Abstract. Let $R$ be a commutative ring with identity and $M$ be an $R$-module. The zero divisor graph of $M$ is denoted by $\Gamma(M)$. In this study, we are going to generalize the zero divisor graph $\Gamma(M)$ to submodule-based zero divisor graph $\Gamma(M, N)$ by replacing elements whose product is zero with elements whose product is in some submodule $N$ of $M$. The main objective of this paper is to study the interplay of the properties of submodule $N$ and the properties of $\Gamma(M, N)$.

Keywords: Zero divisor graph, Submodule-based zero divisor graph, Semisimple module.


1. Introduction

Let $R$ be a commutative ring with identity. The zero divisor graph of $R$, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero divisor of $R$ with two distinct vertices $x$ and $y$ are adjacent by an edge if and only
if $xy = 0$. The idea of a zero divisor graph of a commutative ring was introduced by Beck in [3] where he was mainly interested with colorings of rings. The definition above first is appeared in [2], which contains several fundamental results concerning $Γ(R)$. The zero-divisor graph of a commutative ring is further examined by Anderson, Levy and Shapiro, Mulay in [1, 9]. Also, the ideal-based zero divisor graph of $R$ is defined by Redmond, in [12].

The zero divisor graph for modules over commutative rings has been defined by Behboodi in [4] as a generalization of zero divisor graph of rings. Let $R$ be a commutative ring and $M$ be an $R$-module, for $x \in M$, we denote the annihilator of the factor module $M/Rx$ by $I_x$. An element $x \in M$ is called a zero divisor, if either $x = 0$ or $I_x I_y M = 0$ for some $y \neq 0$ with $I_y \subset R$. The set of zero divisors of $M$ is denoted by $Z(M)$ and the associated graph to $M$ with vertices in $Z^*(M) = Z(M) \setminus \{0\}$ is denoted by $Γ(M)$, such that two different vertices $x$ and $y$ are adjacent provided $I_x I_y M = 0$.

In this paper, we introduce the submodule-based zero divisor graph that is a generalization of zero divisor graph for modules. Let $R$ be a commutative ring, $M$ be an $R$-module and $N$ be a proper submodule of $M$. An element $x \in M$ is called zero divisor with respect to $N$, if either $x \in N$ or $I_x I_y M \subset N$ for some $y \in M \setminus N$ with $I_y \subset R$. We denote $Z(M, N)$ for the set of zero divisors of $M$ with respect to $N$. Also, we denote the associated graph to $M$ with vertices $Z^*(M, N) = Z(M, N) \setminus N$ by $Γ(M, N)$, and two different vertices $x$ and $y$ are adjacent provided $I_x I_y M \subset N$.

In the second section, we define a submodule-based zero divisor graph for a module and we study basic properties of this graph. In the third section, if $M$ is a finitely generated semisimple $R$-module such that its homogenous components are simple and $N$ is a submodule of $M$, we determine some relations between $Γ(M, N)$ and $Γ(M/N)$, where $M/N$ is the quotient module of $M$, we show that the clique number and chromatic number of $Γ(M, N)$ are equal. Also, we determine some submodule of $M$ such that $Γ(M, N)$ is an empty or a complete bipartite graph.

Let $Γ$ be a (undirected) graph. We say that $Γ$ is connected if there is a path between any two distinct vertices. For vertex $x$ the number of graph edges which touch $x$ is called the degree of $x$ and is denoted by deg($x$). For vertices $x$ and $y$ of $Γ$, we define $d(x, y)$ to be the length of a shortest path between $x$ and $y$, if there is no path, then $d(x, y) = ∞$. The diameter of $Γ$ is $\text{diam}(Γ) = \sup \{d(x, y)\}$, $x$ and $y$ are vertices of $Γ$. The girth of $Γ$, denoted by $\text{gr}(Γ)$, is the length of a shortest cycle in $Γ$ ($\text{gr}(Γ) = ∞$ if $Γ$ contains no cycle).

A graph $Γ$ is complete if any two distinct vertices are adjacent. The complete graph with $n$ vertices is denoted by $K^n$ (we allow $n$ to be an infinite cardinal). The clique number, $ω(Γ)$, is the greatest integer $n > 1$ such that $K^n \subseteq Γ$, and $ω(Γ) = ∞$ if $K^n \subseteq Γ$ for all $n \geq 1$. A complete bipartite graph is a graph $Γ$ which may be partitioned into two disjoint nonempty vertex sets $V_1$ and $V_2$. 
such that two distinct vertices are adjacent if and only if they are in different vertex sets. If one of the vertex sets is a singleton, then we call that $\Gamma$ is a star graph. We denote the complete bipartite graph by $K^{m,n}$, where $|V_1| = m$ and $|V_2| = n$ (again, we allow $m$ and $n$ to be infinite cardinals); so a star graph is $K^{1,n}$, for some $n \in \mathbb{N}$.

The chromatic number, $\chi(\Gamma)$, of a graph $\Gamma$ is the minimum number of colors needed to color the vertices of $\Gamma$, so that no two adjacent vertices share the same color. A graph $\Gamma$ is called planar if it can be drawn in such a way that no two edges intersect.

Throughout this study, $R$ is a commutative ring with nonzero identity, $M$ is a unitary $R$-module and $N$ is a proper submodule of $M$. Given any subset $S$ of $M$, the annihilator of $S$ is denoted by $\text{ann}(S) = \{ r \in R | rs = 0 \text{ for all } s \in S \}$ and the cardinal number of $S$ is denoted by $|S|$.

2. Submodule-based Zero Divisor Graph

Recall that $R$ is a commutative ring, $M$ is an $R$-module and $N$ is a proper submodule of $M$. For $x \in M$, we denote $\text{ann}(M/Rx)$ by $I_x$.

Definition 2.1. Let $M$ be an $R$-module and $N$ be a proper submodule of $M$. An $x \in M$ is called a zero divisor with respect to $N$ if $x \in N$ or $I_x \cap N \subseteq N$ for some $y \in M \setminus N$ with $I_y \subseteq R$.

We denote the set of zero divisors of $M$ with respect to $N$ by $Z(M, N)$ and $Z^*(M, N) = Z(M, N) \setminus N$. The submodule-based zero divisor graph of $M$ with respect to $N$, $\Gamma(M, N)$, is an undirected graph with vertices $Z^*(M, N)$ such that distinct vertices $x$ and $y$ are adjacent if and only if $I_x I_y \subseteq N$.

The following example shows that $Z(M/N)$ and $Z(M, N)$ are different from each other.

Example 2.2. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and $N = 2\mathbb{Z} \oplus 0$. Then $I_{(m,n)} = 0$, for all $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$. But $I_{(m,n)+N} = 2n\mathbb{Z}$ whenever $m \in 2\mathbb{Z}$ and $I_{(m,n)+N} = 2\mathbb{Z}$ whenever $m \notin 2\mathbb{Z}$. Thus $(1,0), (1,1) \in Z^*(M, N)$ are adjacent in $\Gamma(M, N)$, but $(1,0) + N, (1,1) + N \notin Z^*(M/N)$.

Proposition 2.3. If $Z^*(M, N) = \emptyset$, then $\text{ann}(M/N)$ is a prime ideal of $R$.

Proof. Suppose that $\text{ann}(M/N)$ is not prime. Then there are ideals $I$ and $J$ of $R$ such that $IJ \subseteq N$ but $IM \not\subseteq N$ and $JM \not\subseteq N$. Let $x \in IM \setminus N$ and $y \in JM \setminus N$. Then $I_x J_y M \subseteq IJM \subseteq N$ and $I_y \subseteq R$. Thus $x \in Z^*(M, N)$, a contradiction. Hence, $\text{ann}(M/N)$ is a prime ideal of $R$. \qed

Lemma 2.4. Let $x, y \in Z^*(M, N)$. If $x - y$ is an edge in $\Gamma(M, N)$, then for each $0 \neq r \in R$, either $ry \in N$ or $x - ry$ is also an edge in $\Gamma(M, N)$.

Proof. Let $x, y \in Z^*(M, N)$ and $r \in R$. Assume that $x - y$ is an edge in $\Gamma(M, N)$ and $ry \not\in N$. Then $I_x I_y M \subseteq N$. It is clear that $I_{rx} \subseteq I_x$. So that $I_x I_y M \subseteq I_x I_y M \subseteq N$ and therefore, $x - ry$ is an edge in $\Gamma(M, N)$. \qed
It is shown that the graphs are defined in [12] and [4], are connected with diameter less than or equal to three. Moreover, it shown that if those graphs contain a cycle, then they have the girth less than or equal to four. In the next theorems, we extend these results to a submodule-based zero divisor graph.

**Theorem 2.5.** $\Gamma(M, N)$ is a connected graph and $\text{diam}(\Gamma(M, N)) \leq 3$.

*Proof.* Let $x$ and $y$ be distinct vertices of $\Gamma(M, N)$. Then, there are $a, b \in Z^*(M, N)$ with $I_a I_x M \subseteq N$ and $I_b I_y M \subseteq N$ (we allow $a, b \in \{x, y\}$). If $I_a I_b M \subseteq N$, then $x - a - b - y$ is a path, thus $d(x, y) \leq 3$. If $I_a I_b M \not\subseteq N$, then $Ra \cap Rb \not\subseteq N$, and for every $d \in (Ra \cap Rb) \setminus N$, $x - d - y$ is a path of length 2, $d(x, y) \leq 2$, by Lemma 2.4. Hence, we conclude that $\text{diam}(\Gamma(M, N)) \leq 3$. □

**Theorem 2.6.** If $\Gamma(M, N)$ contains a cycle, then $\text{gr}(\Gamma(M, N)) \leq 4$.

*Proof.* We have $\text{gr}(\Gamma(M, N)) \leq 7$, by Proposition 1.3.2 in [7] and Theorem 2.5. Assume that $x_1 - x_2 - \cdots - x_7 - x_1$ is a cycle in $\Gamma(M, N)$. If $x_1 = x_4$ then it is clear that $\text{gr}(\Gamma(M, N)) \leq 3$. So, suppose that $x_1 \neq x_4$. Then we have the following two cases:

**Case 1.** If $x_1$ and $x_4$ are adjacent in $\Gamma(M, N)$, then $x_1 - x_2 - x_3 - x_4 - x_1$ is a cycle and $\text{gr}(\Gamma(M, N)) \leq 4$.

**Case 2.** Suppose that $x_1$ and $x_4$ are not adjacent in $\Gamma(M, N)$. Then $I_{x_1} I_{x_4} M \not\subseteq N$ and so there is a $z \in (Rx_1 \cap Rx_4) \setminus N$. If $z = x_1$, then $z \neq x_4$ and $x_3 - x_4 - x_3 - z - x_3$ is a cycle in $\Gamma(M, N)$, by Lemma 2.4. If $z \neq x_1$, then by Lemma 2.4, $x_1 - x_2 - z - x_7 - x_1$ is a cycle and $\text{gr}(\Gamma(M, N)) \leq 4$.

For cycles with length 5 or 6, by using a similar argument as above, one can shows that $\text{gr}(\Gamma(M, N)) \leq 4$. □

**Example 2.7.** Assume that $M = \mathbb{Z}$ and $p, q$ are two prime numbers. If $N = p\mathbb{Z}$, then $\Gamma(M, N) = \emptyset$. If $N = pq\mathbb{Z}$, then $\Gamma(M, N)$ is an infinite complete bipartite graph with vertex set $V_1 \cup V_2$, where $V_1 = p\mathbb{Z} \setminus pq\mathbb{Z}$ and $V_2 = q\mathbb{Z} \setminus pq\mathbb{Z}$ and so $\text{gr}(\Gamma(M, N)) = 4$.

**Corollary 2.8.** If $N$ is a prime submodule of $M$, then $\text{diam}(\Gamma(M, N)) \leq 2$ and $\text{gr}(\Gamma(M, N)) = 3$, whenever it contains a cycle.

*Proof.* Let $x, y$ be two distinct vertices which are not adjacent in $\Gamma(M, N)$. Thus there is an $a \in M \setminus N$ such that $I_a I_x M \subseteq N$. Since $N$ is a prime submodule, then $I_a M \subseteq N$. Thus $I_a I_y M \subseteq N$, and then $x - a - y$ is a path in $\Gamma(M, N)$. Then $\text{diam}(\Gamma(M, N)) \leq 2$. □

**Lemma 2.9.** Let $|\Gamma(M, N)| \geq 3$, $\text{gr}(\Gamma(M, N)) = \infty$ and $x \in Z^*(M, N)$ with $\deg(x) > 1$. Then $Rx = \{0, x\}$ and $\text{ann}(x)$ is a prime ideal of $R$.

*Proof.* First we show that $Rx = \{0, x\}$. Let $u - x - v$ be a path in $\Gamma(M, N)$. Then $u - v$ is not an edge in $\Gamma(M, N)$ since $\text{gr}(\Gamma(M, N)) = \infty$. If $x \neq rx$ for some $r \in R$ and $rx \not\in N$, then by Lemma 2.4, $rx - u - x - v - rx$ is a cycle in
A submodule-based zero divisor graphs for modules

$\Gamma(M, N)$, that is a contradiction. So, for every $r \in R$ either $rx = x$ or $rx \in N$.
If there is an $r \in R$ such that $rx \in N$, then we have either $(1 + r)x \in N$ or
$(1 + r)x = x$. These imply that $x \in N$ or $rx = 0$. Therefore, we have shown that
$Rx = \{0, x\}$.

Let $a, b \in R$ and $abx = 0$. Then $bx = 0$ or $bx = x$. Hence, $bx = 0$ or $ax = 0$.
So, $\text{ann}(x)$ is a prime ideal of $R$. □

**Theorem 2.10.** If $N$ is a nonzero submodule of $M$ and $\text{gr}(\Gamma(M, N)) = \infty$,
then $\Gamma(M, N)$ is a star graph.

*Proof.* Suppose that $\Gamma(M, N)$ is not a star graph. Then there is a path in
$\Gamma(M, N)$ such as $u - x - y - v$. By Lemma 2.9, we have $Ry = \{0, y\}$ and by
assumption $u$ and $y$ are not adjacent, thus $I_yM \neq 0$. So that $I_yM = Ry$. Also,
$x - y - v$ is a path, thus $I_xI_yM \subseteq N$ and $I_xI_yM \subseteq N$. Hence, $I_xRy \subseteq N$ and
$I_yI_x \subseteq N$. On the other hand, for every nonzero $n \in N$, we have

$I_xI_yM \subseteq I_xR(y + n) \subseteq I_y(Ry + N) \subseteq N$

and similarly $I_yI_xnM \subseteq N$. So that $x - y - v - (y + n) - x$ is a cycle in
$\Gamma(M, N)$, a contradiction. Therefore, $\Gamma(M, N)$ is a star graph. □

**Theorem 2.11.** Let $N$ be a nonzero submodule of $M$, $|\Gamma(M, N)| \geq 3$ and
$\Gamma(M, N)$ is a star graph. Then the following statements are true:

(i) If $x$ is the center vertex, then $I_x = \text{ann}(M)$.

(ii) $\Gamma(M, N)$ is a subgraph of $\Gamma(M)$.

*Proof.* (i) By Lemma 2.9, we have $Rx = \{0, x\}$. Thus either $I_xM = 0$ or
$I_xM = Rx$. Assume that $I_xM = Rx$. If $y$ is a vertex of $\Gamma(M, N)$ such
that $y \neq x$, then $\text{deg}(y) = 1$ and $I_xI_yM \subseteq N$. Thus $I_yRx \subseteq N$. Since
$I_xI_yM \subseteq I_yR(x + n) \subseteq N$ for every nonzero element $n \in N$ it concludes
that $y = x + n$. In this case, every other vertices of $\Gamma(M, N)$ are adjacent to $y$,
a contradiction. Hence, $I_xM = 0$ and $I_x = \text{ann}(M)$.

(ii) It is obvious. □

**Theorem 2.12.** If $|N| \geq 3$ and $\Gamma(M, N)$ is a complete bipartite graph which
is not a star graph, then $I_x^2M \not\subseteq N$, for every $x \in Z^*(M, N)$.

*Proof.* Let $Z^*(M, N) = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$. Suppose that $I_x^2M \subseteq N$
for some $x \in Z^*(M, N)$. Without loss of generality, we can assume that $x \in V_1$.
By a similar argument with Lemma 2.9, either $Rx = \{0, x\}$ or there is an $r \in R$ such that $x \neq rx$ and $rx \in N$. If $Rx = \{0, x\}$, then $I_xM = Rx$. Thus
$I_xRx \subseteq N$. Now, for every $y \in V_2$ and $n \in N$ we get

$I_yI_xM \subseteq I_yR(x + n) \subseteq I_y(Rx + N) \subseteq N$

and $I_xI_yM \subseteq N$. Then, $x + n \in V_1 \cap V_2$, a contradiction. So, assume that
$x \neq rx$ and $rx \in N$ for some $r \in R$. Since $I_{rx} \subseteq I_x$, then $I_xI_{rx + x}M \subseteq N$
and for all $y \in V_2$, $I_yI_{rx + x}M \subseteq N$. Thus $rx + x \in V_1 \cap V_2$, a contradiction. □
An $R$-module $X$ is called a multiplication-like module if, for each nonzero submodule $Y$ of $X$, $\text{ann}(X) \subseteq \text{ann}(X/Y)$. Multiplication-like module have been studied in [8, 13].

A vertex $x$ of a connected graph $G$ is a cut-point, if there are vertices $u, v$ of $G$ such that $x$ is in every path from $u$ to $v$ and $x \neq u, x \neq v$. For a connected graph $G$, an edge $E$ of $G$ is defined to be a bridge if $G - \{E\}$ is disconnected, see [6].

**Theorem 2.13.** Let $M$ be a multiplication-like module and $N$ be a nonzero submodule of $M$. Then $\Gamma(M, N)$ has no cut-points.

**Proof.** Suppose that $x$ is a cut-point of $\Gamma(M, N)$. Then there exist vertices $u, v \in M \setminus N$ such that $x$ lies on every path from $u$ to $v$. By Theorem 2.5, the shortest path from $u$ to $v$ has length 2 or 3.

**Case 1.** Suppose that $u-x-v$ is a path of shortest length from $u$ to $v$. Since $x$ is a cut point, $u, v$ aren't in a cycle. By a similar argument to that of Lemma 2.9, we have $Rx = \{0, x\}$. On the other hand, $IxM \subseteq Rx$ and $M$ is a multiplication-like module, so we have $IxM = Rx$. Hence $IuRx \subseteq N$ and $IuRx \subseteq N$. Also, for every nonzero $n \in N$, we have $IuIx_{x+n}M \subseteq Iu(Rx+N) \subseteq N$ and $IuIx_{x+n}M \subseteq N$. Therefore, $u - (x + n) - v$ is a path from $u$ to $v$, a contradiction.

**Case 2.** Suppose that $u-x-y-v$ is a path in $\Gamma(M, N)$. Then, we have $IxM = Rc$ and for every nonzero $n \in N$, we have $IuIx_{x+n}M \subseteq N$ and $IuIx_{x+n}M \subseteq N$. Thus $u - (x + n) - y - v$ is a path from $u$ to $v$, a contradiction. \qed

**Theorem 2.14.** Let $M$ be a multiplication-like module and $N$ be a nonzero submodule of $M$. Then $\Gamma(M, N)$ has a bridge if and only if $\Gamma(M, N)$ is a graph on two vertices.

**Proof.** If $|\Gamma(M, N)| = 3$, then $\Gamma(M, N) = K^3$, by Theorem 2.11, and it has no bridge. Assume that $|\Gamma(M, N)| \geq 4$ and $x - y$ is a bridge. Thus there is not a cycle containing $x - y$. Without loss of generality, we can assume that $\deg(x) > 1$. Thus, there exists a vertex $z \neq y$ such that $z - x$ is an edge of $\Gamma(M, N)$. Then $Rx = \{0, x\}$ and $IxM = Rx$. Hence, for every $n \in N$, $IxIx_{x+n}M \subseteq N$ and $IyIx_{x+n}M \subseteq N$, a contradiction. Therefore, $\Gamma(M, N)$ has not a bridge. The converse is clear. \qed

3. **Submodule-based Zero Divisor Graph of Semisimple Modules**

A nonzero $R$-module $X$ is called simple if its only submodules are $(0)$ and $X$. An $R$-module $X$ is called semisimple if it is a direct sum of simple modules. Also, $X$ is called homogenous semisimple if it is a direct sum of isomorphic simple modules.

In this section, $R$ is a commutative ring and $M$ is a finitely generated semisimple $R$-module such that its homogenous components are simple and...
$N$ is a submodule of $M$. The following theorem has a crucial role in this section.

**Theorem 3.1.** Let $x, y \in M \setminus N$. Then $x, y$ are adjacent in $\Gamma(M, N)$ if and only if $Rx \cap Ry \subseteq N$.

**Proof.** Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$. By assumption $N$ is a submodule of $M$, so there exists a subset $A$ of $I$ such that $M = N \oplus (\bigoplus_{i \in A} M_i)$ and so $\text{ann}(M/N) = \text{ann}(\bigoplus_{i \in A} M_i) = \bigcap_{i \in A} \text{ann}(M_i)$. Assume that $x, y \in M \setminus N$ are adjacent in $\Gamma(M, N)$ and $Rx \cap Ry \not\subseteq N$. Thus there exists $\alpha \in I$ such that $M_\alpha \subseteq (Rx \cap Ry) \setminus N$. Also, there exist subsets $B \subseteq I$ and $C \subseteq I$ such that $M = Rx \oplus (\bigoplus_{i \in B} M_i)$ and $M = Ry \oplus (\bigoplus_{i \in C} M_i)$. Therefore, $I_x = \bigcap_{i \in B} \text{ann}(M_i)$ and $I_y = \bigcap_{i \in C} \text{ann}(M_i)$. Since $I_x I_y M \subseteq N$, we have $I_x I_y \subseteq \text{ann}(M/N)$. For every $i, j \in I$, $\text{ann}(M_i)$ and $\text{ann}(M_j)$ are coprime, then

$$I_x I_y = [\bigcap_{i \in B} \text{ann}(M_i)] [\bigcap_{i \in C} \text{ann}(M_i)] = \prod_{i \in B \cup C} \text{ann}(M_i) \subseteq \bigcap_{i \in A} \text{ann}(M_i) \subseteq \text{ann}(M_r),$$

for all $r \in A$. Thus for any $r \in A$ there exists $j_r \in B \cup C$ such that $\text{ann}(M_{j_r}) \subseteq \text{ann}(M_r)$. So that $\text{ann}(M_{j_r}) = \text{ann}(M_r)$ implies that $M_{j_r} \cong M_r$ and by hypothesis $M_{j_r} = M_r$. Hence,

$$M_\alpha \subseteq \bigoplus_{i \in A} M_i \subseteq \bigoplus_{j \in B \cup C} M_j.$$

Thus there exists $\gamma \in B \cup C$ such that $M_\alpha = M_\gamma$, also

$$M_\alpha \subseteq Rx \cap Ry = (\bigoplus_{i \in I \setminus B} M_i) \cap (\bigoplus_{i \in I \setminus C} M_i).$$

Therefore, $\alpha \in I \setminus (B \cup C)$, a contradiction. The converse is obvious. \qed

**Corollary 3.2.** Let $x, y \in M \setminus N$ be such that $x + N \not\sim y + N$. Then

(i) $x$ and $y$ are adjacent in $\Gamma(M, N)$ if and only if $x + N$ and $y + N$ are adjacent in $\Gamma(M/N)$.

(ii) if $x$ and $y$ are adjacent in $\Gamma(M, N)$, then all distinct elements of $x + N$ and $y + N$ are adjacent in $\Gamma(M, N)$.

**Proof.** (i) Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$. Suppose that $x$ and $y$ are adjacent in $\Gamma(M, N)$, $Rx = \bigoplus_{i \in A} M_i$, $Ry = \bigoplus_{i \in B} M_i$ and $N = \bigoplus_{i \in C} M_i$. Then $Rx + N = \bigoplus_{i \in A \cup C} M_i$ and $Ry + N = \bigoplus_{i \in B \cup C} M_i$. Thus,

$$(Rx + N) \cap (Ry + N) = \bigoplus_{i \in (A \cup C) \cap (B \cup C)} M_i = \bigoplus_{i \in (A \cap B) \cup C} M_i = (Rx \cap Ry) + N.$$

By Theorem 3.1, we have $Rx \cap Ry \subseteq N$ hence,

$$I_x + N I_y + N M \subseteq (Rx + N) \cap (Ry + N) = (Rx \cap Ry) + N = N.$$
Therefore, \(x + N\) and \(y + N\) are adjacent in \(\Gamma(M/N)\). The converse is obvious.

(ii) Let \(x, y \in Z^*(M, N)\) be adjacent in \(\Gamma(M, N)\). Then \(Rx \cap Ry \subseteq N\) by Theorem 3.1. So for every \(n, n' \in N\) we have

\[
I_{x+n}I_{y+n'}M \subseteq R(x + n) \cap R(y + n') \subseteq (Rx + N) \cap (Ry + N) = N.
\]

Hence, \(x + n\) and \(y + n'\) are adjacent in \(\Gamma(M, N)\). \(\square\)

In the following theorem, we prove that the clique number of graphs \(\Gamma(M, N)\) and \(\Gamma(M/N)\) are equal.

**Theorem 3.3.** If \(N\) is a nonzero submodule of \(M\), then \(\omega(\Gamma(M/N)) = \omega(\Gamma(M, N))\).

**Proof.** First we show that \(I_{m+1}^N M \nsubseteq N\) for each \(0 \neq m + N \in M/N\).

Assume that \(N = \oplus_{i \in A} M_i\) and \(m = (m_i)_{i \in I} \in M \setminus N\). Then \(I_{m+1} + N = \bigcap_{i \notin A, m_i = 0} \text{ann}(M_i)\). Hence, \(I_{m+1} + N = I_{m+1}^N\). Thus \(I_{m+1}^N M \nsubseteq N\) since there is at least one \(j \in I \setminus A\) such that \(m_j \neq 0\).

Now, Corollary 3.2 implies that \(\omega(\Gamma(M/N)) \leq \omega(\Gamma(M, N))\). Thus, it is enough to consider the case where \(\omega(\Gamma(M/N)) = d < \infty\). Assume that \(G\) is a complete subgraph of \(\Gamma(M, N)\) with vertices \(m_1, m_2, \cdots, m_{d+1}\), we provide a contradiction. Consider the subgraph \(G_*\) of \(\Gamma(M/N)\) with vertices \(m_1 + N, \cdots, m_{d+1} + N\). By Corollary 3.2, \(G_*\) is a complete subgraph of \(\Gamma(M, N)\). Thus \(m_j + N = m_k + N\) for some \(1 \leq j, k \leq d + 1\) with \(j \neq k\) since \(\omega(\Gamma(M/N)) = d\). We have \(I_{m_j} I_{m_k} M \subseteq N\). Therefore, \(Rm_j \cap Rm_k \subseteq N\) and so \(I_{m_j} + N I_{m_k} + N M \subseteq N\). Hence, \(I_{m_j}^N M \subseteq N\), that is a contradiction. \(\square\)

In the following theorem, we show that there is a relation between \(\omega(\Gamma(M, N))\) and \(\chi(\Gamma(M, N))\).

**Theorem 3.4.** Assume that \(M = \bigoplus_{i \in I} M_i\), where \(M_i\)'s are non-isomorphic simple submodules of \(M\) and \(N = \bigoplus_{i \in A} M_i\) is a submodule of \(M\) for some \(A \subseteq I\). Then \(\omega(\Gamma(M, N)) = \chi(\Gamma(M, N)) = |I| - |A|\).

**Proof.** Suppose that \(I \setminus A = \{1, \cdots, n\}\) so \(M_i, \cdots, M_n \nsubseteq N\). Let for \(1 \leq k \leq n - 1\)

\[
L^k = \{m \in M : m \text{ has } k \text{ nonzero components}\}
\]

and let for \(1 \leq s \leq n\)

\[
L^1_s = \{m \in L^1 : \text{the } s^{th} \text{ component of } m \text{ is nonzero}\}.
\]

If \(m \in L^1_s\) and \(m' \in L^1_t\) for some \(1 \leq s, t \leq n\) with \(s \neq t\), then \(m\) and \(m'\) are adjacent and so \(K^n\) is a subgraph of \(\Gamma(M, N)\). Thus \(\omega(\Gamma(M, N)) \geq n\).

If \((m, m') \in L^1_s\) for some \(1 \leq s \leq n\), then \(m, m'\) are not adjacent because \(\text{ann}(M_s) \nsubseteq I_m I_{m'}\) and so the elements of \(L^1_s\) have same color. On the other hand, if \(x \in L^t\) with \(t > 1\), then there is not a complete subgraph \(K^b\) of \(\Gamma(M, N)\) containing \(x\), such that \(b \geq n\). Thus \(\omega(\Gamma(M, N)) = n \leq \chi(\Gamma(M, N))\).

Also, if \(x \in L^t\) with \(t > 1\), then there is an \(s\) with \(1 \leq s \leq n\) such that \(x\) is not
adjacent to each element of \( L_1^1 \). Thus the color of \( x \) is same as the elements of \( L_1^1 \). Thus \( \chi(\Gamma(M, N)) = n \).

The Kwartowski’s Theorem states: A graph \( G \) is planar if and only if it contains no subgraph homeomorphic to \( K^5 \) or \( K^{3,3} \).

**Theorem 3.5.** Let \( N \) be a nonzero proper submodule of \( M \) such that \( N \) is not prime. Then \( \Gamma(M, N) \) is not planar.

**Proof.** Assume that \( M = \bigoplus_{i \in I} M_i \), where \( M_i \)'s are non-isomorphic simple submodules of \( M \) and \( N = \bigoplus_{i \in A} M_i \) for some \( A \subset I \). Let \( I \setminus A = \{i, j\} \). Then \( \Gamma(M, N) \) is a complete bipartite graph \( K^{n, m} \), where \( n = (|M_i| - 1)(\prod_{k \in I \setminus \{i, j\}} |M_k|) \) and \( m = (|M_j| - 1)(\prod_{k \in I \setminus \{i, j\}} |M_k|) \). By hypotheses \( N \) is a nonzero and \( M_i \)'s are non-isomorphic, so we have \( n, m \geq 3 \). Hence \( \Gamma(M, N) \) has a subgraph homeomorphic to \( K^{3,3} \). The cases \( |I \setminus A| \geq 3 \) are similar to that of the case \( |I \setminus A| = 2 \).

**Theorem 3.6.** A nonzero submodule \( N \) of \( M \) is prime if and only if \( Z^*(M, N) = \emptyset \).

**Proof.** Let \( M = \bigoplus_{i \in I} M_i \), where \( M_i \)'s are non-isomorphic simple submodules of \( M \) and \( N \) is prime. Then \( N = \bigoplus_{i \in I \setminus \{k\}} M_i \), for some \( k \in I \). If \( x \in Z^*(M, N) \), then there exists a \( y \in M \setminus N \) such that \( I_xI_yM \subseteq N \). If \( x \neq y \), then \( Rx \cap Ry \subseteq N \), by Theorem 3.1. Thus either \( M_k \not\subseteq Rx \) or \( M_k \not\subseteq Ry \). Hence, either \( Rx \not\subseteq N \) or \( Ry \not\subseteq N \), a contradiction. Now, suppose that \( x = y \) so by \( I_x^2M \subseteq N \) and hypotheses \( I_xM \subseteq N \). Thus \( I_{x+n}I_xM \subseteq N \) for every \( 0 \neq n \in N \). By a similar argument, we have either \( x \in N \) or \( x + n \in N \), a contradiction. Hence, \( Z^*(M, N) = \emptyset \).

Conversely, assume that \( Z^*(M, N) = \emptyset \). Then \( \text{ann}(M/N) \) is prime ideal of \( R \) by Proposition 2.3 and there exists a \( k \in I \) such that \( \text{ann}(M/N) = \text{ann}(M_k) \). Hence, \( N = \bigoplus_{i \in I \setminus \{k\}} M_i \) is a prime submodule of \( M \).

A proper submodule \( N \) of \( M \) is called 2-absorbing if whenever \( a, b \in R \), \( m \in M \) and \( abm \in N \), then \( am \in N \) or \( bm \in N \) or \( ab \in \text{ann}(M/N) \), see [10, 11]. In the following results, we study the behavior of \( \Gamma(M, N) \) whenever \( N \) is a 2-absorbing submodule of \( M \).

**Theorem 3.7.** A submodule \( N \) of \( M \) is 2-absorbing if and only if at most two components of \( M \) are zero in \( N \).

**Proof.** Let \( M = \bigoplus_{i \in I} M_i \), where \( M_i \)'s are non-isomorphic simple submodules of \( M \). Suppose that \( N \) is a 2-absorbing submodule of \( M \) and \( N = \bigoplus_{i \in A} M_i \), where \( A = I \setminus \{s, t, k\} \). Since for all \( i \in I \), \( \text{ann}(M_i) \) is prime, there are \( a \in \text{ann}(M_s) \setminus (\text{ann}(M_t) \cup \text{ann}(M_k)) \), \( b \in \text{ann}(M_t) \setminus (\text{ann}(M_s) \cup \text{ann}(M_k)) \) and \( c \in \bigcap_{j \in I \setminus \{s, t\}} \text{ann}(M_j) \setminus (\text{ann}(M_k) \cup \text{ann}(M_s) \cup \text{ann}(M_t)) \). Now, \( abc \in \text{ann}(M/N) \) but \( ab \not\in \text{ann}(M/N) \), \( ac \not\in \text{ann}(M/N) \) and \( bc \not\in \text{ann}(M/N) \). This contradicts
Theorem 2.3 in [10]. Thus $|A| \geq |I| - 2$ and at most two components of $M$ are zero in $N$.

Conversely, if one component of $M$ is zero in $N$, then $N$ is a prime submodule of $M$. Suppose that $N = \bigoplus_{i \in A} M_i$, where $A = I \setminus \{i, j\}$. Thus $M_i, M_j \not\subseteq N$. Suppose that $a, b \in R$, $(m_i)_{i \in I} = m \in M \setminus N$ and $abm \in N$. Then either $m_i \neq 0$ or $m_j \neq 0$. If $m_i \neq 0$ and $m_j \neq 0$, then $ab \in \text{ann}(M_i) \cap \text{ann}(M_j) = \text{ann}(M/N)$. If $m_i \neq 0$ and $m_j = 0$, then $ab \in \text{ann}(M_i)$ and so either $a \in \text{ann}(M_i)$ or $b \in \text{ann}(M_j)$. Hence, $am \in N$ or $bm \in N$. The case $m_i = 0$ and $m_j \neq 0$, is similar to the previous case. Therefore, $N$ is a 2-absorbing submodule of $M$. □

Theorem 3.8. $N$ is a 2-absorbing submodule of $M$ if and only if $Z^*(M, N) = \emptyset$ or $\Gamma(M, N)$ is a complete bipartite graph.

Proof. Let $N$ be a 2-absorbing submodule of $M$. If $N$ is prime, then $Z^*(M, N) = \emptyset$, by Theorem 3.6. Now, assume that $N = \bigoplus_{i \in I \setminus \{j, k\}} M_i$ for some $j, k \in I$ and $(m_i)_{i \in I} = m \in M \setminus N$. Thus $I_m = \bigcap_{i \in I, m_i=0} \text{ann}(M_i)$. If $m_i \neq 0$ and $m_k \neq 0$, then $m \notin Z(M, N)$. Let $V_1 = \{(m_i)_{i \in I} \in M \setminus N : m_j = 0\}$ and $V_2 = \{(m_i)_{i \in I} \in M \setminus N : m_k = 0\}$. Thus $m - m'$ is an edge of $\Gamma(M, N)$ for every $m \in V_1$ and $m' \in V_2$. Also, every vertices in $V_1$ and $V_2$ are not adjacent. Hence, $\Gamma(M, N)$ is a complete bipartite graph.

Now, suppose that $\Gamma(M, N)$ is a complete bipartite graph and $N$ is not 2-absorbing. By Theorem 3.7, there are at least three components $M_s, M_t, M_k$ such that $M_s, M_t, M_k \not\subseteq N$. For $i = s, t, k$ let $v_i = (m_i)_{i \in I}$, where $m_i \neq 0$ and $m_j = 0$ for all $j \neq i$. Then $v_s - v_t - v_k - v_s$ is a cycle in $\Gamma(M, N)$. Thus gr$(\Gamma(M, N)) = 3$ and so $\Gamma(M, N)$ is not bipartite graph, by Theorem 1 of Sec. 1.2 in [5]. Hence, $N$ is a 2-absorbing submodule of $M$. □

Example 3.9. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$. Then every nonzero submodule $N$ of $M$ is 2-absorbing. Thus either $Z^*(M, N) = \emptyset$ or $\Gamma(M, N)$ is a complete bipartite graph. In particular, if $N = \mathbb{Z}_7$, then $\Gamma(M, N) = K_{7,28}$.

Acknowledgments

The author is thankful of referees for their valuable comments.

References


