Some Generalizations of Locally Closed Sets

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Abstract. Arenas et al. \cite{1} introduced the notion of \(\lambda\)-closed sets as a generalization of locally closed sets. In this paper, we introduce the notions of \(\lambda\)-locally closed sets, \(\Lambda\)-closed sets and \(\lambda\)\(g\)-closed sets and obtain some decompositions of closed sets and continuity in topological spaces.

Keywords: \(\lambda\)-Open set, \(\lambda\)-Locally closed set, \(\Lambda\)-Closed set, \(\lambda\)\(g\)-Closed set, Decompositions of continuity.

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1. Introduction and Preliminaries

The study of locally closed sets was introduced by Bourbaki \cite{3} in 1966 then the authors Ganster and Reilly \cite{6} have studied it extensively. A subset \(A\) of a topological space \(X\) is called locally closed if \(A = U \cap F\), where \(U\) is open and \(F\) is closed. It is interesting that a locally closed set is a generalization of both open sets and closed sets. The generalization has also been discussed in completely regular Hausdorff spaces \cite{5} and has also been done on algebra with topology in \cite{12} and \cite{2}.

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In this paper we consider a new type of sets in the topological space which is called \( \lambda \)-open sets. A set is said to be \( \lambda \)-open if it contains a nonempty open set. This idea is not a new idea. In literature, semi-open sets [7] and \( \alpha \)-sets [11] are examples of that type of sets although preopen sets [10] is not an example of it. Because: let \( \mathbb{R} \) be the usual real line and \( Q \) the rational numbers. Then \( \text{Cl}(Q) = \mathbb{R} \) and \( Q \subseteq \text{Int}(\text{Cl}(Q)) = \mathbb{R} \) (where ‘Cl’ and ‘Int’ denote the closure and interior operators, respectively). But \( Q \) does not contain nonempty open set. However Dontechev [4] has introduced an \( S \)-space: A topological space \( X \) is called an \( S \)-space if every subset which contains a non-void open subset is open. But the concept of \( \lambda \)-open sets is different from Dontechev’s \( S \)-spaces.

**Definition 1.1.** A subset \( A \) of a topological space \( X \) is said to be \( \lambda \)-open if \( A \) contains a nonempty open set. The complement of a \( \lambda \)-open set is said to be \( \lambda \)-closed.

For a subset \( A \) of a topological space \( X \), \( \text{Int}_\lambda(A) \) and \( \text{Cl}_\lambda(A) \) are defined as follows:

**Definition 1.2.** Let \( X \) be a topological space and \( A \) be a subset of \( X \).

\[
\text{Int}_\lambda(A) = \bigcup\{U : U \subseteq A, U \text{ is } \lambda\text{-open in } X\};
\]

\[
\text{Cl}_\lambda(A) = \bigcap\{F : A \subseteq F, F \text{ is } \lambda\text{-closed in } X\}.
\]

**Lemma 1.3.** Let \( X \) be a topological space and \( A, B \) subsets of \( X \).

1. if \( A \subseteq B \), then \( \text{Int}_\lambda(A) \subseteq \text{Int}_\lambda(B) \) and \( \text{Cl}_\lambda(A) \subseteq \text{Cl}_\lambda(B) \),
2. \( X \setminus \text{Int}_\lambda(A) = \text{Cl}_\lambda(X \setminus A) \),
3. For any index set \( \Delta \), if \( A_\alpha \) is \( \lambda \)-open (resp. \( \lambda \)-closed), then \( \bigcup\{A_\alpha : \alpha \in \Delta\} \) is \( \lambda \)-open (resp. \( \bigcap\{A_\alpha : \alpha \in \Delta\} \) is \( \lambda \)-closed),
4. \( \text{Int}_\lambda(A) \) is \( \lambda \)-open and \( \text{Cl}_\lambda(A) \) is \( \lambda \)-closed.

**Remark 1.4.** The finite intersection of \( \lambda \)-open sets need not be \( \lambda \)-open. Let \( \mathbb{R} \) be the usual real line, \( A = (-1, 0] \) and \( B = [0, 1) \). The \( A \) and \( B \) are \( \lambda \)-open but \( A \cap B = \{0\} \) is not \( \lambda \)-open.

We generalize the locally closed set by using \( \lambda \)-open sets.

2. \( \lambda \)-Locally Closed Sets

**Definition 2.1.** A subset \( A \) of a topological space \( X \) is said to be \( \lambda \)-locally closed if \( A = U \cap F \), where \( U \) is \( \lambda \)-open and \( F \) is closed.

**Corollary 2.2.** Let \( f : X \to Y \) be a continuous function. If \( L \) is a \( \lambda \)-locally closed subset of \( Y \), then \( f^{-1}(L) \) is \( \lambda \)-locally closed in \( X \).

From Definition 1.1 it is obvious that every locally closed set is \( \lambda \)-locally closed. But the converse need not hold in general.

**Example 2.3.** Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}\} \). Then \( C(X) \{\text{all closed sets in } X\} = \{\emptyset, X, \{b, c, d\}\} \). And \( \lambda \)-open sets are: \( \emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}, \{a, c\}, \).
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{a, d}, {a, b, d}, {a, c, d}. Therefore, \( \{d\} = \{a, d\} \cap \{b, c, d\} \) is a \( \lambda \)-locally closed set but it is not a locally closed set in \( X \).

**Remark 2.4.** A subset \( A \) of a topological space \( X \) is \( \lambda \)-locally closed if and only if \( X \setminus A \) is the union of a \( \lambda \)-closed set and an open set.

**Remark 2.5.** For a subset of a topological space, the following hold:

1. Every \( \lambda \)-open set is \( \lambda \)-locally closed.
2. Every closed set is \( \lambda \)-locally closed.

**Theorem 2.6.** For a subset \( A \) of a topological space \( X \), the following are equivalent:

1. \( A \) is \( \lambda \)-locally closed;
2. \( A = U \cap \Cl(A) \) for some \( \lambda \)-open set \( U \);
3. \( A \cup (X \setminus \Cl(A)) \) is \( \lambda \)-open;
4. \( A \subseteq \Int_\lambda[A \cup (X \setminus \Cl(A))] \);
5. \( \Cl(A) \setminus A \) is \( \lambda \)-closed.

**Proof.** (1) \( \Rightarrow \) (2): Suppose \( A \) is \( \lambda \)-locally closed. Then \( A = U \cap F \) where \( U \) is \( \lambda \)-open and \( F \) is closed. Then \( \Cl(A) = \Cl(U \cap F) \subseteq \Cl(F) = F \). Then \( A \subseteq U \cap \Cl(A) \subseteq U \cap F = A \) and hence \( A = U \cap \Cl(A) \).

(2) \( \Rightarrow \) (3): \( X \setminus [A \cup (X \setminus \Cl(A))] = (X \setminus A) \cap \Cl(A) = \Cl(A) \setminus A = \Cl(A) \setminus (U \setminus \Cl(A)) = \Cl(A) \setminus U = \Cl(A) \cap (X \setminus U) \). Since \( U \) is \( \lambda \)-open, \( \Cl(A) \cap (X \setminus U) \) is \( \lambda \)-closed and hence \( A \cup (X \setminus \Cl(A)) \) is \( \lambda \)-open.

(3) \( \Rightarrow \) (4): Since \( A \cup (X \setminus \Cl(A)) \) is a \( \lambda \)-open set containing \( A \), it is obvious that \( A \subseteq \Int_\lambda[A \cup (X \setminus \Cl(A))] \).

(4) \( \Rightarrow \) (1): \( A = A \cap \Cl(A) \subseteq \Int_\lambda[A \cup (X \setminus \Cl(A))] \cap \Cl(A) \subseteq [A \cup (X \setminus \Cl(A))] \cap \Cl(A) = A \cap \Cl(A) = A \). Therefore, \( A = \Int_\lambda[A \cup (X \setminus \Cl(A))] \cap \Cl(A) \) and \( A \) is \( \lambda \)-locally closed.

(3) \( \iff \) (5): It is obvious. 

The union of two \( \lambda \)-locally closed sets need not be \( \lambda \)-locally closed.

**Example 2.7.** Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a, b\}, \{c, d\}\} \). Then \( \Cl(X) = \{\emptyset, X, \{a, d\}\} \) and \( \lambda \)-open sets are: \( \emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c, d\} \). \( \lambda \)-locally closed sets are: \( \emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{c, d\}, \{a\}, \{b\} \). Therefore, \( \{a\} \) and \( \{c\} \) are \( \lambda \)-locally closed sets but their union \( \{a, c\} \) is not a \( \lambda \)-locally closed set.

3. **\( \Lambda_\lambda \)-Closed Sets**

Locally closed sets in a topological space are introduced and investigated in [3] and [6]. As a generalization of locally closed sets, Arenas et al. [1] introduced the notion of \( \lambda \)-closed sets in a topological space. In this section, we introduce the notion of \( \Lambda_\lambda \)-closed sets which is a generalization of \( \lambda \)-closed sets. We obtain some characterizations of \( \Lambda_\lambda \)-closed sets and obtain decompositions of closed sets.
**Definition 3.1.** Let $X$ be a topological space and $A$ a subset of $X$. The subset $\Lambda_\lambda(A)$ is defined as follows: $\Lambda_\lambda(A) = \cap\{U : A \subseteq U, U \text{ is } \lambda\text{-open}\}$.

A subset $A$ is called a $\Lambda_\lambda$-set if $A = \Lambda_\lambda(A)$. If $U$ is open in Definition 3.1, then a $\Lambda_\lambda$-set $A$ is called a $\Lambda$-set [9].

**Lemma 3.2.** For any subsets $A$ and $B$ of a topological space $X$, the following hold:

1. $A \subseteq \Lambda_\lambda(A)$,
2. If $A \subseteq B$, then $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(B)$,
3. $\Lambda_\lambda(\Lambda_\lambda(A)) = \Lambda_\lambda(A)$,
4. $\Lambda_\lambda(\cap_{\alpha \in \Delta} A_\alpha) \subseteq \cap_{\alpha \in \Delta} \Lambda_\lambda(A_\alpha)$ for any index set $\Delta$.

**Lemma 3.3.** For any subset $A$ of a topological space $X$, the following hold:

1. $\Lambda_\lambda(A)$ is a $\Lambda_\lambda$-set,
2. If $A$ is $\lambda$-open, then $A$ is a $\Lambda_\lambda$-set,
3. If $A_\alpha$ is a $\Lambda_\lambda$-set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_\alpha$ is a $\Lambda_\lambda$-set.

**Remark 3.4.** The converse of Lemma 3.3 (2) need not hold as shown by the following example: Let $\mathbb{R}$ be the usual real line and $A = \{0\}$. Then $A$ is a $\Lambda_\lambda$-set but it is not $\lambda$-open. Because $\{0\} \subseteq \Lambda_\lambda(\{0\}) \subseteq (-1,0] \cap [0,1) = \{0\}$ and hence $\Lambda_\lambda(\{0\}) = \{0\}$. Therefore, $A = \{0\}$ is a $\Lambda_\lambda$-set but it is not $\lambda$-open.

**Definition 3.5.** A subset $A$ of a topological space $X$ is said to be $\Lambda_\lambda$-closed (resp. $\lambda$-closed [1]) if $A = L \cap F$, where $L$ is a $\Lambda_\lambda$-set (resp. $\Lambda$-set) and $F$ is a closed set.

**Lemma 3.6.** For a subset of a topological space $X$, the following properties hold:

1. Every $\lambda$-locally closed set is $\Lambda_\lambda$-closed,
2. Every $\lambda$-closed set is $\Lambda_\lambda$-closed.

**Proof.** (1) By Lemma 3.3, every $\lambda$-open set is a $\Lambda_\lambda$-set and (1) holds.

(2) Let $U$ be a $\Lambda$-set. Then,

$$U = \cap\{V : U \subseteq V, V \text{ is open}\} \supseteq \cap\{V : U \subset V, V \text{ is } \lambda\text{-open}\} \supseteq U$$

and hence $U$ is a $\Lambda_\lambda$-set. Therefore, (2) holds. \qed

**Remark 3.7.** By Lemma 3.6, we obtain the following diagram.

<table>
<thead>
<tr>
<th>locally closed</th>
<th>$\Rightarrow$</th>
<th>$\lambda$-locally closed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
<td>$\downarrow$</td>
</tr>
<tr>
<td>$\lambda$-closed</td>
<td>$\Rightarrow$</td>
<td>$\Lambda_\lambda$-closed</td>
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</tbody>
</table>
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Theorem 3.8. For a subset $A$ of a topological space $X$, the following are equivalent:

(1) $A$ is $\Lambda_\lambda$-closed;

(2) $A = U \cap \text{Cl}(A)$ for some $\Lambda_\lambda$-set $U$;

(3) $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$.

Proof. (1) $\Rightarrow$ (2): Let $A$ be a $\Lambda_\lambda$-closed set. Then $A = U \cap F$, where $U$ is a $\Lambda_\lambda$-set and $F$ is a closed set. Thus, we have $A \subseteq U \cap \text{Cl}(A) \subseteq U \cap \text{Cl}(F) = U \cap F = A$. Therefore, $A = U \cap \text{Cl}(A)$.

(2) $\Rightarrow$ (3): Let $A = U \cap \text{Cl}(A)$ for some $\Lambda_\lambda$-set $U$. Since $A \subseteq U$, by Lemma 3.2 $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(U) = U$ and hence $A \subseteq \Lambda_\lambda(A) \cap \text{Cl}(A) \subseteq U \cap \text{Cl}(A) = A$. Therefore, we obtain $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$.

(3) $\Rightarrow$ (1): Let $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$. By Lemma 3.3, $\Lambda_\lambda(A)$ is a $\Lambda_\lambda$-set and $\text{Cl}(A)$ is closed. Therefore, $A$ is $\Lambda_\lambda$-closed. $\square$

Definition 3.9. Let $X$ be a topological space. A subset $A$ of $X$ is said to be $\lambda g$-closed (resp. $g$-closed [8]) if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\lambda$-open (resp. open) set.

Theorem 3.10. For a subset $A$ of a topological space $X$, the following are equivalent:

(1) $A$ is closed;

(2) $A$ is $\lambda$-locally closed and $\lambda g$-closed;

(3) $A$ is $\Lambda_\lambda$-closed and $\lambda g$-closed.

Proof. (1) $\Rightarrow$ (2): Let $A$ be closed in $X$. Since $A = X \cap A$ and $X$ is a $\Lambda_\lambda$-set, $A$ is $\lambda$-locally closed. Let $U$ be any $\lambda$-open set containing $A$. Then $\text{Cl}(A) = A \subseteq U$ and hence $A$ is $\lambda g$-closed.

(2) $\Rightarrow$ (3): By Lemma 3.6, every $\lambda$-locally closed set is $\Lambda_\lambda$-closed.

(3) $\Rightarrow$ (1): Let $A$ be $\Lambda_\lambda$-closed and $\lambda g$-closed. Since $A$ is $\Lambda_\lambda$-closed, $A = P \cap L$, where $P$ is a $\Lambda_\lambda$-set and $L$ is closed in $X$. Let $V$ be any $\lambda$-open set containing $A$. Since $A$ is $\lambda g$-closed, $\text{Cl}(A) \subseteq V$ and hence $\text{Cl}(A) \subseteq \cap \{V : A \subseteq V, V$ is $\lambda$-open $\} = \Lambda_\lambda(A)$. Therefore, $\text{Cl}(A) \subseteq \Lambda_\lambda(A) \subseteq \Lambda_\lambda(P) = P$. On the other hand, $A \subseteq L$ and $\text{Cl}(A) \subseteq \text{Cl}(L) = L$. Therefore, we obtain $\text{Cl}(A) \subseteq P \cap L = A$. Thus $A$ is closed. $\square$

Theorem 3.11. Let $X$ be a topological space. If $A_\alpha$ is a $\Lambda_\lambda$-closed set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_\alpha$ is $\Lambda_\lambda$-closed.

Proof. Let $A_\alpha$ be a $\Lambda_\lambda$-closed set for each $\alpha \in \Delta$. Then $A_\alpha = U_\alpha \cap F_\alpha$, where $U_\alpha$ is a $\Lambda_\lambda$-set and $F_\alpha$ is a closed set for each $\alpha \in \Delta$. By Lemma 3.3, $\cap_{\alpha \in \Delta} U_\alpha$ is a $\Lambda_\lambda$-set, $\cap_{\alpha \in \Delta} F_\alpha$ is closed and $\cap_{\alpha \in \Delta} A_\alpha = (\cap_{\alpha \in \Delta} U_\alpha) \cap (\cap_{\alpha \in \Delta} F_\alpha)$. Therefore, $\cap_{\alpha \in \Delta} A_\alpha$ is $\Lambda_\lambda$-closed. $\square$
4. Decompositions of Continuity

In this section, we obtain the decompositions of continuity.

**Definition 4.1.** A function \( f : X \to Y \) is said to be

1. \( \lambda\)-LC-continuous if \( f^{-1}(V) \) is \( \lambda\)-locally closed in \( X \) for any closed set \( V \) of \( Y \),
2. \( \Lambda\lambda \)-continuous if \( f^{-1}(V) \) is \( \Lambda\lambda \)-closed in \( X \) for any closed set \( V \) of \( Y \),
3. \( \lambda g \)-continuous if \( f^{-1}(V) \) is \( \lambda g \)-closed in \( X \) for any closed set \( V \) of \( Y \).

**Theorem 4.2.** For a function \( f : X \to Y \), the following are equivalent:

1. \( f \) is continuous;
2. \( f \) is \( \lambda\)-LC-continuous and \( \lambda g \)-continuous;
3. \( f \) is \( \Lambda\lambda \)-continuous and \( \lambda g \)-continuous.

**Proof.** This is an immediate consequence of Theorem 3.10 \( \square \)

**Remark 4.3.** The following facts are shown by Examples 4.4 and 4.5 and Remark 4.6:

1. \( \lambda\)-LC-continuity and \( \lambda g \)-continuity are independent of each other,
2. \( \Lambda\lambda \)-continuity and \( \lambda g \)-continuity are independent of each other.

**Example 4.4.** Let \( X = Y = \{a, b, c, d\}, \tau = \sigma = \{\emptyset, X, \{a\}\} \). Then \( C(X) = C(Y) = \{\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\} \) and \( \lambda \)-open sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\). \( \lambda \)-locally closed sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\). Define a function \( f : X \to Y \) by \( f(a) = c, f(b) = b, f(c) = d, f(d) = a \). Then we have the following:

1. Since \( f^{-1}(\{b, c, d\}) = \{a, b, c\} \), then \( f \) is not continuous.
2. Since \( f^{-1}(\{b, c, d\}) = \{a, b, c\} \), then \( f \) is \( \lambda\)-LC-continuous.
3. Since \( Cl(\{a, b, c\}) = Y \) (i.e. \( \{a, b, c\} \) is not \( \lambda g \)-closed), then \( f \) is not \( \lambda g \)-continuous.
4. Since \( \{a, b, c\} \subseteq \bigcap\{U : \{a, b, c\} \subseteq U, U \ is \ \lambda\)-open \} = \{a, b, c\} \) and \( \{a, b, c\} \cap X = \{a, b, c\} \), then \( \{a, b, c\} \) is \( \Lambda\lambda \)-closed. Thus \( f \) is \( \Lambda\lambda \)-continuous.

**Example 4.5.** Let \( X = Y = \{a, b, c, d\}, \tau = \sigma = \{\emptyset, X, \{a\}, \{c\}\} \). Then \( C(X) = C(Y) = \{\emptyset, X, \{a\}, \{c\}\} \) and \( \lambda \)-open sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\). \( \lambda \)-locally closed sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\). Define \( g : X \to Y \) by \( g(a) = c, g(b) = b, g(c) = a, g(d) = d \). Then we have the following:

1. Since \( g^{-1}(\{c, d\}) = \{a\} \), then \( g \) is not a continuous function.
2. Since \( g^{-1}(\{c, d\}) = \{a\} \), it is not a \( \lambda \)-locally closed set in \( X \). Then \( g \) is not a \( \lambda\)-LC-continuous function.
3. Since \( g^{-1}(\{a, b\}) = \{b, c\} \subseteq \bigcap\{U : \{b, c\} \subseteq U, U \ is \ \lambda\)-open in \( X \) =
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\{b, c\} \cap X = \{b, c\} and \(g^{-1}(\{c, d\}) = \{a, d\} = \cap\{U : \{a, d\} \subseteq U, U \text{ is } \lambda\text{-open in } X\}

\{a, d\} \cap X = \{a, d\} are \(\Lambda_{\lambda}\)-closed, then \(\Lambda_{\lambda}\)-continuous.

**Remark 4.6.**

1. If every \(\lambda g\)-continuous function is \(\lambda\)-LC-continuous, then it is continuous from Theorem 4.2. This is not true from Example 4.4(1).

2. If every \(\lambda g\)-continuous function is \(\Lambda_{\lambda}\)-continuous, then it is continuous from Theorem 4.2. This not true from Example 4.5(1).

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**References**