A Submodule-Based Zero Divisor Graph for Modules

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Abstract. Let $R$ be a commutative ring with identity and $M$ be an $R$-module. The zero divisor graph of $M$ is denoted by $\Gamma(M)$. In this study, we are going to generalize the zero divisor graph $\Gamma(M)$ to submodule-based zero divisor graph $\Gamma(M, N)$ by replacing elements whose product is zero with elements whose product is in some submodule $N$ of $M$. The main objective of this paper is to study the interplay of the properties of submodule $N$ and the properties of $\Gamma(M, N)$.

Keywords: Zero divisor graph, Submodule-based zero divisor graph, Semisimple module.


1. Introduction

Let $R$ be a commutative ring with identity. The zero divisor graph of $R$, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero divisor of $R$ with two distinct vertices $x$ and $y$ are adjacent by an edge if and only

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if $xy = 0$. The idea of a zero divisor graph of a commutative ring was introduced by Beck in [3] where he was mainly interested with colorings of rings. The definition above first is appeared in [2], which contains several fundamental results concerning $\Gamma(R)$. The zero-divisor graph of a commutative ring is further examined by Anderson, Levy and Shapiro, Mulay in [1, 9]. Also, the ideal-based zero divisor graph of $R$ is defined by Redmond, in [12].

The zero divisor graph for modules over commutative rings has been defined by Behboodi in [4] as a generalization of zero divisor graph of rings. Let $R$ be a commutative ring and $M$ be an $R$-module, for $x \in M$, we denote the annihilator of the factor module $M/Rx$ by $I_x$. An element $x \in M$ is called a zero divisor, if either $x = 0$ or $I_xI_yM = 0$ for some $y \neq 0$ with $I_y \subset R$. The set of zero divisors of $M$ is denoted by $Z(M)$ and the associated graph to $M$ with vertices in $Z^*(M) = Z(M) \setminus \{0\}$ is denoted by $\Gamma(M)$, such that two different vertices $x$ and $y$ are adjacent provided $I_xI_yM = 0$.

In this paper, we introduce the submodule-based zero divisor graph that is a generalization of zero divisor graph for modules. Let $R$ be a commutative ring, $M$ be an $R$-module and $N$ be a proper submodule of $M$. An element $x \in M$ is called zero divisor with respect to $N$, if either $x \in N$ or $I_xI_yM \subseteq N$ for some $y \in M \setminus N$ with $I_y \subset R$. We denote $Z(M, N)$ for the set of zero divisors of $M$ with respect to $N$. Also, we denote the associated graph to $M$ with vertices $Z^*(M, N) = Z(M, N) \setminus N$ by $\Gamma(M, N)$, and two different vertices $x$ and $y$ are adjacent provided $I_xI_yM \subseteq N$.

In the second section, we define a submodule-based zero divisor graph for a module and we study basic properties of this graph. In the third section, if $M$ is a finitely generated semisimple $R$-module such that its homogenous components are simple and $N$ is a submodule of $M$, we determine some relations between $\Gamma(M, N)$ and $\Gamma(M/N)$, where $M/N$ is the quotient module of $M$, we show that the clique number and chromatic number of $\Gamma(M, N)$ are equal. Also, we determine some submodule of $M$ such that $\Gamma(M, N)$ is an empty or a complete bipartite graph.

Let $\Gamma$ be a (undirected) graph. We say that $\Gamma$ is connected if there is a path between any two distinct vertices. For vertex $x$ the number of graph edges which touch $x$ is called the degree of $x$ and is denoted by $\deg(x)$. For vertices $x$ and $y$ of $\Gamma$, we define $d(x, y)$ to be the length of a shortest path between $x$ and $y$, if there is no path, then $d(x, y) = \infty$. The diameter of $\Gamma$ is $\text{diam}(\Gamma) = \sup\{d(x, y) | x$ and $y$ are vertices of $\Gamma\}$. The girth of $\Gamma$, denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$ ($\text{gr}(\Gamma) = \infty$ if $\Gamma$ contains no cycle).

A graph $\Gamma$ is complete if any two distinct vertices are adjacent. The complete graph with $n$ vertices is denoted by $K^n$ (we allow $n$ to be an infinite cardinal). The clique number, $\omega(\Gamma)$, is the greatest integer $n > 1$ such that $K^n \subseteq \Gamma$, and $\omega(\Gamma) = \infty$ if $K^n \subseteq \Gamma$ for all $n \geq 1$. A complete bipartite graph is a graph $\Gamma$ which may be partitioned into two disjoint nonempty vertex sets $V_1$ and $V_2$.
such that two distinct vertices are adjacent if and only if they are in different vertex sets. If one of the vertex sets is a singleton, then we call that \( \Gamma \) is a star graph. We denote the complete bipartite graph by \( K^{m,n} \), where \( |V_1| = m \) and \( |V_2| = n \) (again, we allow \( m \) and \( n \) to be infinite cardinals); so a star graph is \( K^{1,n} \), for some \( n \in \mathbb{N} \).

The chromatic number, \( \chi(\Gamma) \), of a graph \( \Gamma \) is the minimum number of colors needed to color the vertices of \( \Gamma \), so that no two adjacent vertices share the same color. A graph \( \Gamma \) is called planar if it can be drawn in such a way that no two edges intersect.

Throughout this study, \( R \) is a commutative ring with nonzero identity, \( M \) is a unitary \( R \)-module and \( N \) is a proper submodule of \( M \). Given any subset \( S \) of \( M \), the annihilator of \( S \) is denoted by \( \text{ann}(S) = \{ r \in R | rs = 0 \text{ for all } s \in S \} \) and the cardinal number of \( S \) is denoted by \( |S| \).

2. Submodule-based Zero Divisor Graph

Recall that \( R \) is a commutative ring, \( M \) is an \( R \)-module and \( N \) is a proper submodule of \( M \). For \( x \in M \), we denote \( \text{ann}(M/Rx) \) by \( I_x \).

**Definition 2.1.** Let \( M \) be an \( R \)-module and \( N \) be a proper submodule of \( M \). An \( x \in M \) is called a zero divisor with respect to \( N \) if \( x \in N \) or \( I_xM \subseteq N \) for some \( y \in M \setminus N \) with \( I_y \subseteq R \).

We denote the set of zero divisors of \( M \) with respect to \( N \) by \( Z(M,N) \) and \( Z^*(M,N) = Z(M,N) \setminus N \). The submodule-based zero divisor graph of \( M \) with respect to \( N \), \( \Gamma(M,N) \), is an undirected graph with vertices \( Z^*(M,N) \) such that distinct vertices \( x \) and \( y \) are adjacent if and only if \( I_xI_yM \subseteq N \).

The following example shows that \( Z(M/N) \) and \( Z(M,N) \) are different from each other.

**Example 2.2.** Let \( M = \mathbb{Z} \oplus \mathbb{Z} \) and \( N = 2\mathbb{Z} \oplus 0 \). Then \( I_{(m,n)} = 0 \), for all \( (m,n) \in \mathbb{Z} \oplus \mathbb{Z} \). But \( I_{(m,n)N} = 2n\mathbb{Z} \) whenever \( m \in 2\mathbb{Z} \) and \( I_{(m,n)N} = 2\mathbb{Z} \) whenever \( m \notin 2\mathbb{Z} \). Thus \( (1,0), (1,1) \in Z^*(M,N) \) are adjacent in \( \Gamma(M,N) \), but \( (1,0) + N, (1,1) + N \notin Z^*(M/N) \).

**Proposition 2.3.** If \( Z^*(M,N) = \emptyset \), then \( \text{ann}(M/N) \) is a prime ideal of \( R \).

**Proof.** Suppose that \( \text{ann}(M/N) \) is not prime. Then there are ideals \( I \) and \( J \) of \( R \) such that \( IJ \subseteq N \) but \( IM \nsubseteq N \) and \( JM \nsubseteq N \). Let \( x \in IM \setminus N \) and \( y \in JM \setminus N \). Then \( I_xJ_yM \subseteq IJM \subseteq N \) and \( I_y \subseteq R \). Thus \( x \in Z^*(M,N) \), a contradiction. Hence, \( \text{ann}(M/N) \) is a prime ideal of \( R \). \( \square \)

**Lemma 2.4.** Let \( x, y \in Z^*(M,N) \). If \( x - y \) is an edge in \( \Gamma(M,N) \), then for each \( 0 \neq r \in R \), either \( ry \in N \) or \( x - ry \) is also an edge in \( \Gamma(M,N) \).

**Proof.** Let \( x, y \in Z^*(M,N) \) and \( r \in R \). Assume that \( x - y \) is an edge in \( \Gamma(M,N) \) and \( ry \notin N \). Then \( I_xI_yM \subseteq N \). It is clear that \( I_{rx} \subseteq I_x \). So that \( I_xI_{ry}M \subseteq I_xI_yM \subseteq N \) and therefore, \( x - ry \) is an edge in \( \Gamma(M,N) \). \( \square \)
It is shown that the graphs are defined in [12] and [4], are connected with diameter less than or equal to three. Moreover, it shown that if those graphs contain a cycle, then they have the girth less than or equal to four. In the next theorems, we extend these results to a submodule-based zero divisor graph.

**Theorem 2.5.** $\Gamma(M, N)$ is a connected graph and $\text{diam}(\Gamma(M, N)) \leq 3$.

*Proof.* Let $x$ and $y$ be distinct vertices of $\Gamma(M, N)$. Then, there are $a, b \in Z^+(M, N)$ with $I_aI_xM \subseteq N$ and $I_bI_yM \subseteq N$ (we allow $a, b \in \{x, y\}$). If $I_aI_bM \subseteq N$, then $x - a - b - y$ is a path, thus $d(x, y) \leq 3$. If $I_aI_bM \not\subseteq N$, then $R\cap Rb \not\subseteq N$, and for every $d \in (R\cap Rb) \setminus N$, $x - d - y$ is a path of length 2, $d(x, y) \leq 2$, by Lemma 2.4. Hence, we conclude that $\text{diam}(\Gamma(M, N)) \leq 3$. □

**Theorem 2.6.** If $\Gamma(M, N)$ contains a cycle, then $\text{gr}(\Gamma(M, N)) \leq 4$.

*Proof.* We have $\text{gr}(\Gamma(M, N)) \leq 7$, by Proposition 1.3.2 in [7] and Theorem 2.5. Assume that $x_1 - x_2 - \cdots - x_7 - x_1$ is a cycle in $\Gamma(M, N)$. If $x_1 = x_4$ then it is clear that $\text{gr}(\Gamma(M, N)) \leq 3$. So, suppose that $x_1 \neq x_4$. Then we have the following two cases:

**Case 1.** If $x_1$ and $x_4$ are adjacent in $\Gamma(M, N)$, then $x_1 - x_2 - x_3 - x_4 - x_1$ is a cycle and $\text{gr}(\Gamma(M, N)) \leq 4$.

**Case 2.** Suppose that $x_1$ and $x_4$ are not adjacent in $\Gamma(M, N)$. Then $I_{x_1}I_{x_4}M \not\subseteq N$ and so there is a $z \in (Rx_1 \cap Rx_4) \setminus N$. If $z = x_1$, then $z \neq x_4$ and $x_3 - x_4 - x_3 - z - x_3$ is a cycle in $\Gamma(M, N)$, by Lemma 2.4. If $z \neq x_1$, then by Lemma 2.4, $x_1 - x_2 - z - x_7 - x_1$ is a cycle and $\text{gr}(\Gamma(M, N)) \leq 4$.

For cycles with length 5 or 6, by using a similar argument as above, one can shows that $\text{gr}(\Gamma(M, N)) \leq 4$. □

**Example 2.7.** Assume that $M = \mathbb{Z}$ and $p, q$ are two prime numbers. If $N = pq\mathbb{Z}$, then $\Gamma(M, N) = \emptyset$. If $N = pq\mathbb{Z}$, then $\Gamma(M, N)$ is an infinite complete bipartite graph with vertex set $V_1 \cup V_2$, where $V_1 = pq\mathbb{Z} \setminus pq\mathbb{Z}$ and $V_2 = q\mathbb{Z} \setminus pq\mathbb{Z}$ and so $\text{gr}(\Gamma(M, N)) = 4$.

**Corollary 2.8.** If $N$ is a prime submodule of $M$, then $\text{diam}(\Gamma(M, N)) \leq 2$ and $\text{gr}(\Gamma(M, N)) = 3$, whenever it contains a cycle.

*Proof.* Let $x, y$ be two distinct vertices which are not adjacent in $\Gamma(M, N)$. Thus there is an $a \in M \setminus N$ such that $I_aI_xM \subseteq N$. Since $N$ is a prime submodule, then $I_aM \subseteq N$. Thus $I_aI_yM \subseteq N$, and then $x - a - y$ is a path in $\Gamma(M, N)$. Then $\text{diam}(\Gamma(M, N)) \leq 2$. □

**Lemma 2.9.** Let $|\Gamma(M, N)| \geq 3$, $\text{gr}(\Gamma(M, N)) = \infty$ and $x \in Z^+(M, N)$ with $\text{deg}(x) > 1$. Then $Rx = \{0, x\}$ and $\text{ann}(x)$ is a prime ideal of $R$.

*Proof.* First we show that $Rx = \{0, x\}$. Let $u - x - v$ be a path in $\Gamma(M, N)$. Then $u - v$ is not an edge in $\Gamma(M, N)$ since $\text{gr}(\Gamma(M, N)) = \infty$. If $x \neq rx$ for some $r \in R$ and $rx \not\in N$, then by Lemma 2.4, $rx - u - x - v - rx$ is a cycle in

\[ a, b \in Z^+(M, N) \]
Γ(M, N), that is a contradiction. So, for every \( r \in R \) either \( rx = x \) or \( rx \in N \).

If there is an \( r \in R \) such that \( rx \in N \), then we have either \((1+r)x \in N \) or \((1+r)x = x \). These imply that \( x \in N \) or \( rx = 0 \). Therefore, we have shown that \( Rx = \{0, x\} \).

Let \( a, b \in R \) and \( abx = 0 \). Then \( bx = 0 \) or \( bx = x \). Hence, \( bx = 0 \) or \( ax = 0 \).

So, \( \text{ann}(x) \) is a prime ideal of \( R \). \( \square \)

**Theorem 2.10.** If \( N \) is a nonzero submodule of \( M \) and \( \text{gr}(\Gamma(M, N)) = \infty \), then \( \Gamma(M, N) \) is a star graph.

**Proof.** Suppose that \( \Gamma(M, N) \) is not a star graph. Then there is a path in \( \Gamma(M, N) \) such as \( u - x - y - v \). By Lemma 2.9, we have \( Ry = \{0, y\} \) and by assumption \( u \) and \( v \) are not adjacent, thus \( I_yM \neq 0 \). So that \( I_yM = Ry \). Also, \( x - y - v \) is a path, thus \( I_yI_yM \subseteq N \) and \( I_yI_yM \subseteq N \). Hence, \( I_yRy \subseteq N \) and \( I_yRy \subseteq N \).

On the other hand, for every nonzero \( n \in N \), we have

\[
I_xI_yM \subseteq I_xR(y + n) \subseteq I_y(Ry + N) \subseteq N
\]

and similarly \( I_xI_yM \subseteq N \). So that \( x - y - v - (y + n) - x \) is a cycle in \( \Gamma(M, N) \), a contradiction. Therefore, \( \Gamma(M, N) \) is a star graph. \( \square \)

**Theorem 2.11.** Let \( N \) be a nonzero submodule of \( M \), \( |\Gamma(M, N)| \geq 3 \) and \( \Gamma(M, N) \) is a star graph. Then the following statements are true:

1. If \( x \) is the center vertex, then \( I_x = \text{ann}(M) \).
2. \( \Gamma(M, N) \) is a subgraph of \( \Gamma(M) \).

**Proof.** (i) By Lemma 2.9, we have \( Rx = \{0, x\} \). Thus either \( I_xM = 0 \) or \( I_xM = Rx \). Assume that \( I_xM = Rx \). If \( y \) is a vertex of \( \Gamma(M, N) \) such that \( y \neq x \), then \( \text{deg}(y) = 1 \) and \( I_xI_yM \subseteq N \). Thus \( I_yRx \subseteq N \). Since \( I_x+nI_yM \subseteq I_yR(x + n) \subseteq N \) for every nonzero element \( n \in N \) it concludes that \( y = x + n \). In this case, every other vertices of \( \Gamma(M, N) \) are adjacent to \( y \), a contradiction. Hence, \( I_xM = 0 \) and \( I_x = \text{ann}(M) \).

(ii) It is obvious. \( \square \)

**Theorem 2.12.** If \( |N| \geq 3 \) and \( \Gamma(M, N) \) is a complete bipartite graph which is not a star graph, then \( I^2_xM \subseteq N \), for every \( x \in Z^*(M, N) \).

**Proof.** Let \( Z^*(M, N) = V_1 \cup V_2 \), where \( V_1 \cap V_2 = \emptyset \). Suppose that \( I^2_xM \subseteq N \) for some \( x \in Z^*(M, N) \). Without loss of generality, we can assume that \( x \in V_1 \).

By a similar argument with Lemma 2.9, either \( Rx = \{0, x\} \) or there is an \( r \in R \) such that \( x \neq rx \) and \( rx \in N \). If \( Rx = \{0, x\} \), then \( I_xM = Rx \). Thus \( I_xRx \subseteq N \). Now, for every \( y \in V_2 \) and \( n \in N \) we get

\[
I_yI_{x+n}M \subseteq I_yR(x + n) \subseteq I_y(Rx + N) \subseteq N
\]

and \( I_xI_{x+n}M \subseteq N \). Then, \( x + n \in V_1 \cap V_2 \), a contradiction. So, assume that \( x \neq rx \) and \( rx \in N \) for some \( r \in R \). Since \( I_{rx+x} \subseteq I_x \), then \( I_xI_{rx+x}M \subseteq N \) and for all \( y \in V_2 \), \( I_yI_{rx+x}M \subseteq N \). Thus \( rx + x \in V_1 \cap V_2 \), a contradiction. \( \square \)
An $R$-module $X$ is called a multiplication-like module if, for each nonzero submodule $Y$ of $X$, $\text{ann}(X) \subset \text{ann}(X/Y)$. Multiplication-like module have been studied in [8, 13].

A vertex $x$ of a connected graph $G$ is a cut-point, if there are vertices $u, v$ of $G$ such that $x$ is in every path from $u$ to $v$ and $x \neq u, x \neq v$. For a connected graph $G$, an edge $E$ of $G$ is defined to be a bridge if $G - \{E\}$ is disconnected, see [6].

**Theorem 2.13.** Let $M$ be a multiplication-like module and $N$ be a nonzero submodule of $M$. Then $\Gamma(M, N)$ has no cut-points.

*Proof.* Suppose that $x$ is a cut-point of $\Gamma(M, N)$. Then there exist vertices $u, v \in M \setminus N$ such that $x$ lies on every path from $u$ to $v$. By Theorem 2.5, the shortest path from $u$ to $v$ has length 2 or 3.

**Case 1.** Suppose that $u - x - v$ is a path of shortest length from $u$ to $v$. Since $x$ is a cut point, $u, v$ aren’t in a cycle. By a similar argument to that of Lemma 2.9, we have $Rx = \{0, x\}$. On the other hand, $I_x M \subseteq Rx$ and $M$ is a multiplication-like module, so we have $I_x M = Rx$. Hence $I_u Rx \subseteq N$ and $I_u Rx \subseteq N$. Also, for every nonzero $n \in N$, we have $I_u(I_x+n)M \subseteq I_u(Rx+N) \subseteq N$ and $I_u(I_x+n)M \subseteq N$. Therefore, $u - (x + n) - v$ is a path from $u$ to $v$, a contradiction.

**Case 2.** Suppose that $u - x - y - v$ is a path in $\Gamma(M, N)$. Then, we have $I_x M = Rx$ and for every nonzero $n \in N$, we have $I_y(I_x+n)M \subseteq N$ and $I_y(I_x+n)M \subseteq N$. Thus $u - (x + n) - y - v$ is a path from $u$ to $v$, a contradiction. □

**Theorem 2.14.** Let $M$ be a multiplication-like module and $N$ be a nonzero submodule of $M$. Then $\Gamma(M, N)$ has a bridge if and only if $\Gamma(M, N)$ is a graph on two vertices.

*Proof.* If $|\Gamma(M, N)| = 3$, then $\Gamma(M, N) = K^3$, by Theorem 2.11, and it has no bridge. Assume that $|\Gamma(M, N)| \geq 4$ and $x - y$ is a bridge. Thus there is not a cycle containing $x - y$. Without loss of generality, we can assume that $\deg(x) > 1$. Thus, there exists a vertex $z \neq y$ such that $z - x$ is an edge of $\Gamma(M, N)$. Then $Rx = \{0, x\}$ and $I_x M = Rx$. Hence, for every $n \in N$, $I_z(I_x+n)M \subseteq N$ and $I_y(I_x+n)M \subseteq N$, a contradiction. Therefore, $\Gamma(M, N)$ has not a bridge. The converse is clear. □

3. **Submodule-based Zero Divisor Graph of Semisimple Modules**

A nonzero $R$-module $X$ is called simple if its only submodules are $(0)$ and $X$. An $R$-module $X$ is called semisimple if it is a direct sum of simple modules. Also, $X$ is called homogenous semisimple if it is a direct sum of isomorphic simple modules.

In this section, $R$ is a commutative ring and $M$ is a finitely generated semisimple $R$-module such that its homogenous components are simple and
$N$ is a submodule of $M$. The following theorem has a crucial role in this section.

**Theorem 3.1.** Let $x, y \in M \setminus N$. Then $x, y$ are adjacent in $\Gamma(M, N)$ if and only if $Rx \cap Ry \subseteq N$.

**Proof.** Let $M = \bigoplus_{i \in I} M_i$, where $M_i$'s are non-isomorphic simple submodules of $M$. By assumption $N$ is a submodule of $M$, so there exists a subset $A$ of $I$ such that $M = N \oplus (\bigoplus_{i \in A} M_i)$ and so $\text{ann}(M/N) = \text{ann}(\bigoplus_{i \in A} M_i) = \bigcap_{i \in A} \text{ann}(M_i)$. Assume that $x, y \in M \setminus N$ are adjacent in $\Gamma(M, N)$ and $Rx \cap Ry \not\subseteq N$. Thus there exists $\alpha \in I$ such that $M_\alpha \subseteq (Rx \cap Ry) \setminus N$. Also, there exist subsets $B \subseteq I$ and $C \subseteq I$ such that $M = Rx \oplus (\bigoplus_{i \in B} M_i)$ and $M = Ry \oplus (\bigoplus_{i \in C} M_i)$. Therefore, $I_x = \bigcap_{i \in B} \text{ann}(M_i)$ and $I_y = \bigcap_{i \in C} \text{ann}(M_i)$. Since $I_x I_y M \subseteq N$, we have $I_x I_y \subseteq \text{ann}(M/N)$. For every $i, j \in I$, $\text{ann}(M_i)$ and $\text{ann}(M_j)$ are coprime, then

$$I_x I_y = [\bigcap_{i \in B} \text{ann}(M_i)] [\bigcap_{i \in C} \text{ann}(M_i)] = \prod_{i \in B \cup C} \text{ann}(M_i) \subseteq \bigcap_{i \in A} \text{ann}(M_i) \subseteq \text{ann}(M_r),$$

for all $r \in A$. Thus for any $r \in A$ there exists $j_r \in B \cup C$ such that $\text{ann}(M_{j_r}) \subseteq \text{ann}(M_r)$. So that $\text{ann}(M_{j_r}) = \text{ann}(M_r)$ implies that $M_{j_r} \cong M_r$. Hence,

$$M_\alpha \subseteq \bigoplus_{i \in A} M_i \subseteq \bigoplus_{j \in B \cup C} M_j.$$

Thus there exists $\gamma \in B \cup C$ such that $M_\alpha = M_\gamma$, also

$$M_\alpha \subseteq Rx \cap Ry = (\bigoplus_{i \in I \setminus B} M_i) \cap (\bigoplus_{i \in I \setminus C} M_i).$$

Therefore, $\alpha \in I \setminus (B \cup C)$, a contradiction. The converse is obvious. \qed

**Corollary 3.2.** Let $x, y \in M \setminus N$ be such that $x + N \neq y + N$. Then

(i) $x$ and $y$ are adjacent in $\Gamma(M, N)$ if and only if $x + N$ and $y + N$ are adjacent in $\Gamma(M/N)$.

(ii) if $x$ and $y$ are adjacent in $\Gamma(M, N)$, then all distinct elements of $x + N$ and $y + N$ are adjacent in $\Gamma(M, N)$.

**Proof.** (i) Let $M = \bigoplus_{i \in I} M_i$, where $M_i$'s are non-isomorphic simple submodules of $M$. Suppose that $x$ and $y$ are adjacent in $\Gamma(M, N)$, $Rx = \bigoplus_{i \in A} M_i$, $Ry = \bigoplus_{i \in B} M_i$ and $N = \bigoplus_{i \in C} M_i$. Then $Rx + N = \bigoplus_{i \in A \cup C} M_i$ and $Ry + N = \bigoplus_{i \in B \cup C} M_i$. Thus,

$$(Rx + N) \cap (Ry + N) = \bigoplus_{i \in (A \cup C) \cap (B \cup C)} M_i = \bigoplus_{i \in (A \cap B) \cup C} M_i = (Rx \cap Ry) + N.$$ 

By Theorem 3.1, we have $Rx \cap Ry \subseteq N$ hence,

$$I_x + N I_y + N M \subseteq (Rx + N) \cap (Ry + N) = (Rx \cap Ry) + N = N.$$
Therefore, $x + N$ and $y + N$ are adjacent in $\Gamma(M/N)$. The converse is obvious.

(ii) Let $x, y \in Z^*(M, N)$ be adjacent in $\Gamma(M, N)$. Then $Rx \cap Ry \subseteq N$ by Theorem 3.1. So for every $n, n' \in N$ we have

$$I_{x+n}I_{y+n'}M \subseteq R(x+n) \cap R(y+n') \subseteq (Rx+N) \cap (Ry+N) = N.$$ 

Hence, $x + n$ and $y + n'$ are adjacent in $\Gamma(M, N)$. \qed

In the following theorem, we prove that the clique number of graphs $\Gamma(M, N)$ and $\Gamma(M/N)$ are equal.

**Theorem 3.3.** If $N$ is a nonzero submodule of $M$, then $\omega(\Gamma(M/N)) = \omega(\Gamma(M, N))$.

**Proof.** First we show that $I_{m+N}^2 \not\subseteq N$ for each $0 \neq m + N \in M/N$. Assume that $N = \bigoplus_{i \in A} M_i$ and $m = (m_i)_{i \in I} \in M \setminus N$. Then $I_{m+N} = \bigcap_{i \in A, m_i=0} \text{ann}(M_i)$. Hence, $I_{m+N} = I_{m+N}^\ast$. Thus $I_{m+N}^2 \not\subseteq N$ since there is at least one $j \in I \setminus A$ such that $m_j \neq 0$.

Now, Corollary 3.2 implies that $\omega(\Gamma(M/N)) \leq \omega(\Gamma(M, N))$. Thus, it is enough to consider the case where $\omega(\Gamma(M/N)) = d < \infty$. Assume that $G$ is a complete subgraph of $\Gamma(M, N)$ with vertices $m_1, m_2, \cdots, m_{d+1}$, we provide a contradiction. Consider the subgraph $G_\ast$ of $\Gamma(M/N)$ with vertices $m_1 + N, \cdots, m_{d+1} + N$. By Corollary 3.2, $G_\ast$ is a complete subgraph of $\Gamma(M, N)$. Thus $m_j + N = m_k + N$ for some $1 \leq j, k \leq d+1$ with $j \neq k$ since $\omega(\Gamma(M/N)) = d$. We have $I_{m_j} \cap I_{m_k} \subseteq N$. Therefore, $Rm_j \cap Rm_k \subseteq N$ and so $I_{m_j+N}I_{m_k+N}M \subseteq N$. Hence, $I_{m_j+N}^2 M \subseteq N$, that is a contradiction. \qed

In the following theorem, we show that there is a relation between $\omega(\Gamma(M, N))$ and $\chi(\Gamma(M, N))$.

**Theorem 3.4.** Assume that $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$ and $N = \bigoplus_{i \in A} M_i$ is a submodule of $M$ for some $A \subset I$. Then $\omega(\Gamma(M/N)) = \chi(\Gamma(M, N)) = |I| - |A|$.

**Proof.** Suppose that $I \setminus A = \{1, \cdots, n\}$ so $M_1, \cdots, M_n \not\subseteq N$. Let for $1 \leq k \leq n-1$

$$L_k = \{m \in M : m \text{ has } k \text{ nonzero components}\}$$

and let for $1 \leq s \leq n$

$$L^1_s = \{m \in L^1 : \text{the } s^{th} \text{ component of } m \text{ is nonzero}\}.$$

If $m \in L^1_s$ and $m' \in L^1_t$ for some $1 \leq s, t \leq n$ with $s \neq t$, then $m$ and $m'$ are adjacent and so $K^n$ is a subgraph of $\Gamma(M, N)$. Thus $\omega(\Gamma(M, N)) \geq n$. If $m, m' \in L^1_s$ for some $1 \leq s \leq n$, then $m, m'$ are not adjacent because $\text{ann}(M_s) \not\subseteq I_m \cap I_{m'}$ and so the elements of $L^1_s$ have same color. On the other hand, if $x \in L^1$ with $t > 1$, then there is not a complete subgraph $K^b$ of $\Gamma(M, N)$ containing $x$, such that $b \geq n$. Thus $\omega(\Gamma(M, N)) = n \leq \chi(\Gamma(M, N))$. Also, if $x \in L^1$ with $t > 1$, then there is an $s$ with $1 \leq s \leq n$ such that $x$ is not
adjacent to each element of $L_1$. Thus the color of $x$ is same as the elements of $L_1$. Thus $\chi(\Gamma(M, N)) = n$. □

The Kwartowski’s Theorem states: A graph $G$ is planar if and only if it contains no subgraph homeomorphic to $K^5$ or $K^{3,3}$.

**Theorem 3.5.** Let $N$ be a nonzero proper submodule of $M$ such that $N$ is not prime. Then $\Gamma(M, N)$ is not planar.

**Proof.** Assume that $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$ and $N = \bigoplus_{i \in A} M_i$ for some $A \subseteq I$. Let $I \setminus A = \{i, j\}$. Then $\Gamma(M, N)$ is a complete bipartite graph $K^{n, m}$, where $n = (|M_i| - 1)(\prod_{k \in I \setminus \{i, j\}} |M_k|)$ and $m = (|M_j| - 1)(\prod_{k \in I \setminus \{i, j\}} |M_k|)$. By hypotheses $N$ is a nonzero and $M_i$’s are non-isomorphic, so we have $n, m \geq 3$. Hence $\Gamma(M, N)$ has a subgraph homeomorphic to $K^{3,3}$. The cases $|I \setminus A| \geq 3$ are similar to that of the case $|I \setminus A| = 2$. □

**Theorem 3.6.** A nonzero submodule $N$ of $M$ is prime if and only if $Z^*(M, N) = \emptyset$.

**Proof.** Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$ and $N$ is prime. Then $N = \bigoplus_{i \in I \setminus \{k\}} M_i$, for some $k \in I$. If $x \in Z^*(M, N)$, then there exists a $y \in M \setminus N$ such that $I_x I_y M \subseteq N$. If $x \neq y$, then $Rx \cap Ry \subseteq N$, by Theorem 3.1. Thus either $M_k \not\subseteq Rx$ or $M_k \not\subseteq Ry$. Hence, either $Rx \not\subseteq N$ or $Ry \not\subseteq N$, a contradiction. Now, suppose that $x = y$ so by $I_x^2 M \subseteq N$ and hypotheses $I_x M \subseteq N$. Thus $I_{x+n} I_x M \subseteq N$ for every $0 \neq n \in N$. By a similar argument, we have either $x \in N$ or $x + n \in N$, a contradiction. Hence, $Z^*(M, N) = \emptyset$.

Conversely, assume that $Z^*(M, N) = \emptyset$. Then $\text{ann}(M/N)$ is prime ideal of $R$ by Proposition 2.3 and there exists a $k \in I$ such that $\text{ann}(M/N) = \text{ann}(M_k)$. Hence, $N = M_k$ is a prime submodule of $M$. □

A proper submodule $N$ of $M$ is called 2-absorbing if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in \text{ann}(M/N)$, see [10, 11]. In the following results, we study the behavior of $\Gamma(M, N)$ whenever $N$ is a 2-absorbing submodule of $M$.

**Theorem 3.7.** A submodule $N$ of $M$ is 2-absorbing if and only if at most two components of $M$ are zero in $N$.

**Proof.** Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$. Suppose that $N$ is a 2-absorbing submodule of $M$ and $N = \bigoplus_{i \in A} M_i$, where $A = I \setminus \{s, t, k\}$. Since for all $i \in I$, $\text{ann}(M_i)$ is prime, there are $a \in \text{ann}(M_s) \setminus (\text{ann}(M_t) \cup \text{ann}(M_k))$, $b \in \text{ann}(M_t) \setminus (\text{ann}(M_s) \cup \text{ann}(M_k))$ and $c \in \bigcap_{i \in I \setminus \{s, t\}} \text{ann}(M_i) \setminus (\text{ann}(M_s) \cup \text{ann}(M_k))$. Now, $abc \in \text{ann}(M/N)$ but $ab \not\in \text{ann}(M/N)$, $ac \not\in \text{ann}(M/N)$ and $bc \not\in \text{ann}(M/N)$. This contradicts with
Theorem 2.3 in [10]. Thus $|A| \geq |I| - 2$ and at most two components of $M$ are zero in $N$.

Conversely, if one component of $M$ is zero in $N$, then $N$ is a prime submodule of $M$. Suppose that $N = \bigoplus_{i \in A} M_i$, where $A = I \setminus \{i, j\}$. Thus $M_i, M_j \not\subseteq N$. Suppose that $a, b \in R$, $(m_i)_{i \in I} = m \in M \setminus N$ and $abm \in N$. Then either $m_i \neq 0$ or $m_j \neq 0$. If $m_i \neq 0$ and $m_j \neq 0$, then $ab \in \text{ann}(M_i) \cap \text{ann}(M_j) = \text{ann}(M/N)$. If $m_i \neq 0$ and $m_j = 0$, then $ab \in \text{ann}(M_i)$ and so either $a \in \text{ann}(M_i)$ or $b \in \text{ann}(M_j)$. Hence, $am \in N$ or $bm \in N$. The case $m_i = 0$ and $m_j \neq 0$, is similar to the previous case. Therefore, $N$ is a 2-absorbing submodule of $M$.

\[ \Box \]

**Theorem 3.8.** $N$ is a 2-absorbing submodule of $M$ if and only if $Z^*(M, N) = \emptyset$ or $\Gamma(M, N)$ is a complete bipartite graph.

**Proof.** Let $N$ be a 2-absorbing submodule of $M$. If $N$ is prime, then $Z^*(M, N) = \emptyset$, by Theorem 3.6. Now, assume that $N = \bigoplus_{i \in I} M_i$ for some $j, k \in I$ and $(m_i)_{i \in I} = m \in M \setminus N$. Thus $I_m = \bigcap_{i \in I, m_i = 0} \text{ann}(M_i)$. If $m_i \neq 0$ and $m_k \neq 0$, then $m \not\in Z(M, N)$. Let $V_1 = \{(m_i)_{i \in I} \in M \setminus N : m_j = 0\}$ and $V_2 = \{(m_i)_{i \in I} \in M \setminus N : m_k = 0\}$. Thus $m - m'$ is an edge of $\Gamma(M, N)$ for every $m \in V_1$ and $m' \in V_2$. Also, every vertices in $V_1$ and $V_2$ are not adjacent. Hence, $\Gamma(M, N)$ is a complete bipartite graph.

Now, suppose that $\Gamma(M, N)$ is a complete bipartite graph and $N$ is not 2-absorbing. By Theorem 3.7, there are at least three components $M_s, M_i, M_k$ such that $M_s, M_i, M_k \not\subseteq N$. For $i = s, t, k$ let $v_i = (m_i)_{i \in I}$, where $m_i \neq 0$ and $m_j = 0$ for all $j \neq i$. Then $v_s - v_t - v_k - v_s$ is a cycle in $\Gamma(M, N)$. Thus $\text{gr}(\Gamma(M, N)) = 3$ and so $\Gamma(M, N)$ is not bipartite graph, by Theorem 1 of Sec. 1.2 in [5]. Hence, $N$ is a 2-absorbing submodule of $M$.

\[ \Box \]

**Example 3.9.** Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$. Then every nonzero submodule $N$ of $M$ is 2-absorbing. Thus either $Z^*(M, N) = \emptyset$ or $\Gamma(M, N)$ is a complete bipartite graph. In particular, if $N = \mathbb{Z}_7$, then $\Gamma(M, N) = K_7^{28}$.

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**References**

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