Iranian Journal of Mathematical Sciences and Informatics
Vol. 13, No. 1 (2018), pp 97-102
DOI: 10.7508/ijmsi.2018.1.009

A Shorter and Simple Approach to Study Fixed Point Results via b-Simulation Functions

G. Soleimani Rad[a,*] , S. Radenović[b,c], D. Dolićanin-Đekić[d]

[a] Young Researchers and Elite club, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
[b] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia.
[c] State University of Novi Pazar, Serbia.
[d] Faculty of Technical Sciences, Kneza Miloša 7, 38 220 Kosovska Mitrovica, Serbia.

E-mail: gha.soleimani.sci@iauctb.ac.ir
E-mail: radens@beotel.net
E-mail: diana.dolicanin@pr.ac.rs

Abstract. The purpose of this short note is to consider much shorter and nicer proofs about fixed point results on b-metric spaces via b-simulation function introduced very recently by Demma et al. [M. Demma, R. Saadati, P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, Iranian J. Math. Sci. Infor. 11 (1) (2016) 123-136].

Keywords: b-Metric space, b-Simulation function, Cauchy sequence, Lower semi-continuous.

2000 Mathematics subject classification: 47H10, 54C30, 54H25.

*Corresponding Author

Received 02 November 2016; Accepted 21 January 2017

1. Introduction and Preliminaries

In 2015, Khojasteh et al. [4] gave a new approach to study fixed point results in the framework of metric spaces via simulation function as follows:

A mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\overline{\lim}_{n\to\infty} \zeta(t_n, s_n) < 0$.

Also, they denoted the set of all simulation functions by $\mathcal{Z}$.

It is worth noticing that Argoubi et al. [1] revised the above definition by withdrawing the condition (ζ_1) (also, see [7]). Also, Roldan et al. [8] revised (ζ_3) by taking $t_n < s_n$. Hence, we can say that a mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies:

(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\overline{\lim}_{n\to\infty} \zeta(t_n, s_n) < 0$.

For several examples of simulation functions, see [1, 2, 4, 6, 7, 8].

Definition 1.1. [4] Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. Then a mapping $T : X \to X$ is called a $\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied:

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \qquad \forall x, y \in X. \tag{1.1}$$

Now, it is clear that $\zeta(t, t) < 0$ when $t > 0$; further (1.1) implies that $d(Tx, Ty) < d(x, y)$ when $x \neq y$ for each $x, y \in X$. This means that each $\mathcal{Z}$-contraction with respect to ζ is continuous.

Theorem 1.2. [4] *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then T has a unique fixed point in X and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of T.*

One very important and significant kind of generalized (standard) metric spaces are so-called b-metric spaces (or metric type spaces). Namely, (X, d) is b-metric space if $X \neq \emptyset$ and $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y, z \in X$ hold: $d(x, y) = 0 \Leftrightarrow x = y; d(x, y) = d(y, x)$ and $d(x, y) \leq b(d(x, y) + d(y, z))$ for $b \geq 1$. Then d is called $b-$metric. For more details on b-metric spaces, see [2, 3, 5] and the references contained therein.

Recently, Demma et al. [2] introduced the b-simulation function in the framework of b-metric spaces as follows:

Definition 1.3. Let (X, d) be a b-metric space. A b-simulation function is a function $\zeta : [0, +\infty)^2 \to \mathbb{R}$ satisfying the following:

(ξ_1) $\xi(t,s) < s - t$ for all $t, s > 0$;
(ξ_2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that

$$0 < \lim_{n\to+\infty} t_n \leq \underline{\lim}_{n\to+\infty} s_n \leq \overline{\lim}_{n\to\infty} s_n \leq b \lim_{n\to+\infty} t_n < +\infty, \tag{1.2}$$

then $\overline{\lim_{n\to\infty}} \xi(bt_n, s_n) < 0$.

It is clear if $b = 1$, then b-simulation function is in the fact the simulation function in the framework of (standard) metric spaces.

Example 1.4. [2] Let $\xi : [0, +\infty)^2 \to \mathbb{R}$ be defined by
(i) $\xi(t,s) = \lambda s - t$ for all $t, s \in [0, +\infty)$, where $\lambda \in [0, 1)$.
(ii) $\xi(t,s) = \psi(s) - \varphi(t)$ for all $t, s \in [0, +\infty)$, where $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \varphi(t)$ for all $t > 0$.
(iii) $\xi(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0, +\infty)$, where $f, g : [0, +\infty)^2 \to (0, +\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$.
(iv) $\xi(t,s) = s - \varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
(v) $\xi(t,s) = s\varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, 1)$ is such that $\lim_{t\to r^+} \varphi(t) < 1$ for all $r > 0$.
Each of the function considered in (i)-(v) is a b-simulation function.

The following important and very interesting results are proved in [2].

Lemma 1.5. *Let (X, d) be a b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that following condition holds.*

$$\xi(bd(fx, fy), d(x,y)) \geq 0 \qquad \forall x, y \in X. \tag{1.3}$$

Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0.$$

Lemma 1.6. *Let (X, d) be a b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that* (1.3) *holds. Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a bounded sequence.*

Lemma 1.7. *Let (X, d) be a b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that* (1.3) *holds. Let $\{x_n\}$ be a sequence of Picard of initial at point $x_0 \in X$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.*

Theorem 1.8. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that* (1.3) *holds; that is,*

$$\xi\left(bd\left(fx, fy\right), d\left(x, y\right)\right) \geq 0 \qquad \forall x, y \in X.$$

Then f has a unique fixed point.

For the proof of Theorem 1.8, Demma et al. [2] used Lemmas 1.5-1.7.

2. Main results

In this section we improve the main result from [2]; that is, we prove Theorem 1.8 without using all three lemmas 1.5-1.7. At the first, we quote some well known results from b-metric spaces. The following lemma was used (and proved) in the course of proofs of several fixed point results in the framework of b-metric spaces in [3].

Lemma 2.1. *Let $\{y_n\}$ be a sequence in a b-metric space (X, d) such that*

$$d\left(y_n, y_{n+1}\right) \leq \lambda d\left(y_{n-1}, y_n\right) \tag{2.1}$$

for some λ, $0 \leq \lambda < \frac{1}{b}$ and each $n = 1, 2, \cdots$. Then $\{y_n\}$ is a Cauchy sequence in (X, d).

By utilizing Lemma 2.1, Jovanović et al. [3] proved following result.

Theorem 2.2. *Let (X, d) be a complete b-metric space and $f : X \to X$ be a map such that*

$$d\left(fx, fy\right) \leq \lambda d\left(x, y\right) \tag{2.2}$$

holds for all $x, y \in X$, where $0 \leq \lambda < \frac{1}{b}$. Then f has a unique fixed point z and for every $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to z.

Now we formulate and prove Theorem 1.8 via a shorter and simple approach.

Theorem 2.3. *Let (X, d) be a complete b-metric space and $f : X \to X$ be a mapping. Suppose that there exists a b-simulation function ξ such that* (1.3) *holds; that is,*

$$\xi\left(bd\left(fx, fy\right), d\left(x, y\right)\right) \geq 0 \qquad \forall x, y \in X. \tag{2.3}$$

Then f has a unique fixed point.

Proof. It is enough clear that (2.3) implies

$$bd\left(fx, fy\right) \leq d\left(x, y\right) \qquad \forall x, y \in X. \tag{2.4}$$

Indeed, (2.4) holds if $x = y$. In the case that $x \neq y$ there are two possibilities, either $fx = fy$ or $fx \neq fy$. In the first case we have that $b \cdot d\left(fx, fy\right) = 0 < d\left(x, y\right)$, while in second case the result follows from (ξ_1). This means that (2.3) implies (2.4) for all $x, y \in X$. Further, obviously, (2.4) implies that

$$d\left(f^2 x, f^2 y\right) \leq \frac{1}{b^2} d\left(x, y\right) = \lambda d\left(x, y\right). \tag{2.5}$$

Since $\lambda = \frac{1}{b^2} \in [0, \frac{1}{b})$, then according to Theorem 2.2, f^2 has a unique fixed point (say z) in X. This further means that f has a unique fixed point z in X. Now, the proof of this theorem is complete. $\square$

Obviously, our proof is much shorter than the corresponding ones from Demma et al.'s work [2]. It is very interesting that all four Corollaries 4.1-4.4 from [2] follows immediately according to our easy approach. Thus we have following corollary.

Corollary 2.4. *Let (X, d) be a complete b-metric space and let $f : X \to X$ be a mapping. Suppose that*

(i) $\lambda \in [0, 1)$ *such that* $bd(fx, fy) \leq \lambda d(x, y)$;
(ii) *a lower semi-continuous function* $\varphi : [0, +\infty) \to [0, \infty)$ *with* $\varphi^{-1}(0) = \{0\}$ *such that* $bd(fx, fy) \leq d(x, y) - \varphi(d(x, y))$;
(iii) $\varphi : [0, +\infty) \to [0, 1)$ *with* $\lim\limits_{t \to r^+} \varphi(t) < 1$ *for all* $r > 0$ *such that* $bd(fx, fy) \leq \varphi(d(x, y))\, d(x, y)$;
(iv) $\eta : [0, +\infty) \to [0, \infty)$ *with* $\eta(t) < t$ *for all* $t > 0$ *and* $\eta(0) = 0$ *such that* $bd(fx, fy) \leq \eta(d(x, y))$

for all $x, y \in X$. *Then* f *has a unique fixed point in each one of above condition.*

Proof. Obviously, each one of mentioned conditions implies the condition (2.4) by selecting the appropriate b-simulation function in Example 1.4. Hence, we obtain that $bd(fx, fy) \leq d(x, y)$ for all $x, y \in X$. The result then follows according to Theorem 2.3. $\square$

Example 2.5. Now, we consider Example 4.5 from [2]. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|^2$. Then (X, d) is a complete b-metric space with $b = 2$. Consider a mapping $f : X \to X$ by

$$fx = \frac{ax}{1 + x}$$

for all $x \in X$, where $a \in [0, \frac{1}{\sqrt{2}}]$. Now, we have

$$2d(fx, fy) = 2\left|\frac{ax}{1+x} - \frac{ay}{1+y}\right|^2 = 2a^2 \frac{|x-y|^2}{(1+x)^2(1+y)^2} \leq |x-y|^2 = d(x, y) \tag{2.6}$$

for all $x, y \in X$. Further, (2.6) implies that

$$d\left(f^2 x, f^2 y\right) \leq \frac{1}{4} d(x, y);$$

that is, f^2 has a unique fixed point according to Theorem 2.2. This means that f has a unique fixed point. Here it is $z = 0$.

The next result is probably known, but our proof is very condensed.

Theorem 2.6. *Let (X,d) be a complete b-metric space and let $f : X \to X$ be a mapping such that*

$$d(fx, fy) \leq \lambda d(x,y) \tag{2.7}$$

for all $x, y \in X$, where $\lambda \in [0,1)$. Then f has a unique fixed point (say z) in X and for $x_0 \in X$ the sequence $\{f^n x_0\}_{n\in\mathbb{N}}$ converges to z.

Proof. The condition (2.7) implies that

$$d(f^n x, f^n y) \leq \lambda d\left(f^{n-1}x, f^{n-1}y\right) \leq \cdots \leq \lambda^n d(x,y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $\lambda^n \to 0$ as $n \to \infty$, there is $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{b}$. Therefore, we have

$$d\left(f^{k+1}x, f^{k+1}y\right) \leq \frac{1}{b^2} d(x,y).$$

The result now follows by Theorem 2.2. □

Question 1. Does Theorem 2.3 holds if $\xi(d(fx,fy), d(x,y)) \geq 0$ for all $x, y \in X$, where (X,d) is a given complete b-metric space and $f : X \to X$ be a mapping and ξ a given b-simulation function?

Question 2. Can you obtain this results by considering ordered b-metric spaces or cone b-metric spaces instead of b-metric spaces?

Acknowledgments

The authors are grateful to the editor-in-chief and referee for their accurate reading. Also, the first author would like to thank the Young Researchers and Elite club, Central Tehran Branch of Islamic Azad University, for financial support.

References

1. H. Argoubi, B. Samet, C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, *J. Nonlinear Sci. Appl.*, **8**, (2015), 1082-1094.
2. M. Demma, R. Saadati, P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, *Iranian Journal of Mathematical Sciences and Informatics*, **11**(1), (2016), 123-136.
3. M. Jovanović, Z. Kadelburg, S. Radenović, Common fixed point results in metric-type spaces, *Fixed Point Theory Appl.*, **2010**, 978121, 15 pages.
4. F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theorems via simulation functions, *Filomat*, **29**(6), (2015), 1189-1194.
5. S.K. Mohanta, Coincidence points and common fixed points for expansive type pappings in b-metric spaces, *Iranian Journal of Mathematical Sciences and Informatics*, **11**(1), (2016), 101-113.
6. A. Nastasi, P. Vetro, Existence and uniqueness for a first-order periodic differential problem via fixed point results, *Results. Math.*, (2016), doi: 10.1007/s00025-016-0551-x.
7. A. Nastasi, P. Vetro, Fixed point results on metric and partial metrric spaces via simulations functions, *J. Nonlinear Sci. Appl.*, **8**, (2015), 1059-1069.
8. A. Roldan, E. Karapınar, C. Roldan, J. Martinez-Moreno, Coincidence point theorems on metric spaces via simulation functions, *J. Comut. Appl. Math.*, **275**, (2015), 345-355.