A Shorter and Simple Approach to Study Fixed Point Results via b-Simulation Functions

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\textbf{Abstract.} The purpose of this short note is to consider much shorter and nicer proofs about fixed point results on b-metric spaces via b-simulation function introduced very recently by Demma et al. [M. Demma, R. Saadati, P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, Iranian J. Math. Sci. Infor. 11 (1) (2016) 123-136].

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1. Introduction and Preliminaries

In 2015, Khojasteh et al. [4] gave a new approach to study fixed point results in the framework of metric spaces via simulation function as follows:

A mapping \( \zeta : [0, +\infty)^2 \rightarrow \mathbb{R} \) is called a simulation function if it satisfies the following:

\[
\begin{align*}
(\zeta_1) & \quad \zeta(0,0) = 0; \\
(\zeta_2) & \quad \zeta(t,s) < s - t \text{ for all } t, s > 0; \\
(\zeta_3) & \quad \text{If }\{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that } \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0, \text{ then } \lim_{n \to \infty} \zeta(t_n, s_n) < 0.
\end{align*}
\]

Also, they denoted the set of all simulation functions by \( \mathcal{Z} \).

It is worth noticing that Argoubi et al. [1] revised the above definition by withdrawing the condition \((\zeta_1)\) (also, see [7]). Also, Roldan et al. [8] revised \((\zeta_3)\) by taking \( t_n < s_n \). Hence, we can say that a mapping \( \zeta : [0, +\infty)^2 \rightarrow \mathbb{R} \) is called a simulation function if it satisfies:

\[
\begin{align*}
(\zeta_2') & \quad \zeta(t,s) < s - t \text{ for all } t, s > 0; \\
(\zeta_3') & \quad \text{If }\{t_n\}, \{s_n\} \text{ are sequences in } (0, +\infty) \text{ such that } \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0 \text{ and } t_n < s_n \text{ for all } n \in \mathbb{N}, \text{ then } \lim_{n \to \infty} \zeta(t_n, s_n) < 0.
\end{align*}
\]

For several examples of simulation functions, see [1, 2, 4, 6, 7, 8].

**Definition 1.1.** [4] Let \((X, d)\) be a metric space and \( \zeta \in \mathcal{Z} \). Then a mapping \( T : X \rightarrow X \) is called a \( \mathcal{Z} \)-contraction with respect to \( \zeta \) if the following condition is satisfied:

\[ \zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \forall x, y \in X. \quad (1.1) \]

Now, it is clear that \( \zeta(t,t) < 0 \) when \( t > 0 \); further (1.1) implies that \( d(Tx, Ty) < d(x, y) \) when \( x \neq y \) for each \( x, y \in X \). This means that each \( \mathcal{Z} \)-contraction with respect to \( \zeta \) is continuous.

**Theorem 1.2.** [4] Let \((X, d)\) be a complete metric space and \( T : X \rightarrow X \) be a \( \mathcal{Z} \)-contraction with respect to \( \zeta \). Then \( T \) has a unique fixed point in \( X \) and for every \( x_0 \in X \), the Picard sequence \( \{x_n\} \), where \( x_n = Tx_{n-1} \) for all \( n \in \mathbb{N} \), converges to the fixed point of \( T \).

One very important and significant kind of generalized (standard) metric spaces are so-called b-metric spaces (or metric type spaces). Namely, \((X, d)\) is b-metric space if \( X \neq \emptyset \) and \( d : X \times X \rightarrow [0, +\infty) \) be a mapping such that for all \( x, y, z \in X \) hold:

\[
\begin{align*}
d(x, y) = 0 & \iff x = y; \quad d(x, y) = d(y, x) \quad \text{and} \\
d(x, y) \leq b(d(x, y) + d(y, z)) & \text{ for } b \geq 1.
\end{align*}
\]

Then \( d \) is called b-metric. For more details on b-metric spaces, see [2, 3, 5] and the references contained therein.

Recently, Demma et al. [2] introduced the b-simulation function in the framework of b-metric spaces as follows.

**Definition 1.3.** Let \((X, d)\) be a b-metric space. A b-simulation function is a function \( \zeta : [0, +\infty)^2 \rightarrow \mathbb{R} \) satisfying the following:
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\[(\xi_1)\] \(\xi(t,s) < s - t\) for all \(t, s > 0\);

\[(\xi_2)\] if \(\{t_n\}, \{s_n\}\) are sequences in \((0, +\infty)\) such that

\[
0 < \lim_{n \to +\infty} t_n \leq \lim_{n \to +\infty} s_n \leq \lim_{n \to +\infty} t_n < +\infty,
\]

then \(\lim_{n \to +\infty} \xi(bt_n, s_n) < 0\).

It is clear if \(b = 1\), then b-simulation function is in the fact the simulation function in the framework of (standard) metric spaces.

**Example 1.4.** [2] Let \(\xi : [0, +\infty)^2 \to \mathbb{R}\) be defined by

(i) \(\xi(t, s) = \lambda s - t\) for all \(t, s \in [0, +\infty)\), where \(\lambda \in [0, 1)\).

(ii) \(\xi(t, s) = \psi(s) - \varphi(t)\) for all \(t, s \in [0, +\infty)\), where \(\varphi, \psi : [0, +\infty) \to [0, +\infty)\) are two continuous functions such that \(\psi(t) = \varphi(t) = 0\) if and only if \(t = 0\) and \(\varphi(t) < t \leq \varphi(t)\) for all \(t > 0\).

(iii) \(\xi(t, s) = s - \frac{f(t,s)}{g(t,s)} t\) for all \(t, s \in [0, +\infty)\), where \(f, g : [0, +\infty)^2 \to (0, +\infty)\) are two continuous functions with respect to each variable such that \(f(t, s) > g(t, s)\) for all \(t, s > 0\).

(iv) \(\xi(t, s) = s - \varphi(s) - t\) for all \(t, s \in [0, +\infty)\), where \(\varphi : [0, +\infty) \to [0, +\infty)\) is a lower semi-continuous function such that \(\varphi(t) = 0\) if and only if \(t = 0\).

(v) \(\xi(t, s) = s\varphi(s) - t\) for all \(t, s \in [0, +\infty)\), where \(\varphi : [0, +\infty) \to [0, 1)\) is such that \(\lim_{t \to +\infty} \varphi(t) < 1\) for all \(r > 0\).

Each of the function considered in (i)-(v) is a b-simulation function.

The following important and very interesting results are proved in [2].

**Lemma 1.5.** Let \((X, d)\) be a b-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that following condition holds.

\[\xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X.\]  \(\text{(1.3)}\)

Let \(\{x_n\}\) be a sequence of Picard of initial at point \(x_0 \in X\) and \(x_{n-1} \neq x_n\) for all \(n \in \mathbb{N}\). Then

\[
\lim_{n \to +\infty} d(x_{n-1}, x_n) = 0.
\]

**Lemma 1.6.** Let \((X, d)\) be a b-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that \(\text{(1.3)}\) holds. Let \(\{x_n\}\) be a sequence of Picard of initial at point \(x_0 \in X\) and \(x_{n-1} \neq x_n\) for all \(n \in \mathbb{N}\). Then \(\{x_n\}\) is a bounded sequence.

**Lemma 1.7.** Let \((X, d)\) be a b-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that \(\text{(1.3)}\) holds. Let \(\{x_n\}\) be a sequence of Picard of initial at point \(x_0 \in X\) and \(x_{n-1} \neq x_n\) for all \(n \in \mathbb{N}\). Then \(\{x_n\}\) is a Cauchy sequence.
Theorem 1.8. Let \((X, d)\) be a complete b-metric space and let \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that (1.3) holds; that is,
\[
\xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]
Then \(f\) has a unique fixed point.

For the proof of Theorem 1.8, Demma et al. [2] used Lemmas 1.5-1.7.

2. Main results

In this section we improve the main result from [2]; that is, we prove Theorem 1.8 without using all three lemmas 1.5-1.7. At the first, we quote some well known results from b-metric spaces. The following lemma was used (and proved) in the course of proofs of several fixed point results in the framework of b-metric spaces in [3].

Lemma 2.1. Let \(\{y_n\}\) be a sequence in a b-metric space \((X, d)\) such that
\[
d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \tag{2.1}
\]
for some \(\lambda, 0 \leq \lambda < \frac{1}{b}\) and each \(n = 1, 2, \ldots\). Then \(\{y_n\}\) is a Cauchy sequence in \((X, d)\).

By utilizing Lemma 2.1, Jovanić et al. [3] proved following result.

Theorem 2.2. Let \((X, d)\) be a complete b-metric space and \(f : X \to X\) be a map such that
\[
d(fx, fy) \leq \lambda d(x, y) \tag{2.2}
\]
holds for all \(x, y \in X\), where \(0 \leq \lambda < \frac{1}{b}\). Then \(f\) has a unique fixed point \(z\) and for every \(x_0 \in X\), the sequence \(\{f^n x_0\}\) converges to \(z\).

Now we formulate and prove Theorem 1.8 via a shorter and simple approach.

Theorem 2.3. Let \((X, d)\) be a complete b-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that (1.3) holds; that is,
\[
\xi(bd(fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X. \tag{2.3}
\]
Then \(f\) has a unique fixed point.

Proof. It is enough clear that (2.3) implies
\[
bd(fx, fy) \leq d(x, y) \quad \forall x, y \in X. \tag{2.4}
\]
Indeed, (2.4) holds if \(x = y\). In the case that \(x \neq y\) there are two possibilities, either \(fx = fy\) or \(fx \neq fy\). In the first case we have that \(b \cdot d(fx, fy) = 0 < d(x, y)\), while in second case the result follows from \((\xi_1)\). This means that (2.3) implies (2.4) for all \(x, y \in X\). Further, obviously, (2.4) implies that
\[
d(f^2 x, f^2 y) \leq \frac{1}{b^2} d(x, y) = \lambda d(x, y). \tag{2.5}
\]
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Since \( \lambda = \frac{1}{b^2} \in [0, \frac{1}{b}) \), then according to Theorem 2.2, \( f^2 \) has a unique fixed point (say \( z \)) in \( X \). This further means that \( f \) has a unique fixed point \( z \) in \( X \). Now, the proof of this theorem is complete. \( \square \)

Obviously, our proof is much shorter than the corresponding ones from Demma et al.'s work [2]. It is very interesting that all four Corollaries 4.1-4.4 from [2] follows immediately according to our easy approach. Thus we have following corollary.

**Corollary 2.4.** Let \((X,d)\) be a complete b-metric space and let \( f : X \to X \) be a mapping. Suppose that

(i) \( \lambda \in [0,1) \) such that \( bd(fx,fy) \leq \lambda d(x,y) \);

(ii) a lower semi-continuous function \( \varphi : [0, +\infty) \to [0, \infty) \) with \( \varphi^{-1}(0) = \{0\} \) such that \( bd(fx,fy) \leq \varphi(d(x,y)) d(x,y) \);

(iii) \( \varphi : [0, +\infty) \to [0,1) \) with \( \lim_{t \to r^+} \varphi(t) < 1 \) for all \( r > 0 \) such that \( bd(fx,fy) \leq \varphi(d(x,y)) d(x,y) \);

(iv) \( \eta : [0, +\infty) \to [0, \infty) \) with \( \eta(t) < t \) for all \( t > 0 \) and \( \eta(0) = 0 \) such that \( bd(fx,fy) \leq \eta(d(x,y)) \)

for all \( x,y \in X \). Then \( f \) has a unique fixed point in each one of above condition.

**Proof.** Obviously, each one of mentioned conditions implies the condition (2.4) by selecting the appropriate b-simulation function in Example 1.4. Hence, we obtain that \( bd(fx,fy) \leq d(x,y) \) for all \( x,y \in X \). The result then follows according to Theorem 2.3. \( \square \)

**Example 2.5.** Now, we consider Example 4.5 from [2]. Let \( X = [0,1] \) and \( d : X \times X \to \mathbb{R} \) be defined by \( d(x,y) = |x - y|^2 \). Then \((X,d)\) is a complete b-metric space with \( b = 2 \). Consider a mapping \( f : X \to X \) by

\[
fx = \frac{ax}{1 + x}
\]

for all \( x \in X \), where \( a \in [0, \frac{1}{\sqrt{2}}] \). Now, we have

\[
2d(fx,fy) = 2 \left| \frac{ax}{1 + x} - \frac{ay}{1 + y} \right|^2 = 2a^2 \frac{|x - y|^2}{(1 + x)^2 (1 + y)^2} \leq |x - y|^2 = d(x,y)
\]

(2.6)

for all \( x,y \in X \). Further, (2.6) implies that

\[
d(f^2x, f^2y) \leq \frac{1}{4} d(x,y);
\]

that is, \( f^2 \) has a unique fixed point according to Theorem 2.2. This means that \( f \) has a unique fixed point. Here it is \( z = 0 \).

The next result is probably known, but our proof is very condensed.
Theorem 2.6. Let $(X,d)$ be a complete $b$-metric space and let $f : X \to X$ be a mapping such that
\[
d (fx, fy) \leq \lambda d (x, y)
\]
for all $x, y \in X$, where $\lambda \in [0, 1)$. Then $f$ has a unique fixed point (say $z$) in $X$ and for $x_0 \in X$ the sequence $\{f^n x_0\}_{n \in \mathbb{N}}$ converges to $z$.

Proof. The condition (2.7) implies that
\[
d (f^n x, f^n y) \leq \lambda d (f^{n-1} x, f^{n-1} y) \leq \cdots \leq \lambda^n d (x, y)
\]
for all $x, y \in X$ and $n \in \mathbb{N}$. Since $\lambda^n \to 0$ as $n \to \infty$, there is $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{b}$. Therefore, we have
\[
d (f^{k+1} x, f^{k+1} y) \leq \frac{1}{b^2} d (x, y).
\]
The result now follows by Theorem 2.2. \qed

Question 1. Does Theorem 2.3 holds if $\xi (d (fx, fy), d (x, y)) \geq 0$ for all $x, y \in X$, where $(X,d)$ is a given complete $b$-metric space and $f : X \to X$ be a mapping and $\xi$ a given $b$-simulation function?

Question 2. Can you obtain this results by considering ordered $b$-metric spaces or cone $b$-metric spaces instead of $b$-metric spaces?

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