Some Generalizations of Locally Closed Sets

Shyamapada Modak∗a and Takashi Noiriib

aDepartment of Mathematics, University of Gour Banga
P.O. Mokdumpur, Malda 732 103, India.
b2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi
Kumamoto-ken, 869-5142 JAPAN.

E-mail: spmodak2000@yahoo.co.in
E-mail: t.noiri@nifty.com

Abstract. Arenas et al. [1] introduced the notion of λ-closed sets as a generalization of locally closed sets. In this paper, we introduce the notions of λ-locally closed sets, Λλ-closed sets and λg-closed sets and obtain some decompositions of closed sets and continuity in topological spaces.

Keywords: λ-Open set, λ-Locally closed set, Λλ-Closed set, λg-Closed set, Decompositions of continuity.

2000 Mathematics subject classification: 54A05, 54C08.

1. Introduction and Preliminaries

The study of locally closed sets was introduced by Bourbaki [3] in 1966 then the authors Ganster and Reilly [6] have studied it extensively. A subset A of a topological space X is called locally closed if \( A = U \cap F \), where U is open and F is closed. It is interesting that a locally closed set is a generalization of both open sets and closed sets. The generalization has also been discussed in completely regular Hausdorff spaces [5] and has also been done on algebra with topology in [12] and [2].

∗Corresponding Author

Received 23 November 2016; Accepted 29 March 2017
©2019 Academic Center for Education, Culture and Research TMU
In this paper we consider a new type of sets in the topological space which is called \( \lambda \)-open sets. A set is said to be \( \lambda \)-open if it contains a nonempty open set. This idea is not a new idea. In literature, semi-open sets [7] and \( \alpha \)-sets [11] are examples of that type of sets although preopen sets [10] is not an example of it. Because: let \( \mathbb{R} \) be the usual real line and \( Q \) the rational numbers. Then \( \text{Cl}(Q) = \mathbb{R} \) and \( Q \subseteq \text{Int}(\text{Cl}(Q)) = \mathbb{R} \) (where ‘\( \text{Cl} \)’ and ‘\( \text{Int} \)’ denote the closure and interior operators, respectively). But \( Q \) does not contain nonempty open set. However Dontechev [4] has introduced an \( S \)-space: A topological space \( X \) is called an \( S \)-space if every subset which contains a non-void open subset is open. But the concept of \( \lambda \)-open sets is different from Dontechev’s \( S \)-spaces.

**Definition 1.1.** A subset \( A \) of a topological space \( X \) is said to be \( \lambda \)-open if \( A \) contains a nonempty open set. The complement of a \( \lambda \)-open set is said to be \( \lambda \)-closed.

For a subset \( A \) of a topological space \( X \), \( \text{Int}_\lambda(A) \) and \( \text{Cl}_\lambda(A) \) are defined as follows:

**Definition 1.2.** Let \( X \) be a topological space and \( A \) be a subset of \( X \).

\[
\text{Int}_\lambda(A) = \bigcup \{ U : U \subseteq A, \text{ } U \text{ } \text{is } \lambda\text{-open in } X \};
\]

\[
\text{Cl}_\lambda(A) = \bigcap \{ F : A \subseteq F, F \text{ } \text{is } \lambda\text{-closed in } X \}.
\]

**Lemma 1.3.** Let \( X \) be a topological space and \( A, B \) subsets of \( X \).

1. if \( A \subseteq B \), then \( \text{Int}_\lambda(A) \subseteq \text{Int}_\lambda(B) \) and \( \text{Cl}_\lambda(A) \subseteq \text{Cl}_\lambda(B) \).
2. \( X \setminus \text{Int}_\lambda(A) = \text{Cl}_\lambda(X \setminus A) \).
3. For any index set \( \Delta \), if \( A_\alpha \) is \( \lambda \)-open (resp. \( \lambda \)-closed), then \( \bigcup \{ A_\alpha : \alpha \in \Delta \} \) is \( \lambda \)-open (resp. \( \bigcap \{ A_\alpha : \alpha \in \Delta \} \) is \( \lambda \)-closed).
4. \( \text{Int}_\lambda(A) \) is \( \lambda \)-open and \( \text{Cl}_\lambda(A) \) is \( \lambda \)-closed.

**Remark 1.4.** The finite intersection of \( \lambda \)-open sets need not be \( \lambda \)-open. Let \( \mathbb{R} \) be the usual real line, \( A = (-1, 0] \) and \( B = [0, 1) \). The \( A \) and \( B \) are \( \lambda \)-open but \( A \cap B = \{0\} \) is not \( \lambda \)-open.

We generalize the locally closed set by using \( \lambda \)-open sets.

### 2. \( \lambda \)-Locally Closed Sets

**Definition 2.1.** A subset \( A \) of a topological space \( X \) is said to be \( \lambda \)-locally closed if \( A = U \cap F \), where \( U \) is \( \lambda \)-open and \( F \) is closed.

**Corollary 2.2.** Let \( f : X \to Y \) be a continuous function. If \( L \) is a \( \lambda \)-locally closed subset of \( Y \), then \( f^{-1}(L) \) is \( \lambda \)-locally closed in \( X \).

From Definition 1.1 it is obvious that every locally closed set is \( \lambda \)-locally closed. But the converse need not hold in general.

**Example 2.3.** Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{a\}\} \). Then \( \mathcal{C}(X) \) (all closed sets in \( X \)) = \{\emptyset, X, \{b, c, d\}\}. And \( \lambda \)-open sets are: \( \emptyset, X, \{a\}, \{a, b\}, \{a, b, c\}, \{a, c\} \),
The union of two \(\lambda\)-locally closed sets need not be \(\lambda\)-locally closed.

**Example 2.7.** Let \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, X, \{a, b\}, \{c, d\}\}\). Then \(C(X) = \emptyset, X, \{a, d\}, \{a, b\}\) and \(\lambda\)-open sets are: \(\emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\). \(\lambda\)-locally closed sets are: \(\emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\). Therefore, \(\{a\}\) and \(\{c\}\) are \(\lambda\)-locally closed sets but their union \(\{a, c\}\) is not a \(\lambda\)-locally closed set.

### 3. \(A_\lambda\)-Closed Sets

Locally closed sets in a topological space are introduced and investigated in [3] and [6]. As a generalization of locally closed sets, Arenas et al. [1] introduced the notion of \(\lambda\)-closed sets in a topological space. In this section, we introduce the notion of \(A_\lambda\)-closed sets which is a generalization of \(\lambda\)-closed sets. We obtain some characterizations of \(A_\lambda\)-closed sets and obtain decompositions of closed sets.
Definition 3.1. Let $X$ be a topological space and $A$ a subset of $X$. The subset $\Lambda_\lambda(A)$ is defined as follows: $\Lambda_\lambda(A) = \bigcap \{ U : A \subseteq U, \ U \text{ is } \lambda\text{-open} \}$.

A subset $A$ is called a $\Lambda_\lambda$-set if $A = \Lambda_\lambda(A)$. If $U$ is open in Definition 3.1, then a $\Lambda_\lambda$-set $A$ is called a $\Lambda$-set [9].

Lemma 3.2. For any subsets $A$ and $B$ of a topological space $X$, the following hold:

(1) $A \subseteq \Lambda_\lambda(A)$,
(2) If $A \subseteq B$, then $\Lambda_\lambda(A) \subseteq \Lambda_\lambda(B)$,
(3) $\Lambda_\lambda(\Lambda_\lambda(A)) = \Lambda_\lambda(A)$,
(4) $\Lambda_\lambda(\bigcap_{\alpha \in \Delta} A_\alpha) \subseteq \bigcap_{\alpha \in \Delta} \Lambda_\lambda(A_\alpha)$ for any index set $\Delta$.

Lemma 3.3. For any subset $A$ of a topological space $X$, the following hold:

(1) $\Lambda_\lambda(\Lambda_\lambda(A))$ is a $\Lambda_\lambda$-set,
(2) If $A$ is $\lambda$-open, then $A$ is a $\Lambda_\lambda$-set,
(3) If $A_\alpha$ is a $\Lambda_\lambda$-set for each $\alpha \in \Delta$, then $\bigcap_{\alpha \in \Delta} A_\alpha$ is a $\Lambda_\lambda$-set.

Remark 3.4. The converse of Lemma 3.3 (2) need not hold as shown by the following example: Let $R$ be the usual real line and $A = \{0\}$. Then $A$ is a $\Lambda_\lambda$-set but it is not $\lambda$-open. Because $\{0\} \subseteq \Lambda_\lambda(\{0\}) \subseteq (-1,0] \cap [0,1) = \{0\}$ and hence $\Lambda_\lambda(\{0\}) = \{0\}$. Therefore, $A = \{0\}$ is a $\Lambda_\lambda$-set but it is not $\lambda$-open.

Definition 3.5. A subset $A$ of a topological space $X$ is said to be $\Lambda_\lambda$-closed (resp. $\lambda$-closed [1]) if $A = L \cap F$, where $L$ is a $\Lambda_\lambda$-set (resp. $\Lambda$-set) and $F$ is a closed set.

Lemma 3.6. For a subset of a topological space $X$, the following properties hold:

(1) Every $\lambda$-locally closed set is $\Lambda_\lambda$-closed,
(2) Every $\lambda$-closed set is $\Lambda_\lambda$-closed.

Proof. (1) By Lemma 3.3, every $\lambda$-open set is a $\Lambda_\lambda$-set and (1) holds.
(2) Let $U$ be a $\Lambda$-set. Then,

$$U = \bigcap \{ V : U \subseteq V, V \text{ is open} \} \supseteq \bigcap \{ V : U \subset V, V \text{ is } \lambda\text{-open} \} \supseteq U$$

and hence $U$ is a $\Lambda_\lambda$-set. Therefore, (2) holds. \qed

Remark 3.7. By Lemma 3.6, we obtain the following diagram.

DIAGRAM I

\[
\begin{array}{ccc}
\text{locally closed} & \Rightarrow & \lambda\text{-locally closed} \\
\downarrow & & \downarrow \\
\lambda\text{-closed} & \Rightarrow & \Lambda_\lambda\text{-closed}
\end{array}
\]
Theorem 3.8. For a subset \( A \) of a topological space \( X \), the following are equivalent:

1. \( A \) is \( \Lambda_\lambda \)-closed;
2. \( A = U \cap \text{Cl}(A) \) for some \( \Lambda_\lambda \)-set \( U \);
3. \( A = \Lambda_\lambda (A) \cap \text{Cl}(A) \).

Proof. (1) \( \Rightarrow \) (2): Let \( A \) be a \( \Lambda_\lambda \)-closed set. Then \( A = U \cap F \), where \( U \) is a \( \Lambda_\lambda \)-set and \( F \) is a closed set. Thus, we have \( A \subseteq U \cap \text{Cl}(A) \subseteq U \cap \text{Cl}(F) = U \cap F = A \). Therefore, \( A = U \cap \text{Cl}(A) \).

(2) \( \Rightarrow \) (3): Let \( A = U \cap \text{Cl}(A) \) for some \( \Lambda_\lambda \)-set \( U \). Since \( A \subseteq U \), by Lemma 3.2 \( \Lambda_\lambda (A) \subseteq \Lambda_\lambda (U) = U \) and hence \( A \subseteq \Lambda_\lambda (A) \cap \text{Cl}(A) \subseteq U \cap \text{Cl}(A) = A \).

Therefore, we obtain \( A = \Lambda_\lambda (A) \cap \text{Cl}(A) \).

(3) \( \Rightarrow \) (1): Let \( A = \Lambda_\lambda (A) \cap \text{Cl}(A) \). By Lemma 3.3, \( \Lambda_\lambda (A) \) is a \( \Lambda_\lambda \)-set and \( \text{Cl}(A) \) is closed. Therefore, \( A \) is \( \Lambda_\lambda \)-closed.

\( \Box \)

Definition 3.9. Let \( X \) be a topological space. A subset \( A \) of \( X \) is said to be \( \lambda g \)-closed (resp. \( g \)-closed [8]) if \( \text{Cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a \( \lambda \)-open (resp. open) set.

Theorem 3.10. For a subset \( A \) of a topological space \( X \), the following are equivalent:

1. \( A \) is closed;
2. \( A \) is \( \lambda \)-locally closed and \( \lambda g \)-closed;
3. \( A \) is \( \Lambda_\lambda \)-closed and \( \lambda g \)-closed.

Proof. (1) \( \Rightarrow \) (2): Let \( A \) be closed in \( X \). Since \( A = X \cap A \) and \( A \) is a \( \Lambda_\lambda \)-set, \( A \) is \( \lambda \)-locally closed. Let \( U \) be any \( \lambda \)-open set containing \( A \). Then \( \text{Cl}(A) = A \subseteq U \) and hence \( A \) is \( \lambda g \)-closed.

(2) \( \Rightarrow \) (3): By Lemma 3.6, every \( \lambda \)-locally closed set is \( \Lambda_\lambda \)-closed.

(3) \( \Rightarrow \) (1): Let \( A \) be \( \Lambda_\lambda \)-closed and \( \lambda g \)-closed. Since \( A \) is \( \Lambda_\lambda \)-closed and \( \lambda g \)-closed, \( A = P \cap L \), where \( P \) is a \( \Lambda_\lambda \)-set and \( L \) is closed in \( X \). Let \( V \) be any \( \lambda \)-open set containing \( A \). Since \( A \) is \( \lambda g \)-closed, \( \text{Cl}(A) \subseteq V \) and hence \( \text{Cl}(A) \subseteq \cap \{V : A \subseteq V, V \text{ is } \lambda \text{-open}\} = \Lambda_\lambda (A) \). Therefore, \( \text{Cl}(A) \subseteq \Lambda_\lambda (A) \subseteq \Lambda_\lambda (P) = P \). On the other hand, \( A \subseteq L \) and \( \text{Cl}(A) \subseteq \text{Cl}(L) = L \). Therefore, we obtain \( \text{Cl}(A) \subseteq P \cap L = A \). Thus \( A \) is closed.

\( \Box \)

Theorem 3.11. Let \( X \) be a topological space. If \( A_\alpha \) is a \( \Lambda_\lambda \)-closed set for each \( \alpha \in \Delta \), then \( \cap_{\alpha \in \Delta} A_\alpha \) is \( \Lambda_\lambda \)-closed.

Proof. Let \( A_\alpha \) be a \( \Lambda_\lambda \)-closed set for each \( \alpha \in \Delta \). Then \( A_\alpha = U_\alpha \cap F_\alpha \), where \( U_\alpha \) is a \( \Lambda_\lambda \)-set and \( F_\alpha \) is a closed set for each \( \alpha \in \Delta \). By Lemma 3.3, \( \cap_{\alpha \in \Delta} U_\alpha \) is a \( \Lambda_\lambda \)-set, \( \cap_{\alpha \in \Delta} F_\alpha \) is closed and \( \cap_{\alpha \in \Delta} A_\alpha = (\cap_{\alpha \in \Delta} U_\alpha) \cap (\cap_{\alpha \in \Delta} F_\alpha) \). Therefore, \( \cap_{\alpha \in \Delta} A_\alpha \) is \( \Lambda_\lambda \)-closed.

\( \Box \)
4. Decompositions of Continuity

In this section, we obtain the decompositions of continuity.

**Definition 4.1.** A function \( f : X \to Y \) is said to be

1. \( \lambda \)-LC-continuous if \( f^{-1}(V) \) is \( \lambda \)-locally closed in \( X \) for any closed set \( V \) of \( Y \),
2. \( \Lambda_\lambda \)-continuous if \( f^{-1}(V) \) is \( \Lambda_\lambda \)-closed in \( X \) for any closed set \( V \) of \( Y \),
3. \( \lambda g \)-continuous if \( f^{-1}(V) \) is \( \lambda g \)-closed in \( X \) for any closed set \( V \) of \( Y \).

**Theorem 4.2.** For a function \( f : X \to Y \), the following are equivalent:

1. \( f \) is continuous;
2. \( f \) is \( \lambda \)-LC-continuous and \( \lambda g \)-continuous;
3. \( f \) is \( \Lambda_\lambda \)-continuous and \( \lambda g \)-continuous.

**Proof.** This is an immediate consequence of Theorem 3.10 \( \square \)

**Remark 4.3.** The following facts are shown by Examples 4.4 and 4.5 and Remark 4.6:

1. \( \lambda \)-LC-continuity and \( \lambda g \)-continuity are independent of each other,
2. \( \Lambda_\lambda \)-continuity and \( \lambda g \)-continuity are independent of each other.

**Example 4.4.** Let \( X = Y = \{a, b, c, d\} \), \( \tau = \sigma = \{\emptyset, X, \{a\}\} \). Then \( C(X) = C(Y) = \{\emptyset, \{b, c, d\}\} \) and \( \lambda \)-open sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\} \). \( \lambda \)-locally closed sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\} \).

Define a function \( f : X \to Y \) by \( f(a) = c, f(b) = b, f(c) = d, f(d) = a \). Then we have the following:

1. Since \( f^{-1}(\{a, b\}) = \{a, b\} \), then \( f \) is not continuous.
2. Since \( f^{-1}(\{b, c, d\}) = \{a, b, c\} \), then \( f \) is \( \lambda \)-LC-continuous.
3. Since \( Cl(\{a, b, c\}) = X \) (i.e. \( \{a, b, c\} \) is not \( \lambda g \)-closed), then \( f \) is not \( \lambda g \)-continuous.

**Example 4.5.** Let \( X = Y = \{a, b, c, d\} \), \( \tau = \sigma = \{\emptyset, X, \{a, b\}, \{c, d\}\} \). Then \( C(X) = C(Y) = \{\emptyset, X, \{a, b\}, \{c, d\}\} \) and \( \lambda \)-open sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \). \( \lambda \)-locally closed sets in \( X \) (resp. \( Y \)) are: \( \emptyset, X, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a\}, \{b\}, \{c\}, \{d\} \). Define \( g : X \to Y \) by \( g(a) = c, g(b) = b, g(c) = a, g(d) = d \). Then we have the following:

1. Since \( g^{-1}(\{c, d\}) = \{a, d\} \), then \( g \) is not a continuous function.
2. Since \( g^{-1}(\{c, d\}) = \{a, d\} \), it is not a \( \lambda \)-locally closed set in \( X \). Then \( g \) is not a \( \lambda \)-LC-continuous function.
3. Since \( g^{-1}(\{a, b\}) = \{b, c\} \subseteq \cap \{U : \{b, c\} \subseteq U, U \text{ is } \lambda \text{-open in } X\} = \emptyset \), then \( g \) is not a \( \lambda \)-LC-continuous function.
Some generalizations of locally closed sets

\{b, c\} \cap X = \{b, c\} and \(g^{-1}(\{c, d\}) = \{a, d\} = \cap \{U : \{a, d\} \subseteq U, U \text{ is } \lambda\text{-open in } X\}

= \{a, d\} \cap X = \{a, d\} are \(\Lambda_\lambda\)-closed, then \(\Lambda_\lambda\)-continuous.

Remark 4.6. (1) If every \(\lambda g\)-continuous function is \(\lambda\)-LC-continuous, then it is continuous from Theorem 4.2. This is not true from Example 4.4(1).

(2) If every \(\lambda g\)-continuous function is \(\Lambda_\lambda\)-continuous, then it is continuous from Theorem 4.2. This not true from Example 4.5(1).

Acknowledgments

The authors wish to thank the referees for their valuable comments.

References