \( G \)-graded ring and \( P \) be a graded ideal of \( R \). Then \( P \) is a graded \( r \)-ideal if and only if whenever \( A, B \) are graded ideals of \( R \) such that \( AB \subseteq P \) and \( A \cap r(h(R)) \neq \phi \), then \( B \subseteq P \).

**Proof.** Suppose that \( P \) is a graded \( r \)-ideal of \( R \). Let \( A, B \) be two graded ideals of \( R \) such that \( AB \subseteq P \) and \( A \cap r(h(R)) \neq \phi \). Since \( A \cap r(h(R)) \neq \phi \), there exists \( a \in A \cap r(h(R)) \). Let \( g \in G \) and \( b \in B_g \). Then \( ab \in AB_g \subseteq AB \subseteq P \). Since \( P \) is a graded \( r \)-ideal, \( b \in P \). So, \( B_g \subseteq P \) for all \( g \in G \) which implies that \( B \subseteq P \). Conversely, let \( a, b \in h(R) \) such that \( ab \in P \) and \( \text{Ann}(a) = \{0\} \). Then \( A = \langle a \rangle \) and \( B = \langle b \rangle \) are graded ideals of \( R \) such that \( AB \subseteq P \) and \( a \in A \cap r(h(R)) \). By assumption, \( B \subseteq P \) and then \( b \in P \). Hence, \( P \) is a graded \( r \)-ideal of \( R \).

\[ \square \]

**Theorem 2.7.** If \( R \) is a \( G \)-graded domain, then \( \{0\} \) is a unique graded \( r \)-ideal of \( R \).

**Proof.** Let \( P \) be a nonzero proper graded ideal of \( R \). Then there exists \( 0 \neq a = \sum_{g \in G} a_g \in P \) and then \( a_g \in P \) for all \( g \in G \) since \( P \) is graded. Since \( R \) is...
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a domain, $\text{Ann}(a_g) = \{0\}$ with $1.a_g \in P$. If $P$ is a graded $r$-ideal, then $1 \in P$ which is a contradiction. Hence, $\{0\}$ is the only graded $r$-ideal of $R$. □

Lemma 2.8. If $R$ is a $G$-graded ring, then $R_e$ contains all homogeneous idempotent elements of $R$.

Proof. Let $0 \neq x \in h(R)$ be an idempotent. Then $x \in R_g$ for some $g \in G$ and then $x = x^2 \in R_g \cap R_{g^2}$. Since $0 \neq x \in R_g \cap R_{g^2}$, $g^2 = g(\in G)$ which implies that $g = e$. Hence, $x \in R_e$. □

Theorem 2.9. Let $R$ be a $G$-graded ring. Suppose that $\{x_i : i \in \Gamma\}$ is a set of homogeneous idempotent elements in $R_e$. Then $P = \sum_{i \in \Gamma} R_e x_i$ is an $r$-ideal of $R_e$.

Proof. Let $a, b \in R_e$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$. Let $z = \prod_{k=1}^{n} (1 - x_{i_k})$ where $ab = \sum_{j=1}^{n} r_jx_{i_j}$ for some $r_1 \ldots r_n \in R_e$. Then $abz = 0$. Since $\text{Ann}(a) = \{0\}$, $bz = 0$. On the other hand, there exists $r \in P$ such that $z = 1 - r$ and then $b(1 - r) = 0$ which implies that $b = br \in P$. Hence, $P$ is an $r$-ideal of $R_e$. □

The next lemma is well known and clear; so we omit the proof.

Lemma 2.10. If $P_1$ and $P_2$ are graded ideals of a graded ring $R$, then $P_1 \cap P_2$ is a graded ideal of $R$.

Theorem 2.11. Let $R$ be a $G$-graded ring. If $P_1$ and $P_2$ are graded $r$-ideals of $R$, then $P_1 \cap P_2$ is a graded $r$-ideal of $R$.

Proof. By Lemma 2.10, $P_1 \cap P_2$ is a graded ideal of $R$. Let $a, b \in h(R)$ such that $ab \in P_1 \cap P_2$ and $\text{Ann}(a) = \{0\}$. Then $ab \in P_1$. Since $P_1$ is a graded $r$-ideal, $b \in P_1$. Similarly, $b \in P_2$ and hence $b \in P_1 \cap P_2$. Therefore, $P_1 \cap P_2$ is a graded $r$-ideal of $R$. □

Theorem 2.12. Let $R$ be a $G$-graded ring and $P_1, P_2$ be graded prime ideals of $R$ which are not comparable. If $P_1 \cap P_2$ is a graded $r$-ideal of $R$, then $P_1$ and $P_2$ are graded $r$-ideals of $R$.

Proof. Let $a, b \in h(R)$ such that $ab \in P_1$ and $\text{Ann}(a) = \{0\}$. Suppose that $y \in P_2 - P_1$. Then $aby \in P_1 \cap P_2$. Since $P_1 \cap P_2$ is graded $r$-ideal, $by \in P_1 \cap P_2$ and then $by \in P_1$. Since $P_1$ is graded prime and $y \notin P_1$, $b \in P_1$. Hence, $P_1$ is a graded $r$-ideal of $R$. Similarly, $P_2$ is a graded $r$-ideal of $R$. □

If $P$ is a graded ideal of a $G$-graded ring $R$, then $\sqrt{P}$ need not to be a graded ideal of $R$; see ([4], Exercises 17 and 13 on pp. 127-128). We introduce the following.
Lemma 2.13. If $P$ is a graded ideal of a $\mathbb{Z}$-graded ring $R$, then $\sqrt{P}$ is a graded ideal of $R$.

Proof. Clearly, $\sqrt{P}$ is an ideal of $R$. Let $x \in \sqrt{P}$ and write $x = \sum_{i=1}^{t} x_i$ where $x_i \in R_{n_i}$ and $n_1 < n_2 < \ldots < n_t$. Then $x^k \in P$ for some positive integer $k$. Of course, $x^k = x_1^k + \text{(higher terms)}$ and as $P$ is graded, we should have that $x_1^k \in P$. Thus, $x_1 \in \sqrt{P}$ which implies that $x - x_1 \in \sqrt{P}$. Now, induct on the number of homogeneous components to conclude that $x_i \in \sqrt{P}$ for all $1 \leq i \leq t$. Hence, $\sqrt{P}$ is a graded ideal of $R$. □

Theorem 2.14. Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded pr-ideal of $R$ if and only if $\sqrt{P}$ is a graded r-ideal of $R$.

Proof. Suppose that $P$ is a graded pr-ideal of $R$. By Lemma 2.13, $\sqrt{P}$ is a graded ideal of $R$. Let $a, b \in h(R)$ such that $ab \in \sqrt{P}$ and $\text{Ann}(a) = \{0\}$. Then $a^n b^m = (ab)^n \in P$ for some $n \in \mathbb{N}$. Since $a, b \in h(R)$, there exist $g, h \in G$ such that $a \in R_g$ and $b \in R_h$ and then $a^n \in R_{ng}$ and $b^m \in R_{mh}$ which implies that $a^n b^m \in h(R)$ such that $a^n b^m \in P$. Clearly, $\text{Ann}(a^n) = \{0\}$ and since $P$ is a graded pr-ideal, $b^{nm} = (b^n)^m \in P$ for some $m \in \mathbb{N}$ which implies that $b \in \sqrt{P}$. Hence, $\sqrt{P}$ is a graded r-ideal of $R$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ and $\text{Ann}(a) = \{0\}$. Then $ab \in \sqrt{P}$ and since $\sqrt{P}$ is a graded r-ideal, $b \in \sqrt{P}$ which implies that $b^n \in P$ for some $n \in \mathbb{N}$. Hence, $P$ is a graded pr-ideal of $R$. □

Using Theorem 2.14 and Theorem 2.2, we have the next corollary.

Corollary 2.15. Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded pr-ideal if and only if $a\sqrt{P} = aR \cap \sqrt{P}$ for every $a \in h(R)$ with $\text{Ann}(a) = \{0\}$.

Using Theorem 2.14 and Theorem 2.4, we have the next corollary.

Corollary 2.16. Let $R$ be a $\mathbb{Z}$-graded ring and $P$ be a graded ideal of $R$. Then $P$ is a graded pr-ideal if and only if $\sqrt{P} = (\sqrt{P} : a)$ for all $a \in h(R)$ with $\text{Ann}(a) = \{0\}$.

Theorem 2.17. If $P$ is a graded r-ideal of a $G$-graded ring $R$, then $(P : a)$ is a graded r-ideal of $R$ for all $a \in h(R)$.}

Proof. Let $a \in h(R) - P$. Clearly, $(P : a)$ is an ideal of $R$. Let $x \in (P : a)$. Then $x \in R$ such that $xa \in P$. Since $R$ is graded, $x = \sum_{g \in G} x_g$ where $x_g \in R_g$. Since $a \in h(R)$, $a \in R_h$ for some $h \in G$ and then $x_g a \in R_g R_h \subseteq R_{gh}$, i.e., $x_g a \in h(R)$ for all $g \in G$. Now, $xa = \sum_{g \in G} x_g a \in P$. Since $P$ is a graded, $x_g a \in P$ for all $g \in G$, i.e., $x_g \in (P : a)$ for all $g \in G$. Hence, $(P : a)$ is a graded ideal of $R$.
Let \( b, c \in h(R) \) such that \( bc \in (P : a) \) and \( \text{Ann}(b) = \{0\} \). Then \( bca \in P \). Since \( P \) is a graded \( r \)-ideal, \( ca \in P \) which implies that \( c \in (P : a) \). Therefore, \( (P : a) \) is a graded \( r \)-ideal of \( R \). □

**Theorem 2.18.** Every graded maximal \( r \)-ideal of a graded ring \( R \) is graded prime.

**Proof.** Let \( P \) be a graded maximal \( r \)-ideal of \( R \). Suppose that \( a, b \in h(R) \) such that \( ab \in P \) and \( a \notin P \). Then by Theorem 2.17, \( (P : a) \) is a graded \( r \)-ideal of \( R \). Clearly, \( P \subseteq (P : a) \) and \( b \in (P : a) \). By maximality of \( P \), \( P = (P : a) \) and then \( b \in P \). Hence, \( P \) is a graded prime ideal of \( R \). □

**Definition 2.19.** A graded ring \( R \) is said to be an \( huz \)-ring if every homogeneous element of \( R \) is either a zero divisor or a unit.

The next theorem gives an example on \( huz \)-rings.

**Theorem 2.20.** Every graded finite ring is an \( huz \)-ring.

**Proof.** Let \( R \) be a \( G \)-graded finite ring. Assume that \( a \in h(R) \). Then \( a \in R_g \) for some \( g \in G \). Define \( \phi : R_{g-1} \to R_e \) by \( \phi(x) = ax \). If \( \phi \) is injective, then since \( R \) is finite, \( \phi \) is surjective and as \( 1 \in R_e \), \( 1 = ax \) for some \( x \in R_{g-1} \) and then \( a \) is a unit. Suppose that \( \phi \) is not injective. Then there exist \( x, y \in R_{g-1} \) with \( x \neq y \) such that \( ax = ay \). But then \( a(x - y) = 0 \) and \( x - y \neq 0 \), so \( a \) is a zero divisor. □

If we drop the finite condition in Theorem 2.20, then the result is not true in general. See the following example.

**Example 2.21.** Let \( G = \mathbb{Z} \). Then clearly, the semigroup ring \( R[X; \mathbb{Z}] \) is a \( \mathbb{Z} \)-graded ring. If \( R \) is a field, then \( R[X; \mathbb{Z}] \) is a \( huz \)-ring; and if \( R = \mathbb{Z} \), then \( R[X; \mathbb{Z}] \) is not a \( huz \)-ring.

Finally, we prove that a graded ring \( R \) is an \( huz \)-ring if and only if every proper graded ideal of \( R \) is a graded \( r \)-ideal.

**Theorem 2.22.** A graded ring \( R \) is a \( huz \)-ring if and only if every proper graded ideal of \( R \) is a graded \( r \)-ideal.

**Proof.** Suppose that \( R \) is an \( huz \)-ring. Let \( P \) be a proper graded ideal of \( R \). Assume that \( a, b \in h(R) \) such that \( ab \in P \) and \( \text{Ann}(a) = \{0\} \). Since \( \text{Ann}(a) = \{0\} \), \( a \) is not zero divisor and since \( R \) is \( huz \), \( a \) is a unit and then \( b = a^{-1}(ab) \in P \). Hence, \( P \) is a graded \( r \)-ideal of \( R \). Conversely, let \( a \in h(R) \) such that \( a \) is not a zero divisor. Then \( \text{Ann}(a) = \{0\} \). Suppose that \( P = (a) \). If \( P \) is proper, then \( P \) is a graded \( r \)-ideal of \( R \) by assumption. Let \( b \in h(R) \). Then \( ab \in P \) and then \( b \in P \) since \( P \) is a graded \( r \)-ideal. So, \( h(R) \subseteq P \). Since \( 1 \in R_e \subseteq h(R) \), \( 1 \in P \) which is a contradiction. So, \( P = R \), then \( 1 \in P \) and then \( 1 = xa \) for some \( x \in R \) which implies that \( a \) is a unit and hence \( R \) is an \( huz \)-ring. □
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References