Some Generalizations of Locally Closed Sets

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Abstract. Arenas et al. [1] introduced the notion of \(\lambda\)-closed sets as a generalization of locally closed sets. In this paper, we introduce the notions of \(\lambda\)-locally closed sets, \(\Lambda\)-closed sets and \(\lambda_g\)-closed sets and obtain some decompositions of closed sets and continuity in topological spaces.

Keywords: \(\lambda\)-Open set, \(\lambda\)-Locally closed set, \(\Lambda\)-Closed set, \(\lambda_g\)-Closed set, Decompositions of continuity.

2000 Mathematics subject classification: 54A05, 54C08.

1. Introduction and Preliminaries

The study of locally closed sets was introduced by Bourbaki [3] in 1966 then the authors Ganster and Reilly [6] have studied it extensively. A subset \(A\) of a topological space \(X\) is called locally closed if \(A = U \cap F\), where \(U\) is open and \(F\) is closed. It is interesting that a locally closed set is a generalization of both open sets and closed sets. The generalization has also been discussed in completely regular Hausdorff spaces [5] and has also been done on algebra with topology in [12] and [2].

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Received 23 November 2016; Accepted 29 March 2017
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159
In this paper we consider a new type of sets in the topological space which is called $\lambda$-open sets. A set is said to be $\lambda$-open if it contains a nonempty open set. This idea is not a new idea. In literature, semi-open sets [7] and $\alpha$-sets [11] are examples of that type of sets although preopen sets [10] is not an example of it. Because: let $\mathbb{R}$ be the usual real line and $Q$ the rational numbers. Then $\text{Cl}(Q) = \mathbb{R}$ and $Q \subseteq \text{Int} (\text{Cl}(Q)) = \mathbb{R}$ (where ‘$\text{Cl}$’ and ‘$\text{Int}$’ denote the closure and interior operators, respectively). But $Q$ does not contain nonempty open set. However Dontechev [4] has introduced an $S$-space: A topological space $X$ is called an $S$-space if every subset which contains a non-void open subset is open. But the concept of $\lambda$-open sets is different from Dontechev’s $S$-spaces.

**Definition 1.1.** A subset $A$ of a topological space $X$ is said to be $\lambda$-open if $A$ contains a nonempty open set. The complement of a $\lambda$-open set is said to be $\lambda$-closed.

For a subset $A$ of a topological space $X$, $\text{Int}_{\lambda}(A)$ and $\text{Cl}_{\lambda}(A)$ are defined as follows:

**Definition 1.2.** Let $X$ be a topological space and $A$ be a subset of $X$.

\[
\text{Int}_{\lambda}(A) = \bigcup \{U : U \subseteq A, U \text{ is } \lambda\text{-open in } X\};
\]

\[
\text{Cl}_{\lambda}(A) = \cap \{F : A \subseteq F, F \text{ is } \lambda\text{-closed in } X\}.
\]

**Lemma 1.3.** Let $X$ be a topological space and $A, B$ subsets of $X$.

1. if $A \subseteq B$, then $\text{Int}_{\lambda}(A) \subseteq \text{Int}_{\lambda}(B)$ and $\text{Cl}_{\lambda}(A) \subseteq \text{Cl}_{\lambda}(B)$,

2. $X \setminus \text{Int}_{\lambda}(A) = \text{Cl}_{\lambda}(X \setminus A)$,

3. For any index set $\Delta$, if $A_{\alpha}$ is $\lambda$-open (resp. $\lambda$-closed), then $\bigcup \{A_{\alpha} : \alpha \in \Delta\}$ is $\lambda$-open (resp. $\bigcap \{A_{\alpha} : \alpha \in \Delta\}$ is $\lambda$-closed),

4. $\text{Int}_{\lambda}(A)$ is $\lambda$-open and $\text{Cl}_{\lambda}(A)$ is $\lambda$-closed.

**Remark 1.4.** The finite intersection of $\lambda$-open sets need not be $\lambda$-open. Let $\mathbb{R}$ be the usual real line, $A = (-1, 0]$ and $B = [0, 1)$. The $A$ and $B$ are $\lambda$-open but $A \cap B = \{0\}$ is not $\lambda$-open.

We generalize the locally closed set by using $\lambda$-open sets.

### 2. $\lambda$-Locally Closed Sets

**Definition 2.1.** A subset $A$ of a topological space $X$ is said to be $\lambda$-locally closed if $A = U \cap F$, where $U$ is $\lambda$-open and $F$ is closed.

**Corollary 2.2.** Let $f : X \rightarrow Y$ be a continuous function. If $L$ is a $\lambda$-locally closed subset of $Y$, then $f^{-1}(L)$ is $\lambda$-locally closed in $X$.

From Definition 1.1 it is obvious that every locally closed set is $\lambda$-locally closed. But the converse need not hold in general.

**Example 2.3.** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}\}$. Then $C(X)$(all closed sets in $X$) = $\{\emptyset, X, \{b, c, d\}\}$. And $\lambda$-open sets are: $\emptyset$, $X$, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, c\}$.
\{a, d\}, \{a, b, d\}, \{a, c, d\}. Therefore, \{d\} = \{a, d\} \cap \{b, c, d\} is a \(\lambda\)-locally closed set but it is not a locally closed set in \(X\).

Remark 2.4. A subset \(A\) of a topological space \(X\) is \(\lambda\)-locally closed if and only if \(X \setminus A\) is the union of a \(\lambda\)-closed set and an open set.

Remark 2.5. For a subset of a topological space, the following hold:

(1) Every \(\lambda\)-open set is \(\lambda\)-locally closed,

(2) Every closed set is \(\lambda\)-locally closed.

Theorem 2.6. For a subset \(A\) of a topological space \(X\), the following are equivalent:

(1) \(A\) is \(\lambda\)-locally closed;

(2) \(A = U \cap \text{Cl}(A)\) for some \(\lambda\)-open set \(U\);

(3) \(A \cup (X \setminus \text{Cl}(A))\) is \(\lambda\)-open;

(4) \(A \subseteq \text{Int}_\lambda[A \cup (X \setminus \text{Cl}(A))]\);

(5) \(\text{Cl}(A) \setminus A\) is \(\lambda\)-closed.

Proof. (1) \(\Rightarrow\) (2): Suppose \(A\) is \(\lambda\)-locally closed. Then \(A = U \cap F\) where \(U\) is \(\lambda\)-open and \(F\) is closed. Then \(\text{Cl}(A) = \text{Cl}(U \cap F) \subseteq \text{Cl}(F) = F\). Then \(A \subseteq U \cap \text{Cl}(A) \subseteq U \cap F = A\) and hence \(A = U \cap \text{Cl}(A)\).

(2) \(\Rightarrow\) (3): \(X \setminus [A \cup (X \setminus \text{Cl}(A))] = (X \setminus A) \cap \text{Cl}(A) = \text{Cl}(A) \setminus A = \text{Cl}(A) \setminus (U \cap \text{Cl}(A)) = \text{Cl}(A) \setminus U = \text{Cl}(A) \cap (X \setminus U)\). Since \(U\) is \(\lambda\)-open, \(\text{Cl}(A) \cap (X \setminus U)\) is \(\lambda\)-closed and hence \(A \cup (X \setminus \text{Cl}(A))\) is \(\lambda\)-open.

(3) \(\Rightarrow\) (4): Since \(A \cup (X \setminus \text{Cl}(A))\) is a \(\lambda\)-open set containing \(A\), it is obvious that \(A \subset \text{Int}_\lambda[A \cup (X \setminus \text{Cl}(A))]\).

(4) \(\Rightarrow\) (1): \(A = A \cap \text{Cl}(A) \subseteq \text{Int}_\lambda[A \cup (X \setminus \text{Cl}(A))] \cap \text{Cl}(A) \subseteq [A \cup (X \setminus \text{Cl}(A))] \cap \text{Cl}(A) = A \cap \text{Cl}(A) = A\). Therefore, \(A = \text{Int}_\lambda[A \cup (X \setminus \text{Cl}(A))] \cap \text{Cl}(A)\) and \(A\) is \(\lambda\)-locally closed.

(3) \(\Leftrightarrow\) (5): It is obvious. \(\square\)

The union of two \(\lambda\)-locally closed sets need not be \(\lambda\)-locally closed.

Example 2.7. Let \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}\). Then \(C(X) = \emptyset, X, \{c, d\}, \{a, b\}\) and \(\lambda\)-open sets are: \(\emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\). \(\lambda\)-locally closed sets are: \(\emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{c\}, \{d\}, \{a\}, \{b\}\). Therefore, \(\{a\}\) and \(\{c\}\) are \(\lambda\)-locally closed sets but their union \(\{a, c\}\) is not a \(\lambda\)-locally closed set.

3. \(\Lambda\lambda\)-Closed Sets

Locally closed sets in a topological space are introduced and investigated in [3] and [6]. As a generalization of locally closed sets, Arenas et al. [1] introduced the notion of \(\lambda\)-closed sets in a topological space. In this section, we introduce the notion of \(\Lambda\lambda\)-closed sets which is a generalization of \(\lambda\)-closed sets. We obtain some characterizations of \(\Lambda\lambda\)-closed sets and obtain decompositions of closed sets.
Definition 3.1. Let $X$ be a topological space and $A$ a subset of $X$. The subset $\Lambda_\lambda(A)$ is defined as follows: $\Lambda_\lambda(A) = \cap\{U : A \subseteq U, U \text{ is } \lambda\text{-open}\}$.

A subset $A$ is called a $\Lambda_\lambda$-set if $A = \Lambda_\lambda(A)$. If $U$ is open in Definition 3.1, then a $\Lambda_\lambda$-set $A$ is called a $\Lambda$-set [9].

Lemma 3.2. For any subsets $A$ and $B$ of a topological space $X$, the following hold:

1. $A \subseteq \Lambda_\lambda(A)$,
2. If $A \subseteq B$, then $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(B)$,
3. $\Lambda_\lambda(\Lambda_\lambda(A)) = \Lambda_\lambda(A)$,
4. $\Lambda_\lambda(\cap_{\alpha \in \Delta} A_\alpha) \subseteq \cap_{\alpha \in \Delta} \Lambda_\lambda(A_\alpha)$ for any index set $\Delta$.

Lemma 3.3. For any subset $A$ of a topological space $X$, the following hold:

1. $\Lambda_\lambda(A)$ is a $\Lambda_\lambda$-set,
2. If $A$ is $\lambda$-open, then $A$ is a $\Lambda_\lambda$-set,
3. If $A_\alpha$ is a $\Lambda_\lambda$-set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_\alpha$ is a $\Lambda_\lambda$-set.

Remark 3.4. The converse of Lemma 3.3 (2) need not hold as shown by the following example: Let $\mathbb{R}$ be the usual real line and $A = \{0\}$. Then $A$ is a $\Lambda_\lambda$-set but it is not $\lambda$-open. Because $\{0\} \subseteq \Lambda_\lambda(\{0\}) \subseteq (-1,0] \cap [0,1) = \{0\}$ and hence $\Lambda_\lambda(\{0\}) = \{0\}$. Therefore, $A = \{0\}$ is a $\Lambda_\lambda$-set but it is not $\lambda$-open.

Definition 3.5. A subset $A$ of a topological space $X$ is said to be $\Lambda_\lambda$-closed (resp. $\lambda$-closed [1]) if $A = \Lambda_\lambda(A)$, where $\Lambda_\lambda$ is a $\Lambda_\lambda$-set (resp. $\Lambda$-set) and $F$ is a closed set.

Lemma 3.6. For a subset of a topological space $X$, the following properties hold:

1. Every $\lambda$-locally closed set is $\Lambda_\lambda$-closed,
2. Every $\lambda$-closed set is $\Lambda_\lambda$-closed.

Proof. (1) By Lemma 3.3, every $\lambda$-open set is a $\Lambda_\lambda$-set and (1) holds.

(2) Let $U$ be a $\Lambda$-set. Then,

$$U = \cap\{V : U \subseteq V, V \text{ is open}\} \supseteq \cap\{V : U \subset V, V \text{ is } \lambda\text{-open}\} \supseteq U$$

and hence $U$ is a $\Lambda_\lambda$-set. Therefore, (2) holds. \qed

Remark 3.7. By Lemma 3.6, we obtain the following diagram.

\[\begin{array}{ccc}
\text{locally closed} & \Rightarrow & \lambda\text{-locally closed} \\
\downarrow & & \downarrow \\
\lambda\text{-closed} & \Rightarrow & \Lambda_\lambda\text{-closed}
\end{array}\]
Theorem 3.8. For a subset $A$ of a topological space $X$, the following are equivalent:

1. $A$ is $\Lambda_\lambda$-closed;
2. $A = U \cap \text{Cl}(A)$ for some $\Lambda_\lambda$-set $U$;
3. $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$.

Proof. (1) $\Rightarrow$ (2): Let $A$ be a $\Lambda_\lambda$-closed set. Then $A = U \cap F$, where $U$ is a $\Lambda_\lambda$-set and $F$ is a closed set. Thus, we have $A \subseteq U \cap \text{Cl}(A) \subseteq U \cap \text{Cl}(F) = U \cap F = A$. Therefore, $A = U \cap \text{Cl}(A)$.

(2) $\Rightarrow$ (3): Let $A = U \cap \text{Cl}(A)$ for some $\Lambda_\lambda$-set $U$. Since $A \subseteq U$, by Lemma 3.2 $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(U) = U$ and hence $A \subseteq \Lambda_\lambda(A) \cap \text{Cl}(A) \subseteq U \cap \text{Cl}(A) = A$. Therefore, we obtain $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$.

(3) $\Rightarrow$ (1): Let $A = \Lambda_\lambda(A) \cap \text{Cl}(A)$. By Lemma 3.3, $\Lambda_\lambda(A)$ is a $\Lambda_\lambda$-set and $\text{Cl}(A)$ is closed. Therefore, $A$ is $\Lambda_\lambda$-closed. □

Definition 3.9. Let $X$ be a topological space. A subset $A$ of $X$ is said to be $\lambda g$-closed (resp. $g$-closed [8]) if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\lambda$-open (resp. open) set.

Theorem 3.10. For a subset $A$ of a topological space $X$, the following are equivalent:

1. $A$ is closed;
2. $A$ is $\lambda$-locally closed and $\lambda g$-closed;
3. $A$ is $\Lambda_\lambda$-closed and $\lambda g$-closed.

Proof. (1) $\Rightarrow$ (2): Let $A$ be closed in $X$. Since $A = X \cap A$ and $X$ is a $\Lambda_\lambda$-set, $A$ is $\lambda$-locally closed. Let $U$ be any $\lambda$-open set containing $A$. Then $\text{Cl}(A) = A \subseteq U$ and hence $A$ is $\lambda g$-closed.

(2) $\Rightarrow$ (3): By Lemma 3.6, every $\lambda$-locally closed set is $\Lambda_\lambda$-closed.

(3) $\Rightarrow$ (1): Let $A$ be $\Lambda_\lambda$-closed and $\lambda g$-closed. Since $A$ is $\Lambda_\lambda$-closed, $A = P \cap L$, where $P$ is a $\Lambda_\lambda$-set and $L$ is closed in $X$. Let $V$ be any $\lambda$-open set containing $A$. Since $A$ is $\lambda g$-closed, $\text{Cl}(A) \subseteq V$ and hence $\text{Cl}(A) \subseteq \cap\{V : A \subseteq V, V \text{ is } \lambda\text{-open}\} = \Lambda_\lambda(A)$. Therefore, $\text{Cl}(A) \subseteq \Lambda_\lambda(A) \subseteq \Lambda_\lambda(P) = P$. On the other hand, $A \subseteq L$ and $\text{Cl}(A) \subseteq \text{Cl}(L) = L$. Therefore, we obtain $\text{Cl}(A) \subseteq P \cap L = A$. Thus $A$ is closed. □

Theorem 3.11. Let $X$ be a topological space. If $A_\alpha$ is a $\Lambda_\lambda$-closed set for each $\alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_\alpha$ is $\Lambda_\lambda$-closed.

Proof. Let $A_\alpha$ be a $\Lambda_\lambda$-closed set for each $\alpha \in \Delta$. Then $A_\alpha = U_\alpha \cap F_\alpha$, where $U_\alpha$ is a $\Lambda_\lambda$-set and $F_\alpha$ is a closed set for each $\alpha \in \Delta$. By Lemma 3.3, $\cap_{\alpha \in \Delta} U_\alpha$ is a $\Lambda_\lambda$-set, $\cap_{\alpha \in \Delta} F_\alpha$ is closed and $\cap_{\alpha \in \Delta} A_\alpha = (\cap_{\alpha \in \Delta} U_\alpha) \cap (\cap_{\alpha \in \Delta} F_\alpha)$. Therefore, $\cap_{\alpha \in \Delta} A_\alpha$ is $\Lambda_\lambda$-closed. □
4. Decompositions of Continuity

In this section, we obtain the decompositions of continuity.

**Definition 4.1.** A function \( f : X \to Y \) is said to be

1. \( \lambda\)-LC-continuous if \( f^{-1}(V) \) is \( \lambda\)-locally closed in \( X \) for any closed set \( V \) of \( Y \),
2. \( \Lambda_{\lambda} \)-continuous if \( f^{-1}(V) \) is \( \Lambda_{\lambda} \)-closed in \( X \) for any closed set \( V \) of \( Y \),
3. \( \lambda g \)-continuous if \( f^{-1}(V) \) is \( \lambda g \)-closed in \( X \) for any closed set \( V \) of \( Y \).

**Theorem 4.2.** For a function \( f : X \to Y \), the following are equivalent:

1. \( f \) is continuous;
2. \( f \) is \( \lambda\)-LC-continuous and \( \lambda g \)-continuous;
3. \( f \) is \( \Lambda_{\lambda} \)-continuous and \( \lambda g \)-continuous.

**Proof.** This is an immediate consequence of Theorem 3.10 \( \square \)

**Remark 4.3.** The following facts are shown by Examples 4.4 and 4.5 and Remark 4.6:

1. \( \lambda\)-LC-continuity and \( \lambda g \)-continuity are independent of each other,
2. \( \Lambda_{\lambda} \)-continuity and \( \lambda g \)-continuity are independent of each other.

**Example 4.4.** Let \( X = Y = \{a, b, c, d\} \), \( \tau = \sigma = \{\emptyset, X, \{a\}\} \). Then \( C(X) = C(Y) = \{\emptyset, \{b, c, d\}\} \) and \( \lambda \)-open sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{b, d\} \). \( \lambda \)-locally closed sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{b, d\} \). Define a function \( f : X \to Y \) by \( f(a) = c, f(b) = b, f(c) = d, f(d) = a \). Then we have the following:

1. Since \( f^{-1}(\{b, c, d\}) = \{a, b, c\} \), then \( f \) is not continuous.
2. Since \( f^{-1}(\{b, c, d\}) = \{a, b, c\} \), then \( f \) is \( \lambda\)-LC-continuous.
3. Since \( CL(\{a, b, c\}) = X \) (i.e. \( \{a, b, c\} \) is not \( \lambda g \)-closed), then \( f \) is not \( \lambda g \)-continuous.
4. Since \( \{a, b, c\} \subseteq \cap\{U : \{a, b, c\} \subseteq U, U \text{ is } \lambda \text{-open}\} = \{a, b, c\} \) and \( \{a, b, c\} \cap X = \{a, b, c\} \), then \( \{a, b, c\} \) is \( \Lambda_{\lambda} \)-closed. Thus \( f \) is \( \Lambda_{\lambda} \)-continuous.

**Example 4.5.** Let \( X = Y = \{a, b, c, d\} \), \( \tau = \sigma = \{\emptyset, X, \{a\}, \{c, d\}\} \). Then \( C(X) = C(Y) = \{\emptyset, X, \{a\}, \{c, d\}\} \) and \( \lambda \)-open sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{c, d\}, \{a, b\}, \{a, c, d\}, \{b, c, d\}, \{b, d\} \). And \( \lambda \)-locally closed sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{c, d\}, \{a, b\}, \{a, c, d\}, \{b, c, d\}, \{a\}, \{b\}, \{c\}, \{d\}\). Define \( g : X \to Y \) by \( g(\{a\}) = c, g(\{b\}) = b, g(\{c\}) = a, g(\{d\}) = d \). Then we have the following:

1. Since \( g^{-1}(\{c, d\}) = \{a, d\} \), then \( g \) is not a continuous function.
2. Since \( g^{-1}(\{c, d\}) = \{a, d\} \), it is a \( \lambda \)-locally closed set in \( X \). Then \( g \) is not a \( \lambda\)-LC-continuous function.
3. Since \( g^{-1}(\{a, b\}) = \{b, c\} \subseteq \cap\{U : \{b, c\} \subseteq U, U \text{ is } \lambda \text{-open in } X\} = \)}
Some generalizations of locally closed sets

\{b, c\} \cap X = \{b, c\} and \ g^{-1}(\{c, d\}) = \{a, d\} = \bigcap\{U : \{a, d\} \subseteq U, U \ is \ \lambda\text{-open in } X\}

= \{a, d\} \cap X = \{a, d\} are \Lambda\lambda\text{-closed}, then \Lambda\lambda\text{-continuous.}

Remark 4.6. (1) If every \lambda g\text{-continuous function is } \lambda\text{-LC-continuous, then it is continuous from Theorem 4.2. This is not true from Example 4.4(1).}

(2) If every \lambda g\text{-continuous function is } \Lambda\lambda\text{-continuous, then it is continuous from Theorem 4.2. This not true from Example 4.5(1).}

Acknowledgments

The authors wish to thank the referees for their valuable comments.

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