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Redefined fuzzy subalgebra (with thresholds) of BCK/BCI-algebras

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Abstract. Using the notion of *anti fuzzy points* and its *besideness* to and *non-quasi-coincidence* with a fuzzy set, new concepts of an anti fuzzy subalgebras in BCK/BCI-algebras are introduced and their inter-relations and related properties are investigated in [3]. The notion of the new fuzzy subalgebra with thresholds are introduced and relationship between this notion and the new fuzzy subalgebra of a BCK/BCI-algebra of [3] are studied.

Keywords: Besides to, Non-quasi coincident with, $(\alpha, \beta)^*$-fuzzy subalgebra, Fuzzy subalgebra with thresholds, Fuzzifying subalgebra, t-implication-based subalgebra.

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1. Introduction

Y. Imai and K. Iseki [12] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras [1, 11, 17, 21].

Dedicated to the memory of my dear father, Prof. Javad Borumand Saeid.

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The concept of fuzzy sets was first initiated by Zadeh [20]. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. A new type of fuzzy subgroup, that is the $(\in, \in \vee q)$-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das [2] by using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [16]. In fact the $(\in, \in \vee q)$-fuzzy subgroups is an important generalization of Rosenfeld's fuzzy subgroup. After that this structure was studied by some authors [4, 5, 6, 7, 8, 9, 13, 14, 19].

Fuzzy BCK-algebra is studied in some papers. In [3], author and Y. B. Jun introduced new definition of fuzzy BCK/BCI-algebras by using the notion of anti fuzzy point and two relations besideness and non quasi-coincidence. In this paper, at first we state and prove some theorem in new fuzzy subalgebras as continuation [3]. Then the notion of the new fuzzy subalgebra with thresholds are introduced and we get the results that mentioned in abstract.

2. Preliminaries

Definition 2.1. [15] Let X be a non-empty set with a binary operation "$*$" and a constant "0". Then $(X, *, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(i) $((x * y) * (x * z)) * (z * y) = 0$,
(ii) $(x * (x * y)) * y = 0$,
(iii) $x * x = 0$,
(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$,
for all $x, y, z \in X$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra.

A nonempty subset S of X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [10, 15] for further information regarding BCK/BCI-algebras.

A fuzzy set A in X of the form

$$A(y) := \begin{cases} t \in [0, 1) & \text{if } y = x, \\ 1 & \text{if } y \neq x \end{cases}$$

is called an *anti fuzzy point* with support x and and value t and is denoted by x_t. A fuzzy set A in X is said to be *non-unit* if there exists $x \in X$ such that $A(x) < 1$.

A fuzzy set A in a BCK/BCI-algebra X is called an *anti-fuzzy subalgebra* of X if it satisfies

$$(1) \qquad (\forall x, y \in X)\,(A(x * y) \leq \max\{A(x), A(y)\}).$$

Proposition 2.2. [3] *Let A be a fuzzy set in X. Then A is an anti fuzzy subalgebra of X if and only if $L(A;t) := \{x \in X \mid A(x) \leq t\}$ is a subalgebra of X, for all $t \in [0, 1)$.*

In [16], the authors introduced the notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets with the $\in$ and q respectively. In [3], the notions of " besideness" and "non quasicoincidence" of anti fuzzy points and fuzzy sets was introduced with $\lessdot$ and Υ respectively.

Definition 2.3. [3] An anti-fuzzy point x_t is said to *beside to* (resp. *be non-quasi coincident with*) a fuzzy set A, denoted by $x_t \lessdot A$ (resp. $x_t \Upsilon A$), if $A(x) \leq t$ (resp. $A(x) + t < 1$). We say that $\lessdot$ (resp. Υ) is a *beside to relation* (resp. *non-quasi coincident with*) between anti-fuzzy points and fuzzy sets.

If $x_t \lessdot A$ or $x_t \Upsilon A$ (resp. $x_t \lessdot A$ and $x_t \Upsilon A$), we say that $x_t \lessdot \vee \Upsilon\, A$ (resp. $x_t \lessdot \wedge \Upsilon\, A$).

Proposition 2.4. [3] *Let A be a fuzzy set in a BCK/BCI-algebra X. Then A satisfies the condition* (1) *if and only if it satisfies the following condition.*

(2) $\quad (\forall x, y \in X)\, (\forall t_1, t_2 \in [0, 1))\, (x_{t_1}, y_{t_2} \lessdot A \Rightarrow (x * y)_{\max\{t_1,t_2\}} \lessdot A).$

Note that if A is a fuzzy set in X such that $A(x) \geq 0.5$ for all $x \in X$, then the set $\{x_t \mid x_t \lessdot \wedge \Upsilon\, A\}$ is empty.

In what follows let α and β denote any one of $\lessdot$, Υ, $\lessdot \vee \Upsilon$, or $\lessdot \wedge \Upsilon$ unless otherwise specified. To say that $x_t \overline{\alpha} A$ means that $x_t \alpha A$ does not hold.

Definition 2.5. [3] A fuzzy set A in a BCK/BCI-algebra X is called an $(\alpha, \beta)^*$*-fuzzy subalgebra* of X, where $\alpha \neq \lessdot \wedge \Upsilon$, if it satisfies the following implication:

(3) $\quad (\forall x, y \in X)\, (\forall t_1, t_2 \in [0, 1))\, (x_{t_1} \alpha A, y_{t_2} \alpha A \Rightarrow (x * y)_{\max\{t_1,t_2\}} \beta A).$

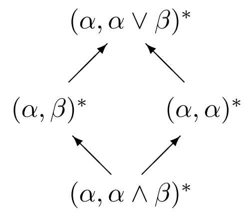
Example 2.6. [3] Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

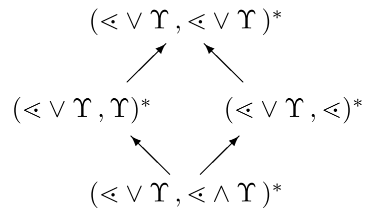
$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let A be a fuzzy set in X defined by $A(0) = 0.4$, $A(a) = 0.3$, and $A(b) = A(c) = 0.7$. It is routine to verify that A is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

The following diagram shows the relation between some types of $(\alpha, \beta)^*$-fuzzy subalgebras of X. For example " $(\alpha, \beta)^* \to (\alpha, \alpha \vee \beta)^*$ " means that any $(\alpha, \beta)^*$-fuzzy subalgebra is a $(\alpha, \alpha \vee \beta)^*$-fuzzy subalgebra of X.

Theorem 2.7. [3] *Let A be a fuzzy set in a BCK/BCI-algebra X. Then the left diagram shows the relationship between $(\alpha,\beta)^*$-fuzzy subalgebras of X, where α,β are one of $\lessdot$ and Υ. Also we have the right diagram.*





For a fuzzy set A in a BCK/BCI-algebra X, we denote

$$X^* := \{x \in X \mid A(x) < 1\}.$$

Theorem 2.8. [3] *Let A be a fuzzy set in a BCK/BCI-algebra X. Then A is a non-unit $(\Upsilon,\Upsilon)^*$-fuzzy subalgebra of X if and only if there exists a subalgebra S of X such that*

$$A(x) := \begin{cases} t \in [0,1) & \text{if } x \in S, \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

Theorem 2.9. [3] *Let S be a subalgebra of a BCK/BCI-algebra X and let A be a fuzzy set in X such that*

(i) $(\forall x \in X \setminus S)\ (A(x) = 1)$,
(ii) $(\forall x \in S)\ (A(x) \leq 0.5)$.

Then A is a $(\Upsilon,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X.

Theorem 2.10. [3] *Let A be a fuzzy set in a BCK/BCI-algebra X. Then A is a $(\lessdot,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X if and only if it satisfies the following inequality.*

$$(\forall x, y \in X)\,(A(x*y) \leq \max\{A(x), A(y), 0.5\}). \tag{5}$$

Theorem 2.11. [3] *Let A be a $(\Upsilon,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of a BCK/BCI-algebra X such that A is not constant on X^*. Then there exists $x \in X$ such that $A(x) \leq 0.5$. Moreover $A(x) \leq 0.5$ for all $x \in X^*$.*

3. Some results on Redefined Fuzzy BCK/BCI-algebra

From now $(A,*,0)$ or simply A is a BCK/BCI-algebra.

Proposition 3.1. *Let A be a fuzzy set in X. If A is a $(\lessdot,\lessdot)^*$-fuzzy subalgebra of X, then $A(0) \leq A(x)$, for all $x \in X$.*

Note. It is clear that the above proposition is valid only for $(\lessdot,\lessdot)^*$-fuzzy subalgebra of X. Since in Example 2.6, A is a $(\lessdot,\lessdot\vee\Upsilon)^*$-fuzzy subalgebra of X, but $A(0) > A(a)$.

Theorem 3.2. *S is a subalgebra of X such if and only if χ_S^c is an $(\alpha,\beta)^*$-fuzzy subalgebra of X where $\chi_S^c : X \to [0,1]$ defined by $\chi_S^c(x) = 1 - \chi_S(x)$ for all $x \in X$ and (α,β) is one of the following forms*

(i) $(\lessdot, \Upsilon)$, *(ii)* $(\lessdot, \lessdot \wedge \Upsilon)$,
(iii) $(\Upsilon, \lessdot)$, *(iv)* $(\Upsilon, \lessdot \wedge \Upsilon)$,
(v) $(\lessdot \vee \Upsilon, \Upsilon)$, *(vi)* $(\lessdot \vee \Upsilon, \lessdot \wedge \Upsilon)$,
(vii) $(\lessdot \vee \Upsilon, \lessdot)$.

Proof. (ii) Let $\mu = \chi_S^c$ and $x_{t_1}, y_{t2} \lessdot \mu$, for $t_1, t_2 \in [0,1)$. Then $\mu(x) \le t_1$ and $\mu(y) \le t_2$. Thus we get that $\chi_S(x) \ge 1 - t_1$ and $\chi_S(y) \ge 1 - t_2$. Hence $x, y \in S$, since S is a subalgebra of X, we get that $x * y \in S$. Then $\mu(x*y) = 0$, therefore $(x*y)_{\max\{(t_1,t_2\}} \lessdot \wedge \Upsilon$, thus χ_S^c is a $(\lessdot, \lessdot \wedge \Upsilon)^*$-fuzzy subalgebra of X.

Conversely, assume that $\mu = \chi_S^c$ is a $(\lessdot, \lessdot \wedge \Upsilon)^*$-fuzzy subalgebra of X and let $x, y \in S$. Then $\mu(x) = 0$ and $\mu(y) = 0$. Hence $x_0, y_0 \lessdot \mu$ which imply that $(x*y)_{\max\{0,0\}} \lessdot \wedge \Upsilon\, \mu$. Thus $\mu(x*y) \le 0$ and $\mu(x*y) + 0 < 1$. If $\mu(x*y) \le 0$, then $\chi(x*y) = 1$. Therefore $x*y \in S$. If $\mu(x*y) + 0 < 1$, then $\chi(x*y) > 0$. Therefore $x*y \in S$.

The proof of other cases is similar, see [3]. □

Proposition 3.3. *Let A be a fuzzy set in X. Then A is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X if and only if for all $t \in [0,1]$, the nonempty level set $L(A;t)$ is a subalgebra of X.*

Proof. The proof follows from Proposition 2.4. □

Theorem 3.4. *Let A be a fuzzy set in X. Then A is a non-unite $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X if and only if $L(A; A(0)) = X^*$ and for all $t \in [0,1]$, the nonempty level set $L(A;t)$ is a subalgebra of X.*

Proof. Let A be a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. Then by Theorem 2.8 we have

$$A(x) = \begin{cases} A(0) & \text{if } x \in X^* \\ 1 & otherwise \end{cases}$$

So it is easy to check that $L(A; A(0)) = X^*$. Let $x, y \in L(A;t)$, for $t \in [0,1]$. Then $A(x) \le t$ and $A(y) \le t$. If $t = 1$, then it is clear that $x*y \in L(A;1)$. Now let $t \in [0,1)$. Then $x, y \in X^*$ and so $x*y \in X^*$. Hence $A(x*y) = A(0) \le t$. Therefore $L(A;t)$ is a subalgebra of X.

Conversely, since $L(A; A(0)) = X^*$ and $0 \in L(A; A(0))$, then X^* is a subalgebra of X and A is non-unit. Now let $x \in X^*$. Then $A(x) \ge A(0)$ and $A(x) > 0$. Since $L(A; A(x)) \neq \emptyset$, so $L(A; A(x))$ is a subalgebra of X. Then $0 \in L(A; A(x))$ implies that $A(0) \ge A(x)$. Hence $A(x) = A(0)$, for all $x \in X^*$. Therefore

$$A(x) = \begin{cases} A(0) & \text{if } x \in X^* \\ 1 & otherwise \end{cases}$$

Hence by Theorem 2.8 we get that A is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. □

Theorem 3.5. *Every* $(\Upsilon, \Upsilon)^*$*-fuzzy subalgebra is an* $(\lessdot, \lessdot)^*$*-fuzzy subalgebra.*

Proof. The proof follows from Theorem 3.4 and Proposition 3.3. □

Note that the converse of the above theorem need not be true in general.

Example 3.6. Consider the BCI-algebra $X = \{0, a, b, c\}$ in Example 2.6. Let A be a fuzzy set in X defined by $A(0) = 0.1$, $A(a) = 0.3$, and $A(b) = A(c) = 0.7$. It is routine to verify that A is a $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X. But A is not a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra, since $a_{065}, c_{0.2} \Upsilon A$, but $A(a * c) + \max(0.65, 0.2) = A(b) + 0.65 > 1$.

Theorem 3.7. *Let* A *be a non-unit* $(\Upsilon, \lessdot \vee \Upsilon)^*$*-fuzzy subalgebra of* X*. Then the nonempty level set* $L(A;t)$ *is a subalgebra of* X*, for all* $t \in [0.5, 1]$*.*

Proof. If A is constant on X^*, then by Theorem 2.8, A is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra. Thus by Theorem 3.4 we have the nonempty level set $L(A;t)$ is a subalgebra of X, for all $t \in [0, 1]$.

If A is not constant on X^*, then by Theorem 2.11, we have

$$A(x) = \begin{cases} \alpha & \text{if } x \in X^* \\ 1 & otherwise \end{cases}$$

where $\alpha \leq 0.5$. Now we show that the nonempty level set $L(A;t)$ is a subalgebra of X for $t \in [0.5, 1]$. If $t = 1$, then it is clear that $L(A;t)$ is a subalgebra of X. Now let $t \in [0.5, 1)$ and $x, y \in L(A;t)$. Then $A(x), A(y) \leq t < 1$ imply that $x, y \in X^*$. Thus $x * y \in X^*$ and so $A(x * y) \leq 0.5 \leq t$. Therefore $x * y \in L(A;t)$. □

Theorem 3.8. *Let* A *be a non-unit fuzzy set of* X*,* $L(A; 0.5) = X^*$ *and the nonempty level set* $L(A;t)$ *is a subalgebra of* X*, for all* $t \in [0, 1]$*. Then* A *is a* $(\Upsilon, \lessdot \vee \Upsilon)^*$*-fuzzy subalgebra of* X*.*

Proof. Since $A \neq 1$ we get that $X^* \neq \emptyset$. Thus by hypothesis we have $L(A; 0.5) \neq \emptyset$ and so X^* is a subalgebra of X. Also $A(x) \leq 0.5$, for all $x \in X^*$ and $A(x) = 1$, if $x \notin X^*$. Therefore by Theorem 2.9, A is a $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. □

Theorem 3.9. *Let* A *be a non-unit fuzzy subset of* X*. Then* A *is a* $(\Upsilon, \lessdot \vee \Upsilon)^*$*-fuzzy subalgebra of* X *if and only if there exists subalgebra* S *of* X *such that*

$$A(x) = \begin{cases} a & \text{if } x \in S \\ 0 & otherwise \end{cases} \quad or \quad A(x) = \begin{cases} \leq 0.5 & \text{if } x \in S \\ 0 & otherwise \end{cases}$$

for some $a \in [0, 1)$*.*

Proof. Let A be an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. If A is constant on X^*, then

$$A(x) = \begin{cases} A(0) & \text{if } x \in X^* \\ 0 & otherwise \end{cases}$$

If A is not constant on X^*, then by Theorem 2.11 we have

$$A(x) = \begin{cases} \leq 0.5 & \text{if } x \in X^* \\ 0 & otherwise \end{cases}$$

Conversely, the proof follows from Theorems 2.7, 2.8 and 2.9. □

Theorem 3.10. *Let A be an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. Then for all $t \in [0.5, 1]$, the nonempty level set $L(A;t)$ is a subalgebra of X. Conversely, if the nonempty level set A is a subalgebra of X, for all $t \in [0,1]$, then A is an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.*

Proof. Let A be an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. If $t = 1$, then $L(A;t)$ is a subalgebra of X. Now let $L(A;t) \neq \emptyset$, $0.5 \leq t < 1$ and $x, y \in L(A;t)$. Then $A(x), A(y) \leq t$. Thus by hypothesis we have $A(x*y) \leq \max(A(x), A(y), 0.5) \leq \max(t, 0.5) \leq t$. Therefore $L(A;t)$ is a subalgebra of X.

Conversely, let $x, y \in X$. Then we have

$$A(x), A(y) \leq \max(A(x), A(y), 0.5) = t_0$$

Hence $x, y \in L(A;t_0)$, for $t_0 \in [0,1]$ and so $x*y \in L(A;t_0)$. Therefore $A(x*y) \leq t_0 = \max(A(x), A(y), 0.5)$, then A is an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X. □

Theorem 3.11. *Let $\{A_i \mid i \in \Lambda\}$ be a family of $(\alpha, \beta)^*$-fuzzy subalgebra of X. Then $A := \bigcap_{i \in \Lambda} A_i$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of X, where (α, β) is one of the following forms*

(i) $(\lessdot, \Upsilon)$, *(ii)* $(\lessdot, \lessdot \wedge \Upsilon)$,
(iii) $(\Upsilon, \lessdot)$, *(iv)* $(\Upsilon, \lessdot \wedge \Upsilon)$,
(v) $(\lessdot \vee \Upsilon, \Upsilon)$, *(vi)* $(\lessdot \vee \Upsilon, \lessdot \wedge \Upsilon)$,
(vii) $(\lessdot \vee \Upsilon, \lessdot)$, *(viii)* $(\Upsilon, \lessdot \vee \Upsilon)$,
(ix) (Υ, Υ).

Proof. We prove theorem for $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra. The proof of the other cases is similar, by using Theorems 3.2 and 3.10.
If there exists $i \in \Lambda$ such that $A_i = 0$, then $A = 0$. So A is a $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra. Let $A_i \neq 0$ for all $i \in \Lambda$. Then by Theorem 2.8, we have

$$A_i(x) = \begin{cases} A_i(0) & \text{if } x \in X_i^* \\ 1 & otherwise \end{cases}$$

for all $i \in \Lambda$. So it is clear that

$$A(x) = \begin{cases} A(0) & \text{if } x \in \bigcap_{i \in \Lambda} X_i^* \\ 1 & otherwise \end{cases}$$

Since $\bigcap_{i \in \Lambda} X_i^*$ is a subalgebra of X, then by Theorem 2.8, A is an $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. □

Lemma 3.12. *Let $f : X \to Y$ be a BCK/BCI-homomorphism and G be a fuzzy set of Y with membership function A_G. Then $x_t \alpha A_{f^{-1}(G)} \Leftrightarrow f(x)_t \alpha A_G$, for all $\alpha \in \{\Upsilon, \lessdot, \lessdot \vee \Upsilon, \lessdot \wedge \Upsilon\}$.*

Proof. Let α be the relation $\lessdot$. We have $x_t \alpha A_{f^{-1}(G)}$ if and only if $A_{f^{-1}(G)}(x) \leq t$ if and only if $A_G(f(x)) \leq t$ and it is equal to $(f(x))_t \alpha A_G$.

The proof of the other cases is similar to above argument. □

Theorem 3.13. *Let $f : X \to Y$ be a BCK/BCI-homomorphism and G be a fuzzy set of Y with membership function A_G.*

(i) If G is an $(\alpha, \beta)^*$-fuzzy subalgebra of Y, then $f^{-1}(G)$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of X,

(ii) Let f be epimorphism. If $f^{-1}(G)$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of X, then G is an $(\alpha, \beta)^*$-fuzzy subalgebra of Y.

Proof. (i) Let $x_t \alpha A_{f^{-1}(G)}$ and $y_r \alpha A_{f^{-1}(G)}$, for $t, r \in [0, 1)$. Then by Lemma 3.12, we conclude that $(f(x))_t \alpha A_G$ and $(f(y))_r \alpha A_G$. Hence by hypothesis we get that $(f(x) * f(y))_{\max(t,r)} \beta A_G$. Then $(f(x * y))_{\max(t,r)} \beta A_G$ and so $(x * y)_{\max(t,r)} \beta A_{f^{-1}(G)}$.

(ii) Let $x, y \in Y$. Then by hypothesis there exist $x', y' \in X$ such that $f(x') = x$ and $f(y') = y$. Now, assume that $x_t \alpha A_G$ and $y_r \alpha A_G$, then $(f(x'))_t \alpha A_G$ and $(f(y'))_r \alpha A_G$.
Thus $x'_t \alpha A_{f^{-1}(G)}$ and $y'_r \alpha A_{f^{-1}(G)}$ and therefore $(x' * y')_{\max(t,r)} \beta A_{f^{-1}(G)}$. If $(f(x' * y'))_{\max(t,r)} \beta A_G$, then $(f(x') * f(y'))_{\max(t,r)} \beta A_G$.
Therefore $(x * y)_{\max(t,r)} \beta A_G$. Hence G is an $(\alpha, \beta)^*$-fuzzy subalgebra of Y. □

Theorem 3.14. *Let $f : X \to Y$ be a BCK/BCI-homomorphism and H be a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X with membership function A_H. If A_H is f-invariant, then $f(H)$ is a $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of Y.*

Proof. Let y_1 and $y_2 \in Y$. If $f^{-1}(y_1)$ or $f^{-1}(y_2) = \emptyset$, then $A_{f(H)}(y_1 * y_2) \leq \max(A_{f(H)}(y_1), A_{f(H)}(y_2), 0.5)$. Now let $f^{-1}(y_1)$ and $f^{-1}(y_2) \neq \emptyset$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus by hypothesis we have

$$\begin{aligned}
A_{f(H)}(y_1 * y_2) &= \sup_{t \in f^{-1}(y_1 * y_2)} A_H(t) \\
&= \sup_{t \in f^{-1}(f(x_1 * x_2))} A_H(t) \\
&= A_H(x_1 * x_2) \\
&\leq \max(A_H(x_1), A_H(x_2), 0.5) \\
&= \max(\sup_{t \in f^{-1}(y_1)} A_H(t), \sup_{t \in f^{-1}(y_2)} A_H(t), 0.5) \\
&= \max(A_{f(H)}(y_1), A_{f(H)}(y_2), 0.5).
\end{aligned}$$

So by Theorem 2.10, $f(H)$ is an $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of Y. □

Lemma 3.15. *Let $f : X \to Y$ be a BCK/BCI-homomorphism.*
(i) If S is a subalgebra of X, then $f(S)$ is a subalgebra of Y.
(ii) If S' is a subalgebra of Y, then $f^{-1}(S')$ is a subalgebra of X.

Proof. The proof is easy. □

Theorem 3.16. *Let $f : X \to Y$ be a BCK/BCI-homomorphism. If H is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X with membership function A_H, then $f(H)$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of Y.*

Proof. Let H be a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of X. Then by Theorem 2.8, we have

$$A_H(x) = \begin{cases} A_H(0) & \text{if } x \in X^* \\ 1 & \textit{otherwise} \end{cases}$$

Now we show that

$$A_{f(H)}(y) = \begin{cases} A_H(0) & \text{if } y \in f(X^*) \\ 1 & \textit{otherwise} \end{cases}$$

Let $y \in Y$. If $y \in f(X^*)$, then there exist $x \in X^*$ such that $f(x) = y$. Thus $A_{f(H)}(y) = \sup_{t \in f^{-1}(y)} A_H(t) = A_H(0)$. If $y \notin f(X^*)$, then it is clear that $A_{f(H)}(y) = 1$. Since X^* is a subalgebra of X, then $f(X^*)$ is a subalgebra of Y. Therefore by Theorem 3.4, $f(H)$ is a non-unit $(\Upsilon, \Upsilon)^*$-fuzzy subalgebra of Y. □

Theorem 3.17. *Let $f : X \to Y$ be a BCK/BCI-homomorphism. If H is an $(\alpha, \beta)^*$-fuzzy subalgebra of X with membership function A_H, then $f(H)$ is an $(\alpha, \beta)^*$-fuzzy subalgebra of Y, where (α, β) is one of the following forms*

(i) $(\lessdot, \Upsilon)$,	*(ii)* $(\lessdot, \lessdot \wedge \Upsilon)$,
(iii) $(\Upsilon, \lessdot)$,	*(iv)* $(\Upsilon, \lessdot \wedge \Upsilon)$,
(v) $(\lessdot \vee \Upsilon, \Upsilon)$,	*(vi)* $(\lessdot \vee \Upsilon, \lessdot \wedge \Upsilon)$,
(vii) $(\lessdot \vee \Upsilon, \lessdot)$,	*(viii)* $(\Upsilon, \lessdot \vee \Upsilon)$.

Proof. The proof is similar to the proof of Theorem 3.16, by using of Theorems 3.2 and 3.10. □

Theorem 3.18. *Let $f : X \to Y$ be a BCK/BCI-homomorphism and H be an $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X with membership function A_H. If A_H is an f-invariant, then $f(H)$ is an $(\lessdot, \lessdot)^*$-fuzzy subalgebra of Y.*

Proof. Let $z_t \lessdot A_{f(H)}$ and $y_r \lessdot A_{f(H)}$, where $t, r \in [0, 1)$. Then $A_{f(H)}(z) \le t$ and $A_{f(H)}(y) \le r$. Thus $f^{-1}(z), f^{-1}(y) \neq \emptyset$ imply that there exists $x_1, x_2 \in X$ such that $f(x_1) = z$ and $f(x_2) = y$. Since A_H is f-invariant, then $A_{f(H)}(z) \le t$

and $A_{f(H)}(y) \leq r$ imply that $A_H(x_1) \leq t$ and $A_H(x_2) \leq r$. So by hypothesis we have

$$\begin{aligned} A_{f(H)}(z * y) &= \sup_{t \in f^{-1}(z*y)} A_H(t) \\ &= \sup_{t \in f^{-1}(f(x_1 * x_2))} A_H(t) \\ &= A_H(x_1 * x_2) \\ &\leq \max(t, r) \end{aligned}$$

Therefore $(z * y)_{\max(t,r)} \in A_{f(H)}$, thus $f(H)$ is an $(\lessdot, \lessdot)^*$-fuzzy subalgebra of Y. □

4. Redefined Fuzzy subalgebra with thresholds

Theorem 4.1. *Let A be a fuzzy subset of X. Then $L(A,t)$ is a subalgebra of X, for all $t \in [0, 0.5)$ if and only if A satisfies in the following*

$$min(A(x * y), 0.5) \leq max(A(x), A(y))$$

for all $x, y \in X$.

Proof. Let A satisfies in the $min(A(x*y), 0.5) \leq max(A(x), A(y))$ for all $x, y \in X$ and $x, y \in L(A,t)$ for $t \in [0, 0.5)$. Then

$$min(A(x * y), 0.5) \leq max(A(x), A(y)) \leq t$$

and so $A(x * y) \leq t$. Hence $x * y \in L(A,t)$, therefore $L(A,t)$ is a subalgebra of X.

Conversely, let A be a fuzzy subset of X such that $L(A,t)$ be a subalgebra of X for all $t \in [0, 0.5)$. If there exist $x, y \in X$ such that $max(A(x), A(y)) = t < min(A(x * y), 0.5)$, then we get that $A(x), A(y) \leq t$, so $x, y \in L(A,t)$ and $t < 0.5$. Since $L(A,t)$ is a subalgebra of X for all $t \in [0, 0.5)$, it follows that $x * y \in L(A,t)$ thus $A(x * y) \leq t$. Which is a contradiction. □

Form the above theorem we get that if a fuzzy subset A in X satisfies in some condition then $L(A,t)$ is a subalgebra of X for some $t \in [0,1)$, but $L(A,t)$ is not a subalgebra of X, for all $t \in [0,1)$. Let

$$\widehat{X} := \{t \in [0,1) \mid L(A,t) \text{ is a subalgebra of } X\}.$$

If $\widehat{X} = [0,1)$, then A is an anti fuzzy subalgebra of X (Proposition 2.2).

If $\widehat{X} = [0, 0.5)$, then A is an $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

Now, we consider the case $\widehat{X} \neq \emptyset$ (for example, $\widehat{X} = [0.5, 1), [r, s)$ where $r, s \in [0,1)$ with $r < s$).

Definition 4.2. Let $r, s \in [0,1]$ and $r < s$. Suppose that A be a fuzzy subset of X. Then A is called an anti fuzzy subalgebra with thresholds (r, s) of X if

$$\min(A(x * y), s) \leq \max(A(x), A(y), r) \quad \text{for all } x, y \in X.$$

Example 4.3. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	1
3	3	2	1	0	2
4	4	4	4	4	0

(i) Let A be a fuzzy set in X defined by $A(0) = 0.3$, $A(1) = 0.2$, $A(3) = A(4) = 0.7$ and $A(2) = 0.8$. It is routine to verify that A is an anti fuzzy subalgebra with thresholds $r = 0.4$ and $s = 0.65$ of X. But

(a) A is not an $(\lessdot, \lessdot)^*$-fuzzy subalgebra of X, since $1_{0.25} \lessdot A$ and $1_{0.22} \lessdot A$, but $(1 * 1)_{\max\{0.25, 0.22\}} = 0_{0.25} \overline{\lessdot} A$.

(b) A is not an $(\Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X, since $3_{0.27} \Upsilon A$ and $4_{0.29} \Upsilon A$, but $(3 * 4)_{\max\{0.27, 0.29\}} = 2_{0.29} \overline{\lessdot \vee \Upsilon} A$ and A is not an $(\lessdot \vee \Upsilon, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra of X.

(ii) Let B be a fuzzy set in X defined by $B(0) = 0.5$, $B(1) = 0.2$, $B(3) = B(4) = 0.6$ and $B(2) = 0.7$. Then B is not an anti fuzzy subalgebra with thresholds $r = 0.4$ and $s = 0.44$ of X since

$$\min(B(1 * 1), 0.44) > \max(B(1), B(1), 0.4).$$

Moreover B is not an anti fuzzy subalgebra of X since

$$B(1 * 1) > \max(B(1), B(1)).$$

But we can check that B is an anti fuzzy subalgebra with thresholds $r = 0.75$ and $s = 0.8$ of X.

Remark 4.4. Let $r, s \in [0, 1]$ with $r < s$. Then we can see that

(i) An anti fuzzy subalgebra (resp. $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra) is an anti fuzzy with some thresholds.

(ii) Any anti fuzzy subalgebra with thresholds $r = 0$ and $s = 1$ is an anti fuzzy subalgebra.

(iii) Any anti fuzzy subalgebra with thresholds $r = 0.5$ and $s = 1$ is an $(\lessdot, \lessdot \vee \Upsilon)^*$-fuzzy subalgebra.

(iv) Any anti fuzzy subalgebra with thresholds $r < A(x) \leq s$ or $r \leq A(x) < s$ for all $x \in X$, is an anti fuzzy subalgebra of X.

(v) If A is a fuzzy set in X and $r \geq A(x)$ for all $x \in X$, then A is an anti fuzzy subalgebra with thresholds r and s of X.

(vi) If A is a fuzzy set in X. In the following cases:

- $A(x) \leq r < s \leq A(0)$, for all $x \in X$.

- $r \leq A(x) < s \leq A(0)$, for all $x \in X$.
- $A(x) \leq r < A(0) \leq s$, for all $x \in X$.
- $r \leq A(x) < A(0) \leq s$, for all $x \in X$.

A can not be an anti fuzzy subalgebra with thresholds r and s of X.

Now, we characterize anti fuzzy subalgebras with thresholds by their level subalgebras.

Theorem 4.5. *A fuzzy subset A of X is an anti fuzzy subalgebra with thresholds (r, s) of X if and only if $L(A, t)(\neq \emptyset)$ is a subalgebra of X for all $t \in [r, s)$.*

Proof. Let A be an anti fuzzy subalgebra with thresholds (r, s) of X and $t \in [r, s)$. Let $x, y \in L(A, t)$. Then $A(x) \leq t$ and $A(y) \leq t$. Consider

$$\min(A(x * y), s) \leq \max(A(x), A(y), r) \leq \max(t, r) \leq t < s.$$

So $x * y \in L(A, t)$. Therefore $L(A, t)$ is a subalgebra of X, for all $t \in (r, s]$.

Conversely, let A be a fuzzy subset of X such that $L(A, t)(\neq \emptyset)$ is a subalgebra of X for all $t \in [r, s)$. If there exist $x, y \in X$ such that $\min(A(x*y), s) > t = \max(A(x), A(y), r)$, then $x, y \in L(A, t)$, where $t \in [r, s)$ and $A(x * y) > t$. Since $L(A, t)$ is a subalgebra of X for all $t \in [r, s)$, we get that $x * y \in L(A, t)$ and so $A(x * y) \leq t$ which is a contradiction. Therefore A is an anti fuzzy subalgebra with thresholds (r, s) of X. □

Theorem 4.6. *Let A be an anti fuzzy subalgebra with thresholds (r, s) of X. Then*

$$\min(A(0), s) \leq \max(A(x), r)$$

for all $x \in X$. In particular, if there exists $y \in X$ such that $A(y) < r$, then $A(0) < s$.

Proof. For all $x \in X$, we have

$$\min(A(0), s) = \min(A(x * x), s) \leq \max(A(x), A(x), r) = \max(A(x), r).$$

If there exists $y \in X$ such that $A(y) < r$, then $\min(A(0), s)r < s$. Hence $A(0) < s$.

□

Theorem 4.7. *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If A is an anti fuzzy subalgebra with thresholds (r, s) of X, then $f(A)$ is an anti fuzzy subalgebra with thresholds (r, s) of Y, where $f(A)(y) := \{\sup A(x) \mid f(x) = y\}$, for all $y \in Y$.*

Proof. For all $y_1, y_2 \in Y$, we have

$$\begin{aligned}
\min(f(A)(y_1 * y_2), s) &= \min(\sup\{A(x_1 * x_2) \mid f(x_1 * x_2) = y_1 * y_2\}, s) \\
&= \sup\{\min(A(x_1 * x_2), s) \mid f(x_1 * x_2) = y_1 * y_2\} \\
&\leq \sup\{\max(A(x_1), A(x_2), r) \mid f(x_1) = y_1, f(x_2) = y_2\} \\
&= \max(\sup\{A(x_1) \mid f(x_1) = y_1\}, \sup\{A(x_2), \mid f(x_2) = y_2\}, r) \\
&= \max(f(A)(y_1), f(A)(y_2), r).
\end{aligned}$$

Hence $f(A)$ is an anti fuzzy subalgebra with thresholds (r, s) of Y. □

Theorem 4.8. *Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If B is an anti fuzzy subalgebra with thresholds (r, s) of Y, then $f^{-1}(B)$ is an anti fuzzy subalgebra with thresholds (r, s) of X, where $f^{-1}(B)(x) := B(f(x))$, for all $x \in X$.*

Proof. For all $x_1, x_2 \in X$, we have

$$\begin{aligned}
\min(f^{-1}(B)(x_1 * x_2), s) &= \min(B(f(x_1 * x_2)), s) \\
&= \min(B(f(x_1) * f(x_2)), s) \\
&\leq \max(B(f(x_1)), B(f(x_2)), r) \\
&= \max(f^{-1}(B)(x_1), f^{-1}(B)(x_2), r).
\end{aligned}$$

Therefore $f^{-1}(B)$ is an anti fuzzy subalgebra with thresholds (r, s) of X. □

5. Implication-based redefined fuzzy subalgebras

An extension of set theoretic multivalued logic is fuzzy logic where the truth values are linguistic variables or terms of the linguistic variable truth. We can define the operators in fuzzy logic by using the truth table and extension principle. In fuzzy logic, the truth value of fuzzy operation Φ is denoted by $[\Phi]$.

[18] For a universe U of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper:

(1) $[x \in A] = A(x)$,
(2) $[x \notin A] = 1 - A(x)$,
(3) $[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\}$,
(4) $[\Phi \vee \Psi] = \max\{[\Phi], [\Psi]\}$,
(5) $[\Phi \to \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\}$,
(6) $[\forall x \Phi(x)] = \inf_{x \in U}\{[\Phi(x)]\}$,
(7) $\vDash \Phi$ if and only if $[\Phi] = 1$, for all valuations.

The truth valuation rules given in (3) are those in the Lukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show some of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(x,y) := \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise} \end{cases}$$

(b) Godel implication operator (I_G):

$$I_G(x,y) := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise} \end{cases}$$

(c) the contraposition of Godel implication operator ($\overline{I}_G$):

$$\overline{I}_G(x,y) := \begin{cases} 1 & \text{if } x \leq y, \\ 1-x & \text{otherwise} \end{cases}$$

(d) Kleene-Dienes operator (I_b):

$$I_b(x,y) = \max\{1-x, y\}.$$

The quality of these implication operator could be evaluated either empirically or axiomatically.

In the following definition we use the definition of implication operator:

Definition 5.1. A fuzzy set A in X is called an anti fuzzifying subalgebra of X if it satisfies the following condition:

$$\models [x \notin A] \wedge [y \notin A] \rightarrow [x * y \notin A],$$

for any $x, y \in X$.

Obviously, above condition is equivalent to the definition of anti subalgebra. Therefore an anti fuzzifying subalgebra is an anti fuzzy subalgebra.

The concept of t-tautology is introduced in [8] as:

$$\models_t \Phi \text{ if and only if } [\Phi] \geq t, \text{ for all valuations.}$$

Definition 5.2. Let A be a fuzzy set in X and $t \in [0,1)$. A is called a t-implication-based subalgebra of X if and only if satisfies

$$(\forall x, y \in X) \models_t [x \notin A] \wedge [y \notin A] \rightarrow [x * y \notin A]). \tag{6}$$

Let I be an implication operator. Then A is a t-implication-based subalgebra of X if and only if it satisfies

$$(\forall x, y \in X)\, (I(\min(1 - A(x), 1 - A(y)), 1 - A(x * y)) \geq t). \tag{7}$$

Theorem 5.3. *Let A be a fuzzy set of X. Then*

(i) If $I = I_{GR}$, then A is a 0.5-implication-based fuzzy subalgebra of X if and and only if A is an anti fuzzy subalgebra with thresholds $r = 1$ and $s = 0$ of X.

(ii) If $I = I_G$, then A is a 0.5-implication-based fuzzy subalgebra of X if and and only if A is an anti fuzzy subalgebra with thresholds $r = 0.5$ and $s = 0.5$ of X.

(iii) If $I = \overline{I}_G$, then A is a 0.5-implication-based fuzzy subalgebra of X if and and only if A is an anti fuzzy subalgebra with thresholds $r = 1$ and $s = 0.5$ of X.

Proof. (i) The proof is clear.

(ii) Let A be a 0.5-implication-based fuzzy subalgebra of X. Then

$$I_G(\min(1 - A(x), 1 - A(y)), 1 - A(x * y)) \geq 0.5,$$

hence $1-A(x*y) \geq \min(1-A(x), 1-A(y))$, or $1-A(x*y) < \min(1-A(x), 1-A(y))$ where $1-A(x*y) \geq 0.5$. Then $\min(A(x*y), 0.5) \leq \max(A(x), A(y), 0.5)$. Hence A is an anti fuzzy subalgebra with thresholds $r = 0.5$ and $s = 0.5$ of X.

Conversely, if A is an anti fuzzy subalgebra with thresholds $r = 0.5$ and $s = 0.5$ of X, then

$$\min(A(x * y), 0.5) \leq \max(A(x), A(y), 0.5).$$

If $\max(A(x), A(y), 0.5) = \max(A(x), A(y))$, then we have the following cases:

case 1) if $\min(A(x * y).0.5) = A(x * y)$,

case 2) if $\min(A(x * y).0.5) = 0.5$.

In both of them we have $I_G(\min(1 - A(x), 1 - A(y)), 1 - A(x * y)) = 1 \geq 0.5$.

Otherwise, again we have the above cases and hence

$$I_G(\min(1 - A(x), 1 - A(y)), 1 - A(x * y)) = 1 \geq 0.5.$$

Therefore A is a 0.5-implication-based fuzzy subalgebra of X.

(iii) Suppose that A is a 0.5-implication-based fuzzy subalgebra of X. Then $\overline{I}_G(\min(1-A(x), 1-A(y)), 1-A(x*y)) \geq 0.5$, thus $\min(1-A(x), 1-A(y)) \leq 1 - A(x * y)$ or $1 - \min(1 - A(x), 1 - A(y) = \max(A(x), A(y)) \geq 0.5$. Then $\min(A(x * y), 1) \leq \max(A(x), A(y), 0.5)$, so A is an anti fuzzy subalgebra with thresholds $r = 1$ and $s = 0.5$ of X.

Conversely, let A be an anti fuzzy subalgebra with thresholds $r = 1$ and $s = 0.5$ of X. Then

$$A(x * y) = \min(A(x * y), 1) \leq \max(A(x), A(y), 0.5).$$

If $\max(A(x), A(y), 0.5) = \max(A(x), A(y))$, then $I_G(\min(1-A(x), 1-A(y)), 1-A(x * y)) = 1 \geq 0.5$. Otherwise, if $\max(A(x), A(y), 0.5) = 0.5$, then $\max(A(x), A(y)) \leq 0.5$. Thus $\min(1 - A(x), 1 - A(y)) \geq 0.5$ and $1 - A(x*y) \geq 0.5$ and so $I_G(\min(1 - A(x), 1 - A(y)), 1 - A(x * y)) \geq 0.5$. Therefore A is a 0.5-implication-based fuzzy subalgebra of X.

□

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