A Shorter and Simple Approach to Study Fixed Point Results via $b$-Simulation Functions

G. Soleimani Rad$^a$,* S. Radenović$^{b,c}$, D. Doličanin-Dekić$^d$

$^a$ Young Researchers and Elite club, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
$^b$ Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia.
$^c$ State University of Novi Pazar, Serbia.
$^d$ Faculty of Technical Sciences, Kneza Mioša 7, 38 220 Kosovska Mitrovica, Serbia.

E-mail: gha.soleimani.sci@iauctb.ac.ir
E-mail: radens@beotel.net
E-mail: diana.dolicanin@pr.ac.rs

Abstract. The purpose of this short note is to consider much shorter and nicer proofs about fixed point results on $b$-metric spaces via $b$-simulation function introduced very recently by Demma et al. [M. Demma, R. Saadati, P. Vetro, Fixed point results on $b$-metric space via Picard sequences and $b$-simulation functions, Iranian J. Math. Sci. Infor. 11 (1) (2016) 123-136].

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1. Introduction and Preliminaries

In 2015, Khojasteh et al. [4] gave a new approach to study fixed point results in the framework of metric spaces via simulation function as follows:

A mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following:

- $(\zeta_1)$ $\zeta(0, 0) = 0$;
- $(\zeta_2)$ $\zeta(t, s) < s - t$ for all $t, s > 0$;
- $(\zeta_3)$ if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0$, then $\lim_{n \to \infty} \zeta(t_n, s_n) < 0$.

Also, they denoted the set of all simulation functions by $Z$.

It is worth noticing that Argoubi et al. [1] revised the above definition by withdrawing the condition $(\zeta_1)$ (also, see [7]). Also, Roldan et al. [8] revised $(\zeta_3)$ by taking $t_n < s_n$. Hence, we can say that a mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies:

- $(\zeta_2)$ $\zeta(t, s) < s - t$ for all $t, s > 0$;
- $(\zeta_3)$ if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\lim_{n \to \infty} \zeta(t_n, s_n) < 0$.

For several examples of simulation functions, see [1, 2, 4, 6, 7, 8].

**Definition 1.1.** [4] Let $(X, d)$ be a metric space and $\zeta \in Z$. Then a mapping $T : X \to X$ is called a $Z$-contraction with respect to $\zeta$ if the following condition is satisfied:

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \forall x, y \in X. \tag{1.1}$$

Now, it is clear that $\zeta(t, t) < 0$ when $t > 0$; further (1.1) implies that $d(Tx, Ty) < d(x, y)$ when $x \neq y$ for each $x, y \in X$. This means that each $Z$-contraction with respect to $\zeta$ is continuous.

**Theorem 1.2.** [4] Let $(X, d)$ be a complete metric space and $T : X \to X$ be a $Z$-contraction with respect to $\zeta$. Then $T$ has a unique fixed point in $X$ and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of $T$.

One very important and significant kind of generalized (standard) metric spaces are so-called b-metric spaces (or metric type spaces). Namely, $(X, d)$ is b-metric space if $X \neq \emptyset$ and $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y, z \in X$ hold: $d(x, y) = 0 \iff x = y; d(x, y) = d(y, x)$ and $d(x, y) \leq b(d(x, z) + d(y, z))$ for $b \geq 1$. Then $d$ is called b-metric. For more details on b-metric spaces, see [2, 3, 5] and the references contained therein.

Recently, Demma et al. [2] introduced the b-simulation function in the framework of b-metric spaces as follows.

**Definition 1.3.** Let $(X, d)$ be a b-metric space. A b-simulation function is a function $\zeta : [0, +\infty)^2 \to \mathbb{R}$ satisfying the following:
\((\xi_1)\) \(\xi(t,s) < s - t\) for all \(t, s > 0\);

\((\xi_2)\) if \(\{t_n\}, \{s_n\}\) are sequences in \((0, +\infty)\) such that

\[
0 < \lim_{n \to +\infty} t_n \leq \lim_{n \to +\infty} s_n \leq \lim_{n \to +\infty} b \lim_{n \to +\infty} t_n < +\infty,
\]

then \(\lim_{n \to +\infty} \xi(bt_n, s_n) < 0\).

It is clear if \(b = 1\), then b-simulation function is in the fact the simulation function in the framework of (standard) metric spaces.

**Example 1.4.** [2] Let \(\xi : [0, +\infty)^2 \to \mathbb{R}\) be defined by

(i) \(\xi(t, s) = \lambda s - t\) for all \(t, s \in [0, +\infty)\), where \(\lambda \in [0, 1)\).

(ii) \(\xi(t, s) = \psi(s) - \varphi(t)\) for all \(t, s \in [0, +\infty)\), where \(\varphi, \psi : [0, +\infty) \to [0, +\infty)\) are two continuous functions such that \(\psi(t) = \varphi(t) = 0\) if and only if \(t = 0\) and \(\psi(t) < t \leq \varphi(t)\) for all \(t > 0\).

(iii) \(\xi(t, s) = s - \frac{f(t, s)}{g(t, s)} t\) for all \(t, s \in [0, +\infty)\), where \(f, g : [0, +\infty)^2 \to (0, +\infty)\) are two continuous functions with respect to each variable such that \(f(t, s) > g(t, s)\) for all \(t, s > 0\).

(iv) \(\xi(t, s) = s - \varphi(s) - t\) for all \(t, s \in [0, +\infty)\), where \(\varphi : [0, +\infty) \to [0, +\infty)\) is a lower semi-continuous function such that \(\varphi(t) = 0\) if and only if \(t = 0\).

(v) \(\xi(t, s) = \varphi(s) - t\) for all \(t, s \in [0, +\infty)\), where \(\varphi : [0, +\infty) \to [0, 1)\) is such that \(\lim_{t \to 0^+} \varphi(t) < 1\) for all \(r > 0\).

Each of the function considered in (i)-(v) is a b-simulation function.

The following important and very interesting results are proved in [2].

**Lemma 1.5.** Let \((X, d)\) be a b-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that the following condition holds.

\[
\xi(bd(f(x, f(y)), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]

(1.3)

Let \(\{x_n\}\) be a sequence of Picard of initial at point \(x_0 \in X\) and \(x_{n-1} \neq x_n\) for all \(n \in \mathbb{N}\). Then

\[
\lim_{n \to \infty} d(x_{n-1}, x_n) = 0.
\]

**Lemma 1.6.** Let \((X, d)\) be a b-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that (1.3) holds. Let \(\{x_n\}\) be a sequence of Picard of initial at point \(x_0 \in X\) and \(x_{n-1} \neq x_n\) for all \(n \in \mathbb{N}\). Then \(\{x_n\}\) is a bounded sequence.

**Lemma 1.7.** Let \((X, d)\) be a b-metric space and \(f : X \to X\) be a mapping. Suppose that there exists a b-simulation function \(\xi\) such that (1.3) holds. Let \(\{x_n\}\) be a sequence of Picard of initial at point \(x_0 \in X\) and \(x_{n-1} \neq x_n\) for all \(n \in \mathbb{N}\). Then \(\{x_n\}\) is a Cauchy sequence.
Theorem 1.8. Let $(X, d)$ be a complete $b$-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a $b$-simulation function $\xi$ such that (1.3) holds; that is,
\[
\xi (bd (fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]
Then $f$ has a unique fixed point.

For the proof of Theorem 1.8, Demma et al. [2] used Lemmas 1.5-1.7.

2. Main results

In this section we improve the main result from [2]; that is, we prove Theorem 1.8 without using all three lemmas 1.5-1.7. At the first, we quote some well known results from $b$-metric spaces. The following lemma was used (and proved) in the course of proofs of several fixed point results in the framework of $b$-metric spaces in [3].

Lemma 2.1. Let $\{y_n\}$ be a sequence in a $b$-metric space $(X, d)$ such that
\[
d (y_n, y_{n+1}) \leq \lambda d (y_{n-1}, y_n)
\]
for some $\lambda$, $0 \leq \lambda < \frac{1}{b}$ and each $n = 1, 2, \ldots$. Then $\{y_n\}$ is a Cauchy sequence in $(X, d)$.

By utilizing Lemma 2.1, Jovanović et al. [3] proved following result.

Theorem 2.2. Let $(X, d)$ be a complete $b$-metric space and $f : X \to X$ be a map such that
\[
d (fx, fy) \leq \lambda d(x, y)
\]
holds for all $x, y \in X$, where $0 \leq \lambda < \frac{1}{b}$. Then $f$ has a unique fixed point $z$ and for every $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to $z$.

Now we formulate and prove Theorem 1.8 via a shorter and simple approach.

Theorem 2.3. Let $(X, d)$ be a complete $b$-metric space and $f : X \to X$ be a mapping. Suppose that there exists a $b$-simulation function $\xi$ such that (1.3) holds; that is,
\[
\xi (bd (fx, fy), d(x, y)) \geq 0 \quad \forall x, y \in X.
\]
Then $f$ has a unique fixed point.

Proof. It is enough clear that (2.3) implies
\[
bd (fx, fy) \leq d(x, y) \quad \forall x, y \in X.
\]
Indeed, (2.4) holds if $x = y$. In the case that $x \neq y$ there are two possibilities, either $fx = fy$ or $fx \neq fy$. In the first case we have that $b \cdot d (fx, fy) = 0 < d(x, y)$, while in second case the result follows from ($\xi_1$). This means that (2.3) implies (2.4) for all $x, y \in X$. Further, obviously, (2.4) implies that
\[
d (f^2 x, f^2 y) \leq \frac{1}{b^2} d(x, y) = \lambda d(x, y).
\]
Since $\lambda = \frac{1}{b^2} \in [0, \frac{1}{b})$, then according to Theorem 2.2, $f^2$ has a unique fixed point (say $z$) in $X$. This further means that $f$ has a unique fixed point $z$ in $X$. Now, the proof of this theorem is complete. □

Obviously, our proof is much shorter than the corresponding ones from Demma et al.’s work [2]. It is very interesting that all four Corollaries 4.1-4.4 from [2] follows immediately according to our easy approach. Thus we have following corollary.

**Corollary 2.4.** Let $(X, d)$ be a complete $b$-metric space and let $f : X \rightarrow X$ be a mapping. Suppose that

1. $\lambda \in [0, 1)$ such that $bd(fx, fy) \leq \lambda d(x, y)$;
2. a lower semi-continuous function $\varphi : [0, +\infty) \rightarrow [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$ such that $bd(fx, fy) \leq d(x, y) - \varphi(d(x, y))$;
3. $\varphi : [0, +\infty) \rightarrow [0, 1)$ with $\lim_{t \rightarrow r^+} \varphi(t) < 1$ for all $r > 0$ such that $bd(fx, fy) \leq \varphi(d(x, y)) d(x, y)$;
4. $\eta : [0, +\infty) \rightarrow [0, \infty)$ with $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$ such that $bd(fx, fy) \leq \eta(d(x, y))$ for all $x, y \in X$. Then $f$ has a unique fixed point in each one of above condition.

**Proof.** Obviously, each one of mentioned conditions implies the condition (2.4) by selecting the appropriate $b$-simulation function in Example 1.4. Hence, we obtain that $bd(fx, fy) \leq d(x, y)$ for all $x, y \in X$. The result then follows according to Theorem 2.3. □

**Example 2.5.** Now, we consider Example 4.5 from [2]. Let $X = [0, 1]$ and $d : X \times X \rightarrow \mathbb{R}$ be defined by $d(x, y) = |x - y|^2$. Then $(X, d)$ is a complete $b$-metric space with $b = 2$. Consider a mapping $f : X \rightarrow X$ by

$$fx = \frac{ax}{1+x}$$

for all $x \in X$, where $a \in [0, \frac{1}{\sqrt{2}}]$. Now, we have

$$2d(fx, fy) = 2 \left| \frac{ax}{1+x} - \frac{ay}{1+y} \right|^2 = 2a^2 \frac{|x-y|^2}{(1+x)^2(1+y)^2} \leq |x-y|^2 = d(x, y)$$

(2.6)

for all $x, y \in X$. Further, (2.6) implies that

$$d(f^2x, f^2y) \leq \frac{1}{4}d(x, y);$$

that is, $f^2$ has a unique fixed point according to Theorem 2.2. This means that $f$ has a unique fixed point. Here it is $z = 0$.

The next result is probably known, but our proof is very condensed.
Theorem 2.6. Let \((X, d)\) be a complete b-metric space and let \(f : X \to X\) be a mapping such that
\[
d (fx, fy) \leq \lambda d (x, y)
\]
for all \(x, y \in X\), where \(\lambda \in [0, 1)\). Then \(f\) has a unique fixed point (say \(z\)) in \(X\) and for \(x_0 \in X\) the sequence \(\{f^n x_0\}_{n \in \mathbb{N}}\) converges to \(z\).

Proof. The condition (2.7) implies that
\[
d (f^n x, f^n y) \leq \lambda d (f^{n-1} x, f^{n-1} y) \leq \cdots \leq \lambda^n d (x, y)
\]
for all \(x, y \in X\) and \(n \in \mathbb{N}\). Since \(\lambda^n \to 0\) as \(n \to \infty\), there is \(k \in \mathbb{N}\) such that \(\lambda^k < \frac{1}{b}\). Therefore, we have
\[
d (f^{k+1} x, f^{k+1} y) \leq \frac{1}{b^2} d (x, y).
\]
The result now follows by Theorem 2.2.

Question 1. Does Theorem 2.3 holds if \(\xi (d (fx, fy), d (x, y)) \geq 0\) for all \(x, y \in X\), where \((X, d)\) is a given complete b-metric space and \(f : X \to X\) be a mapping and \(\xi\) a given b-simulation function?

Question 2. Can you obtain this results by considering ordered b-metric spaces or cone b-metric spaces instead of b-metric spaces?

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References