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Some Generalizations of Locally Closed Sets

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Abstract. Arenas et al. [1] introduced the notion of λ-closed sets as a generalization of locally closed sets. In this paper, we introduce the notions of λ-locally closed sets, Λ_λ-closed sets and λg-closed sets and obtain some decompositions of closed sets and continuity in topological spaces.

Keywords: λ-Open set, λ-Locally closed set, Λ_λ-Closed set, λg-Closed set, Decompositions of continuity.

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1. Introduction and Preliminaries

The study of locally closed sets was introduced by Bourbaki [3] in 1966 then the authors Ganster and Reilly [6] have studied it extensively. A subset A of a topological space X is called locally closed if $A = U \cap F$, where U is open and F is closed. It is interesting that a locally closed set is a generalization of both open sets and closed sets. The generalization has also been discussed in completely regular Hausdorff spaces [5] and has also been done on algebra with topology in [12] and [2].

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In this paper we consider a new type of sets in the topological space which is called λ-open sets. A set is said to be λ-open if it contains a nonempty open set. This idea is not a new idea. In literature, semi-open sets [7] and α-sets [11] are examples of that type of sets although preopen sets [10] is not an example of it. Because: let $\mathbf{R}$ be the usual real line and Q the rational numbers. Then $\mathrm{Cl}(Q) = \mathbf{R}$ and $Q \subseteq \mathrm{Int}(\mathrm{Cl}(Q)) = \mathbf{R}$ (where 'Cl' and 'Int' denote the closure and interior operators, respectively). But Q does not contain nonempty open set. However Dontechev [4] has introduced an S-space: A topological space X is called an S-space if every subset which contains a non-void open subset is open. But the concept of λ-open sets is different from Dontechev's S-spaces.

Definition 1.1. A subset A of a topological space X is said to be λ-*open* if A contains a nonempty open set. The complement of a λ-open set is said to be λ-*closed*.

For a subset A of a topological space X, $\mathrm{Int}_\lambda(A)$ and $\mathrm{Cl}_\lambda(A)$ are defined as follows:

Definition 1.2. Let X be a topological space and A be a subset of X.
$\mathrm{Int}_\lambda(A) = \cup\{U : U \subseteq A, U$ is λ-open in $X\}$;
$\mathrm{Cl}_\lambda(A) = \cap\{F : A \subseteq F, F$ is λ-closed in $X\}$.

Lemma 1.3. *Let X be a topological space and A, B subsets of X.*
(1) if $A \subseteq B$, then $\mathrm{Int}_\lambda(A) \subseteq \mathrm{Int}_\lambda(B)$ *and* $\mathrm{Cl}_\lambda(A) \subseteq \mathrm{Cl}_\lambda(B)$,
(2) $X \setminus \mathrm{Int}_\lambda(A) = \mathrm{Cl}_\lambda(X \setminus A)$,
(3) For any index set Δ, if A_α is λ-open (resp. λ-closed), then $\cup\{A_\alpha : \alpha \in \Delta\}$ is λ-open (resp. $\cap\{A_\alpha : \alpha \in \Delta\}$ is λ-closed),
(4) $\mathrm{Int}_\lambda(A)$ *is λ-open and* $\mathrm{Cl}_\lambda(A)$ *is λ-closed.*

Remark 1.4. The finite intersection of λ-open sets need not be λ-open. Let $\mathbf{R}$ be the usual real line, $A = (-1, 0]$ and $B = [0, 1)$. The A and B are λ-open but $A \cap B = \{0\}$ is not λ-open.

We generalize the locally closed set by using λ-open sets.

2. λ-Locally Closed Sets

Definition 2.1. A subset A of a topological space X is said to be λ-*locally closed* if $A = U \cap F$, where U is λ-open and F is closed.

Corollary 2.2. *Let $f : X \to Y$ be a continuous function. If L is a λ-locally closed subset of Y, then $f^{-1}(L)$ is λ-locally closed in X.*

From Definition 1.1 it is obvious that every locally closed set is λ-locally closed. But the converse need not hold in general.

Example 2.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}\}$. Then $C(X)$(all closed sets in X) $= \{\emptyset, X, \{b, c, d\}\}$. And λ-open sets are: $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, b, c\}$, $\{a, c\}$,

$\{a, d\}$, $\{a, b, d\}, \{a, c, d\}$. Therefore, $\{d\} = \{a, d\} \cap \{b, c, d\}$ is a λ-locally closed set but it is not a locally closed set in X.

Remark 2.4. A subset A of a topological space X is λ-locally closed if and only if $X \setminus A$ is the union of a λ-closed set and an open set.

Remark 2.5. For a subset of a topological space, the following hold:
(1) Every λ-open set is λ-locally closed,
(2) Every closed set is λ-locally closed.

Theorem 2.6. *For a subset A of a topological space X, the following are equivalent:*
(1) A is λ-locally closed;
(2) $A = U \cap \mathrm{Cl}(A)$ for some λ-open set U;
(3) $A \cup (X \setminus \mathrm{Cl}(A))$ is λ-open;
(4) $A \subseteq \mathrm{Int}_\lambda[A \cup (X \setminus \mathrm{Cl}(A))]$;
(5) $\mathrm{Cl}(A) \setminus A$ is λ-closed.

Proof. (1) $\Rightarrow$ (2): Suppose A is λ-locally closed. Then $A = U \cap F$ where U is λ-open and F is closed. Then $\mathrm{Cl}(A) = \mathrm{Cl}(U \cap F) \subseteq \mathrm{Cl}(F) = F$. Then $A \subseteq U \cap \mathrm{Cl}(A) \subseteq U \cap F = A$ and hence $A = U \cap \mathrm{Cl}(A)$.

(2) $\Rightarrow$ (3): $X \setminus [A \cup (X \setminus \mathrm{Cl}(A))] = (X \setminus A) \cap \mathrm{Cl}(A) = \mathrm{Cl}(A) \setminus A = \mathrm{Cl}(A) \setminus (U \cap \mathrm{Cl}(A)) = \mathrm{Cl}(A) \setminus U = \mathrm{Cl}(A) \cap (X \setminus U)$. Since U is λ-open, $\mathrm{Cl}(A) \cap (X \setminus U)$ is λ-closed and hence $A \cup (X \setminus \mathrm{Cl}(A))$ is λ-open.

(3) $\Rightarrow$ (4): Since $A \cup (X \setminus \mathrm{Cl}(A))$ is a λ-open set containing A, it is obvious that $A \subset \mathrm{Int}_\lambda[A \cup (X \setminus \mathrm{Cl}(A))]$.

(4) $\Rightarrow$ (1): $A = A \cap \mathrm{Cl}(A) \subseteq \mathrm{Int}_\lambda[A \cup (X \setminus \mathrm{Cl}(A))] \cap \mathrm{Cl}(A) \subseteq [A \cup (X \setminus \mathrm{Cl}(A))] \cap \mathrm{Cl}(A) = A \cap \mathrm{Cl}(A) = A$. Therefore, $A = \mathrm{Int}_\lambda[A \cup (X \setminus \mathrm{Cl}(A))] \cap \mathrm{Cl}(A)$ and A is λ-locally closed.

(3) $\Leftrightarrow$ (5): It is obvious. □

The union of two λ-locally closed sets need not be λ-locally closed.

Example 2.7. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then $C(X) = \{\emptyset, X, \{c, d\}, \{a, b\}\}$ and λ-open sets are: $\emptyset$, X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. λ-locally closed sets are: $\emptyset$, X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{c\}$, $\{d\}$, $\{a\}$, $\{b\}$. Therefore, $\{a\}$ and $\{c\}$ are λ-locally closed sets but their union $\{a, c\}$ is not a λ-locally closed set.

3. Λ_λ-Closed Sets

Locally closed sets in a topological space are introduced and investigated in [3] and [6]. As a generalization of locally closed sets, Arenas et al. [1] introduced the notion of λ-closed sets in a topological space. In this section, we introduce the notion of Λ_λ-closed sets which is a generalization of λ-closed sets. We obtain some characterizations of Λ_λ-closed sets and obtain decompositions of closed sets.

Definition 3.1. Let X be a topological space and A a subset of X. The subset $\Lambda_\lambda(A)$ is defined as follows: $\Lambda_\lambda(A) = \cap\{U : A \subseteq U,\ U \text{ is } \lambda\text{-open }\}$.

A subset A is called a Λ_λ-*set* if $A = \Lambda_\lambda(A)$. If U is open in Definition 3.1, then a Λ_λ-set A is called a Λ-*set* [9].

Lemma 3.2. *For any subsets A and B of a topological space X, the following hold:*

(1) $A \subseteq \Lambda_\lambda(A)$,
(2) If $A \subseteq B$, then $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(B)$,
(3) $\Lambda_\lambda(\Lambda_\lambda(A)) = \Lambda_\lambda(A)$,
(4) $\Lambda_\lambda(\cap_{\alpha\in\Delta} A_\alpha) \subseteq \cap_{\alpha\in\Delta}\Lambda_\lambda(A_\alpha)$ for any index set Δ.

Lemma 3.3. *For any subset A of a topological space X, the following hold:*
(1) $\Lambda_\lambda(A)$ is a Λ_λ-set,
(2) If A is λ-open, then A is a Λ_λ-set,
(3) If A_α is a Λ_λ-set for each $\alpha \in \Delta$, then $\cap_{\alpha\in\Delta} A_\alpha$ is a Λ_λ-set.

Remark 3.4. The converse of Lemma 3.3 (2) need not hold as shown by the following example: Let $\mathbf{R}$ be the usual real line and $A = \{0\}$. Then A is a Λ_λ-set but it is not λ-open. Because $\{0\} \subseteq \Lambda_\lambda(\{0\}) \subseteq (-1, 0] \cap [0, 1) = \{0\}$ and hence $\Lambda_\lambda(\{0\}) = \{0\}$. Therefore, $A = \{0\}$ is a Λ_λ-set but it is not λ-open.

Definition 3.5. A subset A of a topological space X is said to be Λ_λ-*closed* (resp. λ-*closed* [1]) if $A = L \cap F$, where L is a Λ_λ-set (resp. Λ-set) and F is a closed set.

Lemma 3.6. *For a subset of a topological space X, the following properties hold:*
(1) Every λ-locally closed set is Λ_λ-closed,
(2) Every λ-closed set is Λ_λ-closed.

Proof. (1) By Lemma 3.3, every λ-open set is a Λ_λ-set and (1) holds.

(2) Let U be a Λ-set. Then,

$$U = \cap\{V : U \subseteq V, V \text{ is open }\} \supseteq \cap\{V : U \subset V, V \text{ is } \lambda\text{-open }\} \supseteq U$$

and hence U is a Λ_λ-set. Therefore, (2) holds. □

Remark 3.7. By Lemma 3.6, we obtain the following diagram.

DIAGRAM I

$$\begin{array}{ccc} \text{locally closed} & \Rightarrow & \lambda\text{-locally closed} \\ \Downarrow & & \Downarrow \\ \lambda\text{-closed} & \Rightarrow & \Lambda_\lambda\text{-closed} \end{array}$$

Theorem 3.8. *For a subset A of a topological space X, the following are equivalent:*

(1) A is Λ_λ-closed;
(2) $A = U \cap \mathrm{Cl}(A)$ for some Λ_λ-set U;
(3) $A = \Lambda_\lambda(A) \cap \mathrm{Cl}(A)$.

Proof. (1) $\Rightarrow$ (2): Let A be a Λ_λ-closed set. Then $A = U \cap F$, where U is a Λ_λ-set and F is a closed set. Thus, we have $A \subseteq U \cap \mathrm{Cl}(A) \subseteq U \cap \mathrm{Cl}(F) = U \cap F = A$. Therefore, $A = U \cap \mathrm{Cl}(A)$.

(2) $\Rightarrow$ (3): Let $A = U \cap \mathrm{Cl}(A)$ for some Λ_λ-set U. Since $A \subseteq U$, by Lemma 3.2 $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(U) = U$ and hence $A \subseteq \Lambda_\lambda(A) \cap \mathrm{Cl}(A) \subseteq U \cap \mathrm{Cl}(A) = A$. Therefore, we obtain $A = \Lambda_\lambda(A) \cap \mathrm{Cl}(A)$.

(3) $\Rightarrow$ (1): Let $A = \Lambda_\lambda(A) \cap \mathrm{Cl}(A)$. By Lemma 3.3, $\Lambda_\lambda(A)$ is a Λ_λ-set and $\mathrm{Cl}(A)$ is closed. Therefore, A is Λ_λ-closed. □

Definition 3.9. Let X be a topological space. A subset A of X is said to be *λg-closed* (resp. *g-closed* [8]) if $\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is a λ-open (resp. open) set.

Theorem 3.10. *For a subset A of a topological space X, the following are equivalent:*

(1) A is closed;
(2) A is λ-locally closed and λg-closed;
(3) A is Λ_λ-closed and λg-closed.

Proof. (1) $\Rightarrow$ (2): Let A be closed in X. Since $A = X \cap A$ and X is a Λ_λ-set, A is λ-locally closed. Let U be any λ-open set containing A. Then $\mathrm{Cl}(A) = A \subseteq U$ and hence A is λg-closed.

(2) $\Rightarrow$ (3): By Lemma 3.6, every λ-locally closed set is Λ_λ-closed.

(3) $\Rightarrow$ (1): Let A be Λ_λ-closed and λg-closed. Since A is Λ_λ-closed, $A = P \cap L$, where P is a Λ_λ-set and L is closed in X. Let V be any λ-open set containing A. Since A is λg-closed, $\mathrm{Cl}(A) \subseteq V$ and hence $\mathrm{Cl}(A) \subseteq \cap\{V : A \subseteq V, V \text{ is } \lambda\text{-open}\} = \Lambda_\lambda(A)$. Therefore, $\mathrm{Cl}(A) \subseteq \Lambda_\lambda(A) \subseteq \Lambda_\lambda(P) = P$. On the other hand, $A \subseteq L$ and $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(L) = L$. Therefore, we obtain $\mathrm{Cl}(A) \subseteq P \cap L = A$. Thus A is closed.

□

Theorem 3.11. *Let X be a topological space. If A_α is a Λ_λ-closed set for each $\alpha \in \Delta$, then $\cap_{\alpha\in\Delta} A_\alpha$ is Λ_λ-closed.*

Proof. Let A_α be a Λ_λ-closed set for each $\alpha \in \Delta$. Then $A_\alpha = U_\alpha \cap F_\alpha$, where U_α is a Λ_λ-set and F_α is a closed set for each $\alpha \in \Delta$. By Lemma 3.3, $\cap_{\alpha\in\Delta} U_\alpha$ is a Λ_λ-set, $\cap_{\alpha\in\Delta} F_\alpha$ is closed and $\cap_{\alpha\in\Delta} A_\alpha = (\cap_{\alpha\in\Delta} U_\alpha) \cap (\cap_{\alpha\in\Delta} F_\alpha)$. Therefore, $\cap_{\alpha\in\Delta} A_\alpha$ is Λ_λ-closed.

□

4. Decompositions of Continuity

In this section, we obtain the decompositions of continuity.

Definition 4.1. A function $f : X \to Y$ is said to be
(1) *λ-LC-continuous* if $f^{-1}(V)$ is λ-locally closed in X for any closed set V of Y,
(2) *Λ_λ-continuous* if $f^{-1}(V)$ is Λ_λ-closed in X for any closed set V of Y,
(3) *λg-continuous* if $f^{-1}(V)$ is λg-closed in X for any closed set V of Y.

Theorem 4.2. *For a function $f : X \to Y$, the following are equivalent:*
(1) f is continuous;
(2) f is λ-LC-continuous and λg-continuous;
(3) f is Λ_λ-continuous and λg-continuous.

Proof. This is an immediate consequence of Theorem 3.10 □

Remark 4.3. The following facts are shown by Examples 4.4 and 4.5 and Remark 4.6:
(1) λ-LC-continuity and λg-continuity are independent of each other,
(2) Λ_λ-continuity and λg-continuity are independent of each other.

Example 4.4. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a\}\}$. Then $C(X) = C(Y) = \{\emptyset, \{b, c, d\}\}$ and λ-open sets in X (resp. Y) are: $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$. λ-locally closed sets in X (resp. Y) are: $\emptyset$, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b, d\}$, $\{b, c, d\}$, $\{b, c\}$, $\{c, d\}$, $\{b, d\}$, $\{b\}$, $\{c\}$, $\{d\}$. Define a function $f : X \to Y$ by $f(a) = c$, $f(b) = b$, $f(c) = d$, $f(d) = a$. Then we have the following:
(1) Since $f^{-1}(\{b, c, d\}) = \{a, b, c\}$, then f is not continuous.
(2) Since $f^{-1}(\{b, c, d\}) = \{a, b, c\}$, then f is λ-LC-continuous.
(3) Since $Cl(\{a, b, c\}) = X$ (i.e. $\{a, b, c\}$ is not λg-closed), then f is not λg-continuous.
(4) Since $\{a, b, c\} \subseteq \cap\{U : \{a, b, c\} \subseteq U, U$ is λ-open $\} = \{a, b, c\}$ and $\{a, b, c\} = \{a, b, c\} \cap X = \{a, b.c\}$, then $\{a, b, c\}$ is Λ_λ-closed. Thus f is Λ_λ-continuous.

Example 4.5. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a, b\}, \{c, d\}\}$. Then $C(X) = C(Y) = \{\emptyset, X, \{a, b\}, \{c, d\}\}$ and λ-open sets in X (resp. Y) are: $\emptyset$, X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$. And λ-locally closed sets in X (resp. Y) are: $\emptyset$, X, $\{a, b\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$. Define $g : X \to Y$ by $g(a) = c$, $g(b) = b$, $g(c) = a$, $g(d) = d$. Then we have the following:
(1) Since $g^{-1}(\{c, d\}) = \{a, d\}$, then g is not a continuous function.
(2) Since $g^{-1}(\{c, d\}) = \{a, d\}$, it is not a λ-locally closed set in X. Then g is not a λ-LC-continuous function.
(3) Since $g^{-1}(\{a, b\}) = \{b, c\} \subseteq \cap\{U : \{b, c\} \subseteq U, U$ is λ -open in $X\} =$

$\{b, c\} \cap X = \{b, c\}$ and $g^{-1}(\{c, d\}) = \{a, d\} = \cap\{U : \{a, d\} \subseteq U, U$ is λ-open in $X\}$ $= \{a, d\} \cap X = \{a, d\}$ are Λ_λ-closed, then Λ_λ-continuous.

Remark 4.6. (1) If every λg-continuous function is λ-LC-continuous, then it is continuous from Theorem 4.2 This is not true from Example 4.4(1).

(2) If every λg-continuous function is Λ_λ-continuous, then it is continuous from Theorem 4.2. This not true from Example 4.5(1).

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