A Submodule-Based Zero Divisor Graph for Modules

Sakineh Babaei\textsuperscript{a}, Shiroyeh Payrovi\textsuperscript{∗,a}, Esra Sengelen Sevim\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Imam Khomeini International University, Postal Code: 34149-1-6818, Qazvin, Iran.
\textsuperscript{b}Department of Mathematics, Istanbul Bilgi University, Kazim Karabekir Cad. No: 2/13, 34060 Eyup-Istanbul, Turkey.

E-mail: sbabaei@edu.ikiu.ac.ir
E-mail: shpayrovi@sci.ikiu.ac.ir
E-mail: esra.sengelen@bilgi.edu.tr

Abstract. Let $R$ be a commutative ring with identity and $M$ be an $R$-module. The zero divisor graph of $M$ is denoted by $\Gamma(M)$. In this study, we are going to generalize the zero divisor graph $\Gamma(M)$ to submodule-based zero divisor graph $\Gamma(M, N)$ by replacing elements whose product is zero with elements whose product is in some submodule $N$ of $M$. The main objective of this paper is to study the interplay of the properties of submodule $N$ and the properties of $\Gamma(M, N)$.

Keywords: Zero divisor graph, Submodule-based zero divisor graph, Semisimple module.


1. Introduction

Let $R$ be a commutative ring with identity. The zero divisor graph of $R$, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero divisor of $R$ with two distinct vertices $x$ and $y$ are adjacent by an edge if and only

\textsuperscript{∗}Corresponding Author

Received 08 September 2016; Accepted 18 December 2016
©2019 Academic Center for Education, Culture and Research TMU
The idea of a zero divisor graph of a commutative ring was introduced by Beck in [3] where he was mainly interested with colorings of rings. The definition above first is appeared in [2], which contains several fundamental results concerning $\Gamma(R)$. The zero-divisor graph of a commutative ring is further examined by Anderson, Levy and Shapiro, Mulay in [1, 9]. Also, the ideal-based zero divisor graph of R is defined by Redmond, in [12].

The zero divisor graph for modules over commutative rings has been defined by Behboodi in [4] as a generalization of zero divisor graph of rings. Let $R$ be a commutative ring and $M$ be an $R$-module, for $x \in M$, we denote the annihilator of the factor module $M/Rx$ by $I_x$. An element $x \in M$ is called a zero divisor, if either $x = 0$ or $I_xI_yM = 0$ for some $y \neq 0$ with $I_y \subset R$. The set of zero divisors of $M$ is denoted by $Z(M)$ and the associated graph to $M$ with vertices in $Z^*(M) = Z(M) \setminus \{0\}$ is denoted by $\Gamma(M)$, such that two different vertices $x$ and $y$ are adjacent provided $I_xI_yM = 0$.

In this paper, we introduce the submodule-based zero divisor graph that is a generalization of zero divisor graph for modules. Let $R$ be a commutative ring, $M$ be an $R$-module and $N$ be a proper submodule of $M$. An element $x \in M$ is called zero divisor with respect to $N$, if either $x \in N$ or $I_xI_yM \subseteq N$ for some $y \in M \setminus N$ with $I_y \subset R$. We denote $Z(M, N)$ for the set of zero divisors of $M$ with respect to $N$. Also, we denote the associated graph to $M$ with vertices $Z^*(M, N) = Z(M, N) \setminus N$ by $\Gamma(M, N)$, and two different vertices $x$ and $y$ are adjacent provided $I_xI_yM \subseteq N$.

In the second section, we define a submodule-based zero divisor graph for a module and we study basic properties of this graph. In the third section, if $M$ is a finitely generated semisimple $R$-module such that its homogenous components are simple and $N$ is a submodule of $M$, we determine some relations between $\Gamma(M, N)$ and $\Gamma(M/N)$, where $M/N$ is the quotient module of $M$, we show that the clique number and chromatic number of $\Gamma(M, N)$ are equal. Also, we determine some submodule of $M$ such that $\Gamma(M, N)$ is an empty or a complete bipartite graph.

Let $\Gamma$ be a (undirected) graph. We say that $\Gamma$ is connected if there is a path between any two distinct vertices. For vertex $x$ the number of graph edges which touch $x$ is called the degree of $x$ and is denoted by $\deg(x)$. For vertices $x$ and $y$ of $\Gamma$, we define $d(x, y)$ to be the length of a shortest path between $x$ and $y$, if there is no path, then $d(x, y) = \infty$. The diameter of $\Gamma$ is $\text{diam}(\Gamma) = \sup\{d(x, y)|x \text{ and } y \text{ are vertices of } \Gamma\}$. The girth of $\Gamma$, denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$ ($\text{gr}(\Gamma) = \infty$ if $\Gamma$ contains no cycle).

A graph $\Gamma$ is complete if any two distinct vertices are adjacent. The complete graph with $n$ vertices is denoted by $K^n$ (we allow $n$ to be an infinite cardinal). The clique number, $\omega(\Gamma)$, is the greatest integer $n > 1$ such that $K^n \subseteq \Gamma$, and $\omega(\Gamma) = \infty$ if $K^n \subseteq \Gamma$ for all $n \geq 1$. A complete bipartite graph is a graph $\Gamma$ which may be partitioned into two disjoint nonempty vertex sets $V_1$ and $V_2$.
such that two distinct vertices are adjacent if and only if they are in different vertex sets. If one of the vertex sets is a singleton, then we call that $\Gamma$ is a star graph. We denote the complete bipartite graph by $K^{m,n}$, where $|V_1| = m$ and $|V_2| = n$ (again, we allow $m$ and $n$ to be infinite cardinals); so a star graph is $K^{1,n}$, for some $n \in \mathbb{N}$.

The chromatic number, $\chi(\Gamma)$, of a graph $\Gamma$ is the minimum number of colors needed to color the vertices of $\Gamma$, so that no two adjacent vertices share the same color. A graph $\Gamma$ is called planar if it can be drawn in such a way that no two edges intersect.

Throughout this study, $R$ is a commutative ring with nonzero identity, $M$ is a unitary $R$-module and $N$ is a proper submodule of $M$. Given any subset $S$ of $M$, the annihilator of $S$ is denoted by $\text{ann}(S) = \{r \in R | rs = 0 \text{ for all } s \in S\}$ and the cardinal number of $S$ is denoted by $|S|$.

2. Submodule-based Zero Divisor Graph

Recall that $R$ is a commutative ring, $M$ is an $R$-module and $N$ is a proper submodule of $M$. For $x \in M$, we denote $\text{ann}(M/Rx)$ by $I_x$.

Definition 2.1. Let $M$ be an $R$-module and $N$ be a proper submodule of $M$. An $x \in M$ is called a zero divisor with respect to $N$ if $x \in N$ or $I_xI_yM \subseteq N$ for some $y \in M \setminus N$ with $I_y \subseteq R$.

We denote the set of zero divisors of $M$ with respect to $N$ by $Z(M,N)$ and $Z^*(M,N) = Z(M,N) \setminus N$. The submodule-based zero divisor graph of $M$ with respect to $N$, $\Gamma(M,N)$, is an undirected graph with vertices $Z^*(M,N)$ such that distinct vertices $x$ and $y$ are adjacent if and only if $I_xI_yM \subseteq N$.

The following example shows that $Z(M/N)$ and $Z(M,N)$ are different from each other.

Example 2.2. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and $N = 2\mathbb{Z} \oplus 0$. Then $I_{(m,n)} = 0$, for all $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$. But $I_{(m,n) + N} = 2n\mathbb{Z}$ whenever $m \in 2\mathbb{Z}$ and $I_{(m,n) + N} = 2\mathbb{Z}$ whenever $m \not\in 2\mathbb{Z}$. Thus $(1,0), (1,1) \in Z^*(M,N)$ are adjacent in $\Gamma(M,N)$, but $(1,0) + N, (1,1) + N \not\in Z^*(M/N)$.

Proposition 2.3. If $Z^*(M,N) = \emptyset$, then $\text{ann}(M/N)$ is a prime ideal of $R$.

Proof. Suppose that $\text{ann}(M/N)$ is not prime. Then there are ideals $I$ and $J$ of $R$ such that $IJM \subset N$ but $IM \not\subset N$ and $JM \not\subset N$. Let $x \in IM \setminus N$ and $y \in JM \setminus N$. Then $I_xJ_yM \subseteq IJM \subseteq N$ and $I_y \subseteq R$. Thus $x \in Z^*(M,N)$, a contradiction. Hence, $\text{ann}(M/N)$ is a prime ideal of $R$. \qed

Lemma 2.4. Let $x, y \in Z^*(M,N)$. If $x - y$ is an edge in $\Gamma(M,N)$, then for each $0 \neq r \in R$, either $ry \in N$ or $x - ry$ is also an edge in $\Gamma(M,N)$.

Proof. Let $x, y \in Z^*(M,N)$ and $r \in R$. Assume that $x - y$ is an edge in $\Gamma(M,N)$ and $ry \not\in N$. Then $I_xI_yM \subseteq N$. It is clear that $I_{rx} \subseteq I_x$. So that $I_xI_{ry}M \subseteq I_xI_yM \subseteq N$ and therefore, $x - ry$ is an edge in $\Gamma(M,N)$. \qed
It is shown that the graphs are defined in [12] and [4], are connected with diameter less than or equal to three. Moreover, it shown that if those graphs contain a cycle, then they have the girth less than or equal to four. In the next theorems, we extend these results to a submodule-based zero divisor graph.

**Theorem 2.5.** $\Gamma(M, N)$ is a connected graph and $\text{diam}(\Gamma(M, N)) \leq 3$.

**Proof.** Let $x$ and $y$ be distinct vertices of $\Gamma(M, N)$. Then, there are $a, b \in Z^*(M, N)$ with $I_aI_xM \subseteq N$ and $I_yI_bM \subseteq N$ (we allow $a, b \in \{x, y\}$). If $I_aI_bM \subseteq N$, then $x - a - b - y$ is a path, thus $d(x, y) \leq 3$. If $I_aI_bM \not\subseteq N$, then $Ra \cap Rb \not\subseteq N$, and for every $d \in (Ra \cap Rb) \setminus N$, $x - d - y$ is a path of length 2, $d(x, y) \leq 2$, by Lemma 2.4. Hence, we conclude that $\text{diam}(\Gamma(M, N)) \leq 3$. □

**Theorem 2.6.** If $\Gamma(M, N)$ contains a cycle, then $\text{gr}(\Gamma(M, N)) \leq 4$.

**Proof.** We have $\text{gr}(\Gamma(M, N)) \leq 7$, by Proposition 1.3.2 in [7] and Theorem 2.5. Assume that $x_1 - x_2 - \cdots - x_7 - x_1$ is a cycle in $\Gamma(M, N)$. If $x_1 = x_4$ then it is clear that $\text{gr}(\Gamma(M, N)) \leq 3$. So, suppose that $x_1 \neq x_4$. Then we have the following two cases:

**Case 1.** If $x_1$ and $x_4$ are adjacent in $\Gamma(M, N)$, then $x_1 - x_2 - x_3 - x_4 - x_1$ is a cycle and $\text{gr}(\Gamma(M, N)) \leq 4$.

**Case 2.** Suppose that $x_1$ and $x_4$ are not adjacent in $\Gamma(M, N)$. Then $I_{x_1}I_{x_4}M \not\subseteq N$ and so there is a $z \in (Rx_1 \cap Rx_4) \setminus N$. If $z = x_1$, then $z \neq x_4$ and $x_1 - x_2 - z - x_7 - x_1$ is a cycle in $\Gamma(M, N)$, by Lemma 2.4. If $z \neq x_1$, then by Lemma 2.4, $x_1 - x_2 - z - x_7 - x_1$ is a cycle and $\text{gr}(\Gamma(M, N)) \leq 4$.

For cycles with length 5 or 6, by using a similar argument as above, one can shows that $\text{gr}(\Gamma(M, N)) \leq 4$. □

**Example 2.7.** Assume that $M = \mathbb{Z}$ and $p, q$ are two prime numbers. If $N = p\mathbb{Z}$, then $\Gamma(M, N) = \emptyset$. If $N = pq\mathbb{Z}$, then $\Gamma(M, N)$ is an infinite complete bipartite graph with vertex set $V_1 \cup V_2$, where $V_1 = p\mathbb{Z} \setminus pq\mathbb{Z}$ and $V_2 = q\mathbb{Z} \setminus pq\mathbb{Z}$ and so $\text{gr}(\Gamma(M, N)) = 4$.

**Corollary 2.8.** If $N$ is a prime submodule of $M$, then $\text{diam}(\Gamma(M, N)) \leq 2$ and $\text{gr}(\Gamma(M, N)) = 3$, whenever it contains a cycle.

**Proof.** Let $x, y$ be two distinct vertices which are not adjacent in $\Gamma(M, N)$. Thus there is an $a \in M \setminus N$ such that $I_aI_xM \subseteq N$. Since $N$ is a prime submodule, then $I_aM \subseteq N$. Thus $I_aI_bM \subseteq N$, and then $x - a - y$ is a path in $\Gamma(M, N)$. Then $\text{diam}(\Gamma(M, N)) \leq 2$. □

**Lemma 2.9.** Let $|\Gamma(M, N)| \geq 3$, $\text{gr}(\Gamma(M, N)) = \infty$ and $x \in Z^*(M, N)$ with $\text{deg}(x) > 1$. Then $Rx = \{0, x\}$ and $\text{ann}(x)$ is a prime ideal of $R$.

**Proof.** First we show that $Rx = \{0, x\}$. Let $u - x - v$ be a path in $\Gamma(M, N)$. Then $u - v$ is not an edge in $\Gamma(M, N)$ since $\text{gr}(\Gamma(M, N)) = \infty$. If $x \neq rx$ for some $r \in R$ and $rx \not\in N$, then by Lemma 2.4, $rx - u - x - v - rx$ is a cycle in
Γ(M, N), that is a contradiction. So, for every r ∈ R either rx = x or rx ∈ N. If there is an r ∈ R such that rx ∈ N, then we have either (1 + r)x ∈ N or (1 + r)x = x. These imply that x ∈ N or rx = 0. Therefore, we have shown that Rx = {0, x}.

Let a, b ∈ R and abx = 0. Then bx = 0 or bx = x. Hence, bx = 0 or ax = 0. So, ann(x) is a prime ideal of R.

□

Theorem 2.10. If N is a nonzero submodule of M and gr(Γ(M, N)) = ∞, then Γ(M, N) is a star graph.

Proof. Suppose that Γ(M, N) is not a star graph. Then there is a path in Γ(M, N) such as u − x − y − v. By Lemma 2.9, we have Rv = {0, y} and by assumption u and y are not adjacent, thus Iy M ≠ 0. So that Iy M = Ry. Also, x − y − v is a path, thus Ix Iy M ⊆ N and Ix Iy M ⊆ N. Hence, Iy Ry ⊆ N and Ix Ry ⊆ N. On the other hand, for every nonzero n ∈ N, we have

Iu Iy+n M ⊆ Iu Rx(y+n) ⊆ Iu(Ry+n) ⊆ N

and similarly Iy Ix+n M ⊆ N. So that x − y − v − (y + n) − x is a cycle in Γ(M, N), a contradiction. Therefore, Γ(M, N) is a star graph. □

Theorem 2.11. Let N be a nonzero submodule of M, |Γ(M, N)| ≥ 3 and Γ(M, N) is a star graph. Then the following statements are true:

(i) If x is the center vertex, then Ix = ann(M).

(ii) Γ(M, N) is a subgraph of Γ(M).

Proof. (i) By Lemma 2.9, we have Rx = {0, x}. Thus either Ix M = 0 or Ix M = Rx. Assume that Ix M = Rx. If y is a vertex of Γ(M, N) such that y ≠ x, then deg(y) = 1 and Ix Iy M ⊆ N. Thus Iy Rx ⊆ N. Since Ix+n Iy M ⊆ Iy Rx(x+n) ⊆ N for every nonzero element n ∈ N it concludes that y = x + n. In this case, every other vertices of Γ(M, N) are adjacent to y, a contradiction. Hence, Ix M = 0 and Ix = ann(M).

(ii) It is obvious. □

Theorem 2.12. If |N| ≥ 3 and Γ(M, N) is a complete bipartite graph which is not a star graph, then Ix2 M ⊆ N, for every x ∈ Z∗(M, N).

Proof. Let Z∗(M, N) = V1 ∪ V2, where V1 ∩ V2 = ∅. Suppose that Ix2 M ⊆ N for some x ∈ Z∗(M, N). Without loss of generality, we can assume that x ∈ V1. By a similar argument with Lemma 2.9, either Rx = {0, x} or there is an r ∈ R such that x ≠ rx and rx ∈ N. If Rx = {0, x}, then Ix M = Rx. Thus Ix Rx ⊆ N. Now, for every y ∈ V2 and n ∈ N we get

Iy Ix+n M ⊆ Iy Rx(x+n) ⊆ Iy(Rx+N) ⊆ N

and Ix Ix+n M ⊆ N. Then, x + n ∈ V1 ∩ V2, a contradiction. So, assume that x ≠ rx and rx ∈ N for some r ∈ R. Since Irx+x ⊆ Ix, then Ix Irx+x M ⊆ N and for all y ∈ V2, Iy Irx+x M ⊆ N. Thus rx+x ∈ V1 ∩ V2, a contradiction. □
An $R$-module $X$ is called a multiplication-like module if, for each nonzero submodule $Y$ of $X$, $\text{ann}(X) \subset \text{ann}(X/Y)$. Multiplication-like module have been studied in [8, 13].

A vertex $x$ of a connected graph $G$ is a cut-point, if there are vertices $u, v$ of $G$ such that $x$ is in every path from $u$ to $v$ and $x \neq u, x \neq v$. For a connected graph $G$, an edge $E$ of $G$ is defined to be a bridge if $G - \{E\}$ is disconnected, see [6].

**Theorem 2.13.** Let $M$ be a multiplication-like module and $N$ be a nonzero submodule of $M$. Then $\Gamma(M, N)$ has no cut-points.

**Proof.** Suppose that $x$ is a cut-point of $\Gamma(M, N)$. Then there exist vertices $u, v \in M \setminus N$ such that $x$ lies on every path from $u$ to $v$. By Theorem 2.5, the shortest path from $u$ to $v$ has length 2 or 3.

**Case 1.** Suppose that $u-x-v$ is a path of shortest length from $u$ to $v$. Since $x$ is a cut point, $u, v$ aren’t in a cycle. By a similar argument to that of Lemma 2.9, we have $Rx = \{0, x\}$. On the other hand, $I_x M \subseteq Rx$ and $M$ is a multiplication-like module, so we have $I_x M = Rx$. Hence $I_x Rx \subseteq N$ and $I_x M \subseteq N$. Also, for every nonzero $u \in N$, we have $I_u I_{z+n} M \subseteq I_u (Rx + N) \subseteq N$ and $I_u I_{z+n} M \subseteq N$. Therefore, $u - (x + n) - v$ is a path from $u$ to $v$, a contradiction.

**Case 2.** Suppose that $u-x-y-v$ is a path in $\Gamma(M, N)$. Then, we have $I_x M = Rx$ and for every nonzero $n \in N$, we have $I_y I_{z+n} M \subseteq N$ and $I_y I_{z+n} M \subseteq N$. Thus $u - (x + n) - y - v$ is a path from $u$ to $v$, a contradiction. □

**Theorem 2.14.** Let $M$ be a multiplication-like module and $N$ be a nonzero submodule of $M$. Then $\Gamma(M, N)$ has a bridge if and only if $\Gamma(M, N)$ is a graph on two vertices.

**Proof.** If $|\Gamma(M, N)| = 3$, then $\Gamma(M, N) = K^3$, by Theorem 2.11, and it has no bridge. Assume that $|\Gamma(M, N)| \geq 4$ and $x - y$ is a bridge. Thus there is not a cycle containing $x - y$. Without loss of generality, we can assume that $\text{deg}(x) > 1$. Thus, there exists a vertex $z \neq y$ such that $z - x$ is an edge of $\Gamma(M, N)$. Then $Rx = \{0, x\}$ and $I_x M = Rx$. Hence, for every $n \in N$, $I_z I_{z+n} M \subseteq N$ and $I_y I_{z+n} M \subseteq N$, a contradiction. Therefore, $\Gamma(M, N)$ has not a bridge. The converse is clear. □

3. **Submodule-based Zero Divisor Graph of Semisimple Modules**

A nonzero $R$-module $X$ is called simple if its only submodules are (0) and $X$. An $R$-module $X$ is called semisimple if it is a direct sum of simple modules. Also, $X$ is called homogenous semisimple if it is a direct sum of isomorphic simple modules.

In this section, $R$ is a commutative ring and $M$ is a finitely generated semisimple $R$-module such that its homogenous components are simple and
$N$ is a submodule of $M$. The following theorem has a crucial role in this section.

**Theorem 3.1.** Let $x, y \in M \setminus N$. Then $x, y$ are adjacent in $\Gamma(M, N)$ if and only if $Rx \cap Ry \subseteq N$.

**Proof.** Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$. By assumption $N$ is a submodule of $M$, so there exists a subset $A$ of $I$ such that $M = N \oplus \left( \bigoplus_{i \in A} M_i \right)$ and so $\text{ann}(M/N) = \text{ann}(\bigoplus_{i \in A} M_i) = \bigcap_{i \in A} \text{ann}(M_i)$. Assume that $x, y \in M \setminus N$ are adjacent in $\Gamma(M, N)$ and $Rx \cap Ry \not\subseteq N$. Thus there exists $\alpha \in I$ such that $M_\alpha \subseteq (Rx \cap Ry) \setminus N$. Also, there exist subsets $B \subseteq I$ and $C \subseteq I$ such that $M = Rx \oplus \left( \bigoplus_{i \in B} M_i \right)$ and $M = Ry \oplus \left( \bigoplus_{i \in C} M_i \right)$. Therefore, $I_x = \bigcap_{i \in B} \text{ann}(M_i)$ and $I_y = \bigcap_{i \in C} \text{ann}(M_i)$. Since $I_x I_y M \subseteq N$, we have $I_x I_y \subseteq \text{ann}(M/N)$. For every $i, j \in I$, $\text{ann}(M_i)$ and $\text{ann}(M_j)$ are coprime, then

$$I_x I_y = \left( \bigcap_{i \in B} \text{ann}(M_i) \right) \left( \bigcap_{i \in C} \text{ann}(M_i) \right) = \prod_{i \in B \cup C} \text{ann}(M_i) \subseteq \bigcap_{i \in A} \text{ann}(M_i) \subseteq \text{ann}(M_r),$$

for all $r \in A$. Thus for any $r \in A$ there exists $j_r \in B \cup C$ such that $\text{ann}(M_{j_r}) \subseteq \text{ann}(M_r)$. So that $\text{ann}(M_{j_r}) = \text{ann}(M_r)$ implies that $M_{j_r} \cong M_r$. Hence, $M_\alpha \subseteq \bigoplus_{i \in A} M_i \subseteq \bigoplus_{j \in B \cup C} M_j$.

Thus there exists $\gamma \in B \cup C$ such that $M_\alpha = M_\gamma$, also

$$M_\alpha \subseteq Rx \cap Ry = \left( \bigoplus_{i \in I \setminus \gamma} M_i \right) \cap \left( \bigoplus_{i \in \gamma} M_i \right).$$

Therefore, $\alpha \in I \setminus (B \cup C)$, a contradiction. The converse is obvious. $\square$

**Corollary 3.2.** Let $x, y \in M \setminus N$ be such that $x + N \neq y + N$. Then

(i) $x$ and $y$ are adjacent in $\Gamma(M, N)$ if and only if $x + N$ and $y + N$ are adjacent in $\Gamma(M/N)$.

(ii) if $x$ and $y$ are adjacent in $\Gamma(M, N)$, then all distinct elements of $x + N$ and $y + N$ are adjacent in $\Gamma(M, N)$.

**Proof.** (i) Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$. Suppose that $x$ and $y$ are adjacent in $\Gamma(M, N)$, $Rx = \bigoplus_{i \in A} M_i$, $Ry = \bigoplus_{i \in B} M_i$ and $N = \bigoplus_{i \in C} M_i$. Then $Rx + N = \bigoplus_{i \in A \cup C} M_i$ and $Ry + N = \bigoplus_{i \in B \cup C} M_i$. Thus,

$$(Rx + N) \cap (Ry + N) = \bigoplus_{i \in (A \cup C) \cap (B \cup C)} M_i = \bigoplus_{i \in (A \cap B) \cup C} M_i = (Rx \cap Ry) + N.$$ 

By Theorem 3.1, we have $Rx \cap Ry \subseteq N$ hence,

$$I_x + I_y + NM \subseteq (Rx + N) \cap (Ry + N) = (Rx \cap Ry) + N = N.$$
Therefore, \( x + N \) and \( y + N \) are adjacent in \( \Gamma(M/N) \). The converse is obvious.

(ii) Let \( x, y \in Z^*(M, N) \) be adjacent in \( \Gamma(M, N) \). Then \( Rx \cap Ry \subseteq N \) by Theorem 3.1. So for every \( n, n' \in N \) we have

\[
I_{x+n} I_{y+n'} M \subseteq R(x+n) \cap R(y+n') \subseteq (Rx+N) \cap (Ry+N) = N.
\]

Hence, \( x + n \) and \( y + n' \) are adjacent in \( \Gamma(M, N) \).

In the following theorem, we prove that the clique number of graphs \( \Gamma(M, N) \) and \( \Gamma(M/N) \) are equal.

**Theorem 3.3.** If \( N \) is a nonzero submodule of \( M \), then \( \omega(\Gamma(M/N)) = \omega(\Gamma(M, N)) \).

**Proof.** First we show that \( I_{m+n}^2 M \not\subseteq N \) for each \( 0 \neq m + n \in M/N \).

Assume that \( N = \oplus_{i \in A} M_i \) and \( m = (m_i)_{i \in I} \in M \setminus N \). Then \( I_{m+n} = \bigcap_{i \notin A, m_i = 0} \text{ann}(M_i) \). Hence, \( I_{m+n} = I_{m+n}^2 \). Thus \( I_{m+n}^2 M \notsubseteq N \) since there is at least one \( j \in I \setminus A \) such that \( m_j \neq 0 \).

Now, Corollary 3.2 implies that \( \omega(\Gamma(M/N)) \leq \omega(\Gamma(M, N)) \). Thus, it is enough to consider the case where \( \omega(\Gamma(M/N)) = d < \infty \). Assume that \( G \) is a complete subgraph of \( \Gamma(M, N) \) with vertices \( m_1, m_2, \ldots, m_{d+1} \), we provide a contradiction. Consider the subgraph \( G_* \) of \( \Gamma(M/N) \) with vertices \( m_1, m_2, \ldots, m_{d+1} + N \). By Corollary 3.2, \( G_* \) is a complete subgraph of \( \Gamma(M, N) \). Thus \( m_j + N = m_k + N \) for some \( 1 \leq j, k \leq d+1 \) with \( j \neq k \) since \( \omega(\Gamma(M/N)) = d \). We have \( I_{m_j} I_{m_k} M \subseteq N \). Therefore, \( Rm_j \cap Rm_k \subseteq N \) and so \( I_{m_j} + I_{m_k} + N M \subseteq N \). Hence, \( I_{m_j}^2 + N M \subseteq N \), that is a contradiction.

In the following theorem, we show that there is a relation between \( \omega(\Gamma(M, N)) \) and \( \chi(\Gamma(M, N)) \).

**Theorem 3.4.** Assume that \( M = \bigoplus_{i \in I} M_i \), where \( M_i \)’s are non-isomorphic simple submodules of \( M \) and \( N = \bigoplus_{i \in A} M_i \) is a submodule of \( M \) for some \( A \subset I \). Then \( \omega(\Gamma(M/N)) = \chi(\Gamma(M, N)) = |I| - |A| \).

**Proof.** Suppose that \( I \setminus A = \{1, \cdots, n\} \) so \( M_1, \cdots, M_n \notsubseteq N \). Let for \( 1 \leq k \leq n-1 \)

\[
L^k = \{m \in M : m \text{ has } k \text{ nonzero components}\}
\]

and let for \( 1 \leq s \leq n \)

\[
L^s = \{m \in L^1 : \text{the } s^{th} \text{ component of } m \text{ is nonzero}\}.
\]

If \( m \in L^1 \) and \( m' \in L^1 \) for some \( 1 \leq s, t \leq n \) with \( s \neq t \), then \( m \) and \( m' \) are adjacent and so \( K^n \) is a subgraph of \( \Gamma(M, N) \). Thus \( \omega(\Gamma(M/N)) \geq n \).

If \( m, m' \in L^s \) for some \( 1 \leq s \leq n \), then \( m, m' \) are not adjacent because \( \text{ann}(M_s) \not\subseteq I_{m}I_{m'} \) and so the elements of \( L^s \) have same color. On the other hand, if \( x \in L^t \) with \( t > 1 \), then there is not a complete subgraph \( K^b \) of \( \Gamma(M, N) \) containing \( x \), such that \( b \geq n \). Thus \( \omega(\Gamma(M, N)) = n \leq \chi(\Gamma(M, N)) \).

Also, if \( x \in L^t \) with \( t > 1 \), then there is an \( s \) with \( 1 \leq s \leq n \) such that \( x \) is not
The Kuartowski’s Theorem states: A graph $G$ is planar if and only if it contains no subgraph homeomorphic to $K^5$ or $K^{3,3}$.

**Theorem 3.5.** Let $N$ be a nonzero proper submodule of $M$ such that $N$ is not prime. Then $\Gamma(M, N)$ is not planar.

*Proof.* Assume that $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$ and $N = \bigoplus_{i \in A} M_i$ for some $A \subseteq I$. Let $I \setminus A = \{i, j\}$. Then $\Gamma(M, N)$ is a complete bipartite graph $K^{n, m}$, where $n = (|M_i| - 1)(\prod_{k \in I \setminus \{i, j\}} |M_k|)$ and $m = (|M_j| - 1)(\prod_{k \in I \setminus \{i, j\}} |M_k|)$. By hypotheses $N$ is a nonzero and $M_i$’s are non-isomorphic, so we have $n, m \geq 3$. Hence $\Gamma(M, N)$ has a subgraph homeomorphic to $K^{3,3}$. The cases $|I \setminus A| \geq 3$ are similar to that of the case $|I \setminus A| = 2$. □

**Theorem 3.6.** A nonzero submodule $N$ of $M$ is prime if and only if $Z^*(M, N) = \emptyset$.

*Proof.* Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$ and $N$ is prime. Then $N = \bigoplus_{i \in \Gamma \setminus \{k\}} M_i$, for some $k \in I$. If $x \in Z^*(M, N)$, then there exists a $y \in M \setminus N$ such that $I_x I_y M \subseteq N$. If $x \neq y$, then $Rx \cap Ry \subseteq N$, by Theorem 3.1. Thus either $M_k \not\subseteq Rx$ or $M_k \not\subseteq Ry$. Hence, either $Rx \subseteq N$ or $Ry \subseteq N$, a contradiction. Now, suppose that $x = y$ so by $I_x^2 M \subseteq N$ and hypotheses $I_x M \subseteq N$. Thus $I_x I_{x+n} M \subseteq N$ for every $0 \neq n \in N$. By a similar argument, we have either $x \in N$ or $x + n \in N$, a contradiction. Hence, $Z^*(M, N) = \emptyset$.

Conversely, assume that $Z^*(M, N) = \emptyset$. Then $\text{ann}(M/N)$ is prime ideal of $R$ by Proposition 2.3 and there exists a $k \in I$ such that $\text{ann}(M/N) = \text{ann}(M_k)$. Hence, $N = \bigoplus_{i \in \Gamma \setminus \{k\}} M_i$ is a prime submodule of $M$. □

A proper submodule $N$ of $M$ is called 2-absorbing if whenever $a, b \in R$, $m \in M$ and $am \in N$, then $am \in N$ or $bm \in N$ or $ab \in \text{ann}(M/N)$, see [10, 11]. In the following results, we study the behavior of $\Gamma(M, N)$ whenever $N$ is a 2-absorbing submodule of $M$.

**Theorem 3.7.** A submodule $N$ of $M$ is 2-absorbing if and only if at most two components of $M$ are zero in $N$.

*Proof.* Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$. Suppose that $N$ is a 2-absorbing submodule of $M$ and $N = \bigoplus_{i \in A} M_i$, where $A = I \setminus \{s, t, k\}$. Since for all $i \in I$, $\text{ann}(M_i)$ is prime, there are $a \in \text{ann}(M_s) \setminus (\text{ann}(M_t) \cup \text{ann}(M_k))$, $b \in \text{ann}(M_t) \setminus (\text{ann}(M_s) \cup \text{ann}(M_k))$ and $c \in \bigcap_{i \in A \setminus \{s,t\}} \text{ann}(M_i) \setminus (\text{ann}(M_s) \cup \text{ann}(M_t))$. Now, $abc \in \text{ann}(M/N)$ but $ab \not\in \text{ann}(M/N)$, $ac \not\in \text{ann}(M/N)$ and $bc \not\in \text{ann}(M/N)$. This contradicts with
Theorem 2.3 in [10]. Thus \(|A| \geq |I| - 2\) and at most two components of \(M\) are zero in \(N\).

Conversely, if one component of \(M\) is zero in \(N\), then \(N\) is a prime submodule of \(M\). Suppose that \(N = \bigoplus_{A \in A} M_i\), where \(A = I \setminus \{i, j\}\). Thus \(M_i, M_j \not\subseteq N\).

Suppose that \(a, b \in R, (m_i)_{i \in I} = (m) \in M \setminus N\) and \(abm \in N\). Then either \(m_i \neq 0\) or \(m_j \neq 0\). If \(m_i \neq 0\) and \(m_j \neq 0\), then \(ab \in \text{ann}(M_i) \cap \text{ann}(M_j) = \text{ann}(M/N)\). If \(m_i \neq 0\) and \(m_j = 0\), then \(ab \in \text{ann}(M_i)\) and so either \(a \in \text{ann}(M_i)\) or \(b \in \text{ann}(M_i)\). Hence, \(am \in N\) or \(bm \in N\). The case \(m_i = 0\) and \(m_j \neq 0\), is similar to the previous case. Therefore, \(N\) is a 2-absorbing submodule of \(M\).

\[\square\]

**Theorem 3.8.** \(N\) is a 2-absorbing submodule of \(M\) if and only if \(Z^*(M, N) = \emptyset\) or \(\Gamma(M, N)\) is a complete bipartite graph.

**Proof.** Let \(N\) be a 2-absorbing submodule of \(M\). If \(N\) is prime, then \(Z^*(M, N) = \emptyset\), by Theorem 3.6. Now, assume that \(N = \bigoplus_{i \in I \setminus \{j, k\}} M_i\) for some \(j, k \in I\) and \((m_i)_{i \in I} = (m) \in M \setminus N\). Thus \(I_m = \bigcap_{\{i \in I : m_i = 0\}} \text{ann}(M_i)\). If \(m_j \neq 0\) and \(m_k \neq 0\), then \(m \not\in Z(M, N)\). Let \(V_1 = \{(m_i)_{i \in I} \in M \setminus N : m_j = 0\}\) and \(V_2 = \{(m_i)_{i \in I} \in M \setminus N : m_k = 0\}\). Thus \(m - m'\) is an edge of \(\Gamma(M, N)\) for every \(m \in V_1\) and \(m' \in V_2\). Also, every vertices in \(V_1\) and \(V_2\) are not adjacent. Hence, \(\Gamma(M, N)\) is a complete bipartite graph.

Now, suppose that \(\Gamma(M, N)\) is a complete bipartite graph and \(N\) is not 2-absorbing. By Theorem 3.7, there are at least three components \(M_s, M_t, M_k\) such that \(M_s, M_t, M_k \not\subseteq N\). For \(i = s, t, k\) let \(v_i = (m_i)_{i \in I}\), where \(m_i \neq 0\) and \(m_j = 0\) for all \(j \neq i\). Then \(v_s - v_t - v_k - v_s\) is a cycle in \(\Gamma(M, N)\). Thus \(\text{gr}(\Gamma(M, N)) = 3\) and so \(\Gamma(M, N)\) is not bipartite graph, by Theorem 1 of Sec. 1.2 in [5]. Hence, \(N\) is a 2-absorbing submodule of \(M\). \[\square\]

**Example 3.9.** Let \(M = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7\). Then every nonzero submodule \(N\) of \(M\) is 2-absorbing. Thus either \(Z^*(M, N) = \emptyset\) or \(\Gamma(M, N)\) is a complete bipartite graph. In particular, if \(N = \mathbb{Z}_7\), then \(\Gamma(M, N) = K^7, 28\).

**Acknowledgments**

The author is thankful of referees for their valuable comments.

**References**

