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A Submodule-Based Zero Divisor Graph for Modules

Sakineh Babaei[a], Shiroyeh Payrovi[*,a], Esra Sengelen Sevim[b]

[a]Department of Mathematics, Imam Khomeini International University, Postal Code: 34149-1-6818, Qazvin, Iran.
[b]Department of Mathematics, Istanbul Bilgi University, Kazim Karabekir Cad. No: 2/13, 34060 Eyup-Istanbul, Turkey.

E-mail: sbabaei@edu.ikiu.ac.ir
E-mail: shpayrovi@sci.ikiu.ac.ir
E-mail: esra.sengelen@bilgi.edu.tr

Abstract. Let R be a commutative ring with identity and M be an R-module. The zero divisor graph of M is denoted by $\Gamma(M)$. In this study, we are going to generalize the zero divisor graph $\Gamma(M)$ to submodule-based zero divisor graph $\Gamma(M,N)$ by replacing elements whose product is zero with elements whose product is in some submodule N of M. The main objective of this paper is to study the interplay of the properties of submodule N and the properties of $\Gamma(M,N)$.

Keywords: Zero divisor graph, Submodule-based zero divisor graph, Semisimple module.

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1. Introduction

Let R be a commutative ring with identity. The zero divisor graph of R, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero divisor of R with two distinct vertices x and y are adjacent by an edge if and only

*Corresponding Author

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if $xy = 0$. The idea of a zero divisor graph of a commutative ring was introduced by Beck in [3] where he was mainly interested with colorings of rings. The definition above first is appeared in [2], which contains several fundamental results concerning $\Gamma(R)$. The zero-divisor graph of a commutative ring is further examined by Anderson, Levy and Shapiro, Mulay in [1, 9]. Also, the ideal-based zero divisor graph of R is defined by Redmond, in [12].

The zero divisor graph for modules over commutative rings has been defined by Behboodi in [4] as a generalization of zero divisor graph of rings. Let R be a commutative ring and M be an R-module, for $x \in M$, we denote the annihilator of the factor module M/Rx by I_x. An element $x \in M$ is called a zero divisor, if either $x = 0$ or $I_xI_yM = 0$ for some $y \neq 0$ with $I_y \subset R$. The set of zero divisors of M is denoted by $Z(M)$ and the associated graph to M with vertices in $Z^*(M) = Z(M) \setminus \{0\}$ is denoted by $\Gamma(M)$, such that two different vertices x and y are adjacent provided $I_xI_yM = 0$.

In this paper, we introduce the submodule-based zero divisor graph that is a generalization of zero divisor graph for modules. Let R be a commutative ring, M be an R-module and N be a proper submodule of M. An element $x \in M$ is called zero divisor with respect to N, if either $x \in N$ or $I_xI_yM \subseteq N$ for some $y \in M \setminus N$ with $I_y \subset R$. We denote $Z(M,N)$ for the set of zero divisors of M with respect to N. Also, we denote the associated graph to M with vertices $Z^*(M,N) = Z(M,N) \setminus N$ by $\Gamma(M,N)$, and two different vertices x and y are adjacent provided $I_xI_yM \subseteq N$.

In the second section, we define a submodule-based zero divisor graph for a module and we study basic properties of this graph. In the third section, if M is a finitely generated semisimple R-module such that its homogenous components are simple and N is a submodule of M, we determine some relations between $\Gamma(M,N)$ and $\Gamma(M/N)$, where M/N is the quotient module of M, we show that the clique number and chromatic number of $\Gamma(M,N)$ are equal. Also, we determine some submodule of M such that $\Gamma(M,N)$ is an empty or a complete bipartite graph.

Let Γ be a (undirected) graph. We say that Γ is *connected* if there is a path between any two distinct vertices. For vertex x the number of graph edges which touch x is called the degree of x and is denoted by $\deg(x)$. For vertices x and y of Γ, we define $\mathrm{d}(x,y)$ to be the length of a shortest path between x and y, if there is no path, then $\mathrm{d}(x,y) = \infty$. The *diameter* of Γ is $\mathrm{diam}(\Gamma) = \sup\{\mathrm{d}(x,y) | x$ and y are vertices of $\Gamma\}$. The *girth* of Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ ($\mathrm{gr}(\Gamma) = \infty$ if Γ contains no cycle).

A graph Γ is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices is denoted by K^n (we allow n to be an infinite cardinal). The *clique number*, $\omega(\Gamma)$, is the greatest integer $n > 1$ such that $K^n \subseteq \Gamma$, and $\omega(\Gamma) = \infty$ if $K^n \subseteq \Gamma$ for all $n \geq 1$. A *complete bipartite* graph is a graph Γ which may be partitioned into two disjoint nonempty vertex sets V_1 and V_2

such that two distinct vertices are adjacent if and only if they are in different vertex sets. If one of the vertex sets is a singleton, then we call that Γ is a *star graph*. We denote the complete bipartite graph by $K^{m,n}$, where $|V_1| = m$ and $|V_2| = n$ (again, we allow m and n to be infinite cardinals); so a star graph is $K^{1,n}$, for some $n \in \mathbb{N}$.

The *chromatic number*, $\chi(\Gamma)$, of a graph Γ is the minimum number of colors needed to color the vertices of Γ, so that no two adjacent vertices share the same color. A graph Γ is called *planar* if it can be drawn in such a way that no two edges intersect.

Throughout this study, R is a commutative ring with nonzero identity, M is a unitary R-module and N is a proper submodule of M. Given any subset S of M, the annihilator of S is denoted by $\text{ann}(S) = \{r \in R | rs = 0 \text{ for all } s \in S\}$ and the cardinal number of S is denoted by $|S|$.

2. Submodule-based Zero Divisor Graph

Recall that R is a commutative ring, M is an R-module and N is a proper submodule of M. For $x \in M$, we denote $\text{ann}(M/Rx)$ by I_x.

Definition 2.1. Let M be an R-module and N be a proper submodule of M. An $x \in M$ is called a zero divisor with respect to N if $x \in N$ or $I_x I_y M \subseteq N$ for some $y \in M \setminus N$ with $I_y \subset R$.

We denote the set of zero divisors of M with respect to N by $Z(M, N)$ and $Z^*(M, N) = Z(M, N) \setminus N$. The submodule-based zero divisor graph of M with respect to N, $\Gamma(M, N)$, is an undirected graph with vertices $Z^*(M, N)$ such that distinct vertices x and y are adjacent if and only if $I_x I_y M \subseteq N$.

The following example shows that $Z(M/N)$ and $Z(M, N)$ are different from each other.

Example 2.2. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and $N = 2\mathbb{Z} \oplus 0$. Then $I_{(m,n)} = 0$, for all $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$. But $I_{(m,n)+N} = 2n\mathbb{Z}$ whenever $m \in 2\mathbb{Z}$ and $I_{(m,n)+N} = 2\mathbb{Z}$ whenever $m \notin 2\mathbb{Z}$. Thus $(1, 0), (1, 1) \in Z^*(M, N)$ are adjacent in $\Gamma(M, N)$, but $(1, 0) + N, (1, 1) + N \notin Z^*(M/N)$.

Proposition 2.3. *If* $Z^*(M, N) = \emptyset$, *then* ann(M/N) *is a prime ideal of* R.

Proof. Suppose that $\text{ann}(M/N)$ is not prime. Then there are ideals I and J of R such that $IJM \subset N$ but $IM \not\subseteq N$ and $JM \not\subseteq N$. Let $x \in IM \setminus N$ and $y \in JM \setminus N$. Then $I_x J_y M \subseteq IJM \subseteq N$ and $I_y \subset R$. Thus $x \in Z^*(M, N)$, a contradiction. Hence, $\text{ann}(M/N)$ is a prime ideal of R. □

Lemma 2.4. *Let* $x, y \in Z^*(M, N)$. *If* $x - y$ *is an edge in* $\Gamma(M, N)$, *then for each* $0 \neq r \in R$, *either* $ry \in N$ *or* $x - ry$ *is also an edge in* $\Gamma(M, N)$.

Proof. Let $x, y \in Z^*(M, N)$ and $r \in R$. Assume that $x - y$ is an edge in $\Gamma(M, N)$ and $ry \notin N$. Then $I_x I_y M \subseteq N$. It is clear that $I_{rx} \subseteq I_x$. So that $I_x I_{ry} M \subseteq I_x I_y M \subseteq N$ and therefore, $x - ry$ is an edge in $\Gamma(M, N)$. □

It is shown that the graphs are defined in [12] and [4], are connected with diameter less than or equal to three. Moreover, it shown that if those graphs contain a cycle, then they have the girth less than or equal to four. In the next theorems, we extend these results to a submodule-based zero divisor graph.

Theorem 2.5. *$\Gamma(M,N)$ is a connected graph and* $\mathrm{diam}(\Gamma(\mathrm{M},\mathrm{N})) \leq 3$.

Proof. Let x and y be distinct vertices of $\Gamma(M,N)$. Then, there are $a,b \in Z^*(M,N)$ with $I_aI_xM \subseteq N$ and $I_bI_yM \subseteq N$ (we allow $a,b \in \{x,y\}$). If $I_aI_bM \subseteq N$, then $x-a-b-y$ is a path, thus $\mathrm{d}(x,y) \leq 3$. If $I_aI_bM \nsubseteq N$, then $Ra \cap Rb \nsubseteq N$, and for every $d \in (Ra \cap Rb) \setminus N$, $x-d-y$ is a path of length 2, $\mathrm{d}(x,y) \leq 2$, by Lemma 2.4. Hence, we conclude that $\mathrm{diam}(\Gamma(M,N)) \leq 3$. □

Theorem 2.6. *If $\Gamma(M,N)$ contains a cycle, then* $\mathrm{gr}(\Gamma(\mathrm{M},\mathrm{N})) \leq 4$.

Proof. We have $\mathrm{gr}(\Gamma(M,N)) \leq 7$, by Proposition 1.3.2 in [7] and Theorem 2.5. Assume that $x_1 - x_2 - \cdots - x_7 - x_1$ is a cycle in $\Gamma(M,N)$. If $x_1 = x_4$ then it is clear that $\mathrm{gr}(\Gamma(M,N)) \leq 3$. So, suppose that $x_1 \neq x_4$. Then we have the following two cases:
Case 1. If x_1 and x_4 are adjacent in $\Gamma(M,N)$, then $x_1 - x_2 - x_3 - x_4 - x_1$ is a cycle and $\mathrm{gr}(\Gamma(M,N)) \leq 4$.
Case 2. Suppose that x_1 and x_4 are not adjacent in $\Gamma(M,N)$. Then $I_{x_1}I_{x_4}M \nsubseteq N$ and so there is a $z \in (Rx_1 \cap Rx_4) \setminus N$. If $z = x_1$, then $z \neq x_4$ and $x_3 - x_4 - x_5 - z - x_3$ is a cycle in $\Gamma(M,N)$, by Lemma 2.4. If $z \neq x_1$, then by Lemma 2.4, $x_1 - x_2 - z - x_7 - x_1$ is a cycle and $\mathrm{gr}(\Gamma(M,N)) \leq 4$.

For cycles with length 5 or 6, by using a similar argument as above, one can shows that $\mathrm{gr}(\Gamma(M,N)) \leq 4$. □

Example 2.7. Assume that $M = \mathbb{Z}$ and p,q are two prime numbers. If $N = p\mathbb{Z}$, then $\Gamma(M,N) = \emptyset$. If $N = pq\mathbb{Z}$, then $\Gamma(M,N)$ is an infinite complete bipartite graph with vertex set $V_1 \cup V_2$, where $V_1 = p\mathbb{Z} \setminus pq\mathbb{Z}$ and $V_2 = q\mathbb{Z} \setminus pq\mathbb{Z}$ and so $\mathrm{gr}(\Gamma(M,N)) = 4$.

Corollary 2.8. *If N is a prime submodule of M, then* $\mathrm{diam}(\Gamma(M,N)) \leq 2$ *and* $\mathrm{gr}(\Gamma(M,N)) = 3$*, whenever it contains a cycle.*

Proof. Let x,y be two distinct vertices which are not adjacent in $\Gamma(M,N)$. Thus there is an $a \in M \setminus N$ such that $I_aI_xM \subseteq N$. Since N is a prime submodule, then $I_aM \subseteq N$. Thus $I_aI_yM \subseteq N$, and then $x-a-y$ is a path in $\Gamma(M,N)$. Then $\mathrm{diam}(\Gamma(M,N)) \leq 2$. □

Lemma 2.9. *Let* $|\Gamma(M,N)| \geq 3$, $\mathrm{gr}(\Gamma(M,N)) = \infty$ *and $x \in Z^*(M,N)$ with* $\deg(x) > 1$*. Then $Rx = \{0,\ x\}$ and* $\mathrm{ann}(x)$ *is a prime ideal of R.*

Proof. First we show that $Rx = \{0,x\}$. Let $u-x-v$ be a path in $\Gamma(M,N)$. Then $u-v$ is not an edge in $\Gamma(M,N)$ since $\mathrm{gr}(\Gamma(M,N)) = \infty$. If $x \neq rx$ for some $r \in R$ and $rx \notin N$, then by Lemma 2.4, $rx-u-x-v-rx$ is a cycle in

$\Gamma(M, N)$, that is a contradiction. So, for every $r \in R$ either $rx = x$ or $rx \in N$. If there is an $r \in R$ such that $rx \in N$, then we have either $(1 + r)x \in N$ or $(1 + r)x = x$. These imply that $x \in N$ or $rx = 0$. Therefore, we have shown that $Rx = \{0, x\}$.
Let $a, b \in R$ and $abx = 0$. Then $bx = 0$ or $bx = x$. Hence, $bx = 0$ or $ax = 0$. So, $\mathrm{ann}(x)$ is a prime ideal of R. □

Theorem 2.10. *If N is a nonzero submodule of M and $\mathrm{gr}(\Gamma(M, N)) = \infty$, then $\Gamma(M, N)$ is a star graph.*

Proof. Suppose that $\Gamma(M, N)$ is not a star graph. Then there is a path in $\Gamma(M, N)$ such as $u - x - y - v$. By Lemma 2.9, we have $Ry = \{0, y\}$ and by assumption u and y are not adjacent, thus $I_y M \neq 0$. So that $I_y M = Ry$. Also, $x - y - v$ is a path, thus $I_v I_y M \subseteq N$ and $I_x I_y M \subseteq N$. Hence, $I_v Ry \subseteq N$ and $I_x Ry \subseteq N$. On the other hand, for every nonzero $n \in N$, we have

$$I_v I_{y+n} M \subseteq I_v R(y + n) \subseteq I_v(Ry + N) \subseteq N$$

and similarly $I_x I_{y+n} M \subseteq N$. So that $x - y - v - (y + n) - x$ is a cycle in $\Gamma(M, N)$, a contradiction. Therefore, $\Gamma(M, N)$ is a star graph. □

Theorem 2.11. *Let N be a nonzero submodule of M, $|\Gamma(M, N)| \geq 3$ and $\Gamma(M, N)$ is a star graph. Then the following statements are true:*

(i) *If x is the center vertex, then $I_x = \mathrm{ann}(M)$.*
(ii) *$\Gamma(M, N)$ is a subgraph of $\Gamma(M)$.*

Proof. (i) By Lemma 2.9, we have $Rx = \{0, x\}$. Thus either $I_x M = 0$ or $I_x M = Rx$. Assume that $I_x M = Rx$. If y is a vertex of $\Gamma(M, N)$ such that $y \neq x$, then $\deg(y) = 1$ and $I_x I_y M \subseteq N$. Thus $I_y Rx \subseteq N$. Since $I_{x+n} I_y M \subseteq I_y R(x + n) \subseteq N$ for every nonzero element $n \in N$ it concludes that $y = x + n$. In this case, every other vertices of $\Gamma(M, N)$ are adjacent to y, a contradiction. Hence, $I_x M = 0$ and $I_x = \mathrm{ann}(M)$.

(ii) It is obvious. □

Theorem 2.12. *If $|N| \geq 3$ and $\Gamma(M, N)$ is a complete bipartite graph which is not a star graph, then $I_x^2 M \nsubseteq N$, for every $x \in Z^*(M, N)$.*

Proof. Let $Z^*(M, N) = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$. Suppose that $I_x^2 M \subseteq N$ for some $x \in Z^*(M, N)$. Without loss of generality, we can assume that $x \in V_1$. By a similar argument with Lemma 2.9, either $Rx = \{0, x\}$ or there is an $r \in R$ such that $x \neq rx$ and $rx \in N$. If $Rx = \{0, x\}$, then $I_x M = Rx$. Thus $I_x Rx \subseteq N$. Now, for every $y \in V_2$ and $n \in N$ we get

$$I_y I_{x+n} M \subseteq I_y R(x + n) \subseteq I_y(Rx + N) \subseteq N$$

and $I_x I_{x+n} M \subseteq N$. Then, $x + n \in V_1 \cap V_2$, a contradiction. So, assume that $x \neq rx$ and $rx \in N$ for some $r \in R$. Since $I_{rx+x} \subseteq I_x$, then $I_x I_{rx+x} M \subseteq N$ and for all $y \in V_2$, $I_y I_{rx+x} M \subseteq N$. Thus $rx + x \in V_1 \cap V_2$, a contradiction. □

An R-module X is called a *multiplication-like* module if, for each nonzero submodule Y of X, $\mathrm{ann}(X) \subset \mathrm{ann}(X/Y)$. Multiplication-like module have been studied in [8, 13].

A vertex x of a connected graph G is a *cut-point*, if there are vertices u, v of G such that x is in every path from u to v and $x \neq u$, $x \neq v$. For a connected graph G, an edge E of G is defined to be a *bridge* if $G - \{E\}$ is disconnected, see [6].

Theorem 2.13. *Let M be a multiplication-like module and N be a nonzero submodule of M. Then $\Gamma(M, N)$ has no cut-points.*

Proof. Suppose that x is a cut-point of $\Gamma(M, N)$. Then there exist vertices $u, v \in M \setminus N$ such that x lies on every path from u to v. By Theorem 2.5, the shortest path from u to v has length 2 or 3.
Case 1. Suppose that $u{-}x{-}v$ is a path of shortest length from u to v. Since x is a cut point x, u, v aren't in a cycle. By a similar argument to that of Lemma 2.9, we have $Rx = \{0, x\}$. On the other hand, $I_xM \subseteq Rx$ and M is a multiplication-like module, so we have $I_xM = Rx$. Hence $I_uRx \subseteq N$ and $I_vRx \subseteq N$. Also, for every nonzero $n \in N$, we have $I_uI_{x+n}M \subseteq I_u(Rx{+}N) \subseteq N$ and $I_vI_{x+n}M \subseteq N$. Therefore, $u - (x + n) - v$ is a path from u to v, a contradiction.
Case 2. Suppose that $u{-}x{-}y{-}v$ is a path in $\Gamma(M, N)$. Then, we have $I_xM = Rx$ and for every nonzero $n \in N$, we have $I_yI_{x+n}M \subseteq N$ and $I_uI_{x+n}M \subseteq N$. Thus $u - (x + n) - y - v$ is a path from u to v, a contradiction. □

Theorem 2.14. *Let M be a multiplication-like module and N be a nonzero submodule of M. Then $\Gamma(M, N)$ has a bridge if and only if $\Gamma(M, N)$ is a graph on two vertices.*

Proof. If $|\Gamma(M, N)| = 3$, then $\Gamma(M, N) = K^3$, by Theorem 2.11, and it has no bridge. Assume that $|\Gamma(M, N)| \geq 4$ and $x - y$ is a bridge. Thus there is not a cycle containing $x - y$. Without loss of generality, we can assume that $\deg(x) > 1$. Thus, there exists a vertex $z \neq y$ such that $z - x$ is an edge of $\Gamma(M, N)$. Then $Rx = \{0, x\}$ and $I_xM = Rx$. Hence, for every $n \in N$, $I_zI_{x+n}M \subseteq N$ and $I_yI_{x+n}M \subseteq N$, a contradiction. Therefore, $\Gamma(M, N)$ has not a bridge. The converse is clear. □

3. Submodule-based Zero Divisor Graph of Semisimple Modules

A nonzero R-module X is called simple if its only submodules are (0) and X. An R-module X is called semisimple if it is a direct sum of simple modules. Also, X is called homogenous semisimple if it is a direct sum of isomorphic simple modules.

In this section, R is a commutative ring and M is a finitely generated semisimple R-module such that its homogenous components are simple and

N is a submodule of M. The following theorem has a crucial role in this section.

Theorem 3.1. *Let $x, y \in M \setminus N$. Then x, y are adjacent in $\Gamma(M, N)$ if and only if $Rx \cap Ry \subseteq N$.*

Proof. Let $M = \bigoplus_{i\in I} M_i$, where M_i's are non-isomorphic simple submodules of M. By assumption N is a submodule of M, so there exists a subset A of I such that $M = N \oplus (\bigoplus_{i\in A} M_i)$ and so $\mathrm{ann}(M/N) = \mathrm{ann}(\bigoplus_{i\in A} M_i) = \bigcap_{i\in A} \mathrm{ann}(M_i)$. Assume that $x, y \in M \setminus N$ are adjacent in $\Gamma(M, N)$ and $Rx \cap Ry \nsubseteq N$. Thus there exists $\alpha \in I$ such that $M_\alpha \subseteq (Rx \cap Ry) \setminus N$. Also, there exist subsets $B \subset I$ and $C \subset I$ such that $M = Rx \oplus (\bigoplus_{i\in B} M_i)$ and $M = Ry \oplus (\bigoplus_{i\in C} M_i)$. Therefore, $I_x = \bigcap_{i\in B} \mathrm{ann}(M_i)$ and $I_y = \bigcap_{i\in C} \mathrm{ann}(\mathrm{M_i})$. Since $I_x I_y M \subseteq N$, we have $I_x I_y \subseteq \mathrm{ann}(M/N)$. For every $i, j \in I$, $\mathrm{ann}(M_i)$ and $\mathrm{ann}(M_j)$ are coprime, then

$$\begin{aligned} I_x I_y = [\bigcap_{i\in B} \mathrm{ann}(M_i)][\bigcap_{i\in C} \mathrm{ann}(M_i)] &= \prod_{i\in B\cup C} \mathrm{ann}(M_i) \\ &\subseteq \bigcap_{i\in A} \mathrm{ann}(M_i) \subseteq \mathrm{ann}(M_r), \end{aligned}$$

for all $r \in A$. Thus for any $r \in A$ there exists $j_r \in B \cup C$ such that $\mathrm{ann}(M_{j_r}) \subseteq \mathrm{ann}(M_r)$. So that $\mathrm{ann}(M_{j_r}) = \mathrm{ann}(M_r)$ implies that $M_{j_r} \cong M_r$ and by hypothesis $M_{j_r} = M_r$. Hence,

$$M_\alpha \subseteq \bigoplus_{i\in A} M_i \subseteq \bigoplus_{j\in B\cup C} M_j.$$

Thus there exists $\gamma \in B \cup C$ such that $M_\alpha = M_\gamma$, also

$$M_\alpha \subseteq Rx \cap Ry = (\bigoplus_{i\in I\setminus B} M_i) \cap (\bigoplus_{i\in I\setminus C} M_i).$$

Therefore, $\alpha \in I \setminus (B \cup C)$, a contradiction. The converse is obvious. □

Corollary 3.2. *Let $x, y \in M \setminus N$ be such that $x + N \neq y + N$. Then*

(i) *x and y are adjacent in $\Gamma(M, N)$ if and only if $x + N$ and $y + N$ are adjacent in $\Gamma(M/N)$.*

(ii) *if x and y are adjacent in $\Gamma(M, N)$, then all distinct elements of $x + N$ and $y + N$ are adjacent in $\Gamma(M, N)$.*

Proof. (i) Let $M = \bigoplus_{i\in I} M_i$, where M_i's are non-isomorphic simple submodules of M. Suppose that x and y are adjacent in $\Gamma(M, N)$, $Rx = \bigoplus_{i\in A} M_i$, $Ry = \bigoplus_{i\in B} M_i$ and $N = \bigoplus_{i\in C} M_i$. Then $Rx + N = \bigoplus_{i\in A\cup C} M_i$ and $Ry + N = \bigoplus_{i\in B\cup C} M_i$. Thus,

$$(Rx + N) \cap (Ry + N) = \bigoplus_{i\in(A\cup C)\cap(B\cup C)} M_i = \bigoplus_{i\in(A\cap B)\cup C} M_i = (Rx \cap Ry) + N.$$

By Theorem 3.1, we have $Rx \cap Ry \subseteq N$ hence,

$$I_{x+N} I_{y+N} M \subseteq (Rx + N) \cap (Ry + N) = (Rx \cap Ry) + N = N.$$

Therefore, $x+N$ and $y+N$ are adjacent in $\Gamma(M/N)$. The converse is obvious.

(ii) Let $x, y \in Z^*(M,N)$ be adjacent in $\Gamma(M,N)$. Then $Rx \cap Ry \subseteq N$ by Theorem 3.1. So for every $n, n' \in N$ we have

$$I_{x+n}I_{y+n'}M \subseteq R(x+n) \cap R(y+n') \subseteq (Rx+N) \cap (Ry+N) = N.$$

Hence, $x+n$ and $y+n'$ are adjacent in $\Gamma(M,N)$. □

In the following theorem, we prove that the clique number of graphs $\Gamma(M,N)$ and $\Gamma(M/N)$ are equal.

Theorem 3.3. *If N is a nonzero submodule of M, then $\omega(\Gamma(M/N)) = \omega(\Gamma(M,N))$.*

Proof. First we show that $I^2_{m+N}M \not\subseteq N$ for each $0 \neq m+N \in M/N$. Assume that $N = \oplus_{i\in A}M_i$ and $m = (m_i)_{i\in I} \in M \setminus N$. Then $I_{m+N} = \bigcap_{i\notin A, m_i=0} \mathrm{ann(M_i)}$. Hence, $I_{m+N} = I^2_{m+N}$. Thus $I^2_{m+N}M \not\subseteq N$ since there is at least one $j \in I \setminus A$ such that $m_j \neq 0$.

Now, Corollary 3.2 implies that $\omega(\Gamma(M/N)) \leq \omega(\Gamma(M,N))$. Thus, it is enough to consider the case where $\omega(\Gamma(M/N)) = d < \infty$. Assume that G is a complete subgraph of $\Gamma(M,N)$ with vertices $m_1, m_2, \cdots, m_{d+1}$, we provide a contradiction. Consider the subgraph G_* of $\Gamma(M/N)$ with vertices $m_1+N, \cdots, m_{d+1}+N$. By Corollary 3.2, G_* is a complete subgraph of $\Gamma(M,N)$. Thus $m_j+N = m_k+N$ for some $1 \leq j,k \leq d+1$ with $j \neq k$ since $\omega(\Gamma(M/N)) = d$. We have $I_{m_j}I_{m_k}M \subseteq N$. Therefore, $Rm_j \cap Rm_k \subseteq N$ and so $I_{m_j+N}I_{m_k+N}M \subseteq N$. Hence, $I^2_{m_j+N}M \subseteq N$, that is a contradiction. □

In the following theorem, we show that there is a relation between $\omega(\Gamma(M,N))$ and $\chi(\Gamma(M,N))$.

Theorem 3.4. *Assume that $M = \bigoplus_{i\in I} M_i$, where M_i's are non-isomorphic simple submodules of M and $N = \bigoplus_{i\in A} M_i$ is a submodule of M for some $A \subset I$. Then $\omega(\Gamma(M,N)) = \chi(\Gamma(M,N)) = |I| - |A|$.*

Proof. Suppose that $I \setminus A = \{1, \cdots, n\}$ so $M_1, \cdots, M_n \not\subseteq N$. Let for $1 \leq k \leq n-1$

$$L^k = \{m \in M : m \text{ has } k \text{ nonzero components }\}$$

and let for $1 \leq s \leq n$

$$L^1_s = \{m \in L^1 : \text{the } s^{\text{th}} \text{ component of } m \text{ is nonzero}\}.$$

If $m \in L^1_s$ and $m' \in L^1_t$ for some $1 \leq s,t \leq n$ with $s \neq t$, then m and m' are adjacent and so K^n is a subgraph of $\Gamma(M,N)$. Thus $\omega(\Gamma(M,N)) \geq n$. If $m, m' \in L^1_s$ for some $1 \leq s \leq n$, then m, m' are not adjacent because $\mathrm{ann}(M_s) \not\subseteq I_mI_{m'}$ and so the elements of L^1_s have same color. On the other hand, if $x \in L^t$ with $t > 1$, then there is not a complete subgraph K^h of $\Gamma(M,N)$ containing x, such that $h \geq n$. Thus $\omega(\Gamma(M,N)) = n \leq \chi(\Gamma(M,N))$. Also, if $x \in L^t$ with $t > 1$, then there is an s with $1 \leq s \leq n$ such that x is not adjacent to each element of L_s^1. Thus the color of x is same as the elements of L_s^1. Thus $\chi(\Gamma(M,N)) = n$. □

The Kuartowski's Theorem states: A graph G is planar if and only if it contains no subgraph homeomorphic to K^5 or $K^{3,3}$.

Theorem 3.5. *Let N be a nonzero proper submodule of M such that N is not prime. Then $\Gamma(M,N)$ is not planar.*

Proof. Assume that $M = \bigoplus_{i\in I} M_i$, where M_i's are non-isomorphic simple submodules of M and $N = \bigoplus_{i\in A} M_i$ for some $A \subset I$. Let $I \setminus A = \{i,j\}$. Then $\Gamma(M,N)$ is a complete bipartite graph $K^{n,m}$, where $n = (|M_i|-1)(\prod_{k\in I-\{i,j\}}|M_k|)$ and $m = (|M_j|-1)(\prod_{k\in I-\{i,j\}}|M_k|)$. By hypotheses N is a nonzero and M_i's are non-isomorphic, so we have $n,m \geq 3$. Hence $\Gamma(M,N)$ has a subgraph homeomorphic to $K^{3,3}$. The cases $|I \setminus A| \geq 3$ are similar to that of the case $|I \setminus A| = 2$. □

Theorem 3.6. *A nonzero submodule N of M is prime if and only if $Z^*(M,N) = \emptyset$.*

Proof. Let $M = \bigoplus_{i\in I} M_i$, where M_i's are non-isomorphic simple submodules of M and N is prime. Then $N = \bigoplus_{i\in I\setminus\{k\}} M_i$, for some $k \in I$. If $x \in Z^*(M,N)$, then there exists a $y \in M \setminus N$ such that $I_x I_y M \subseteq N$. If $x \neq y$, then $Rx \cap Ry \subseteq N$, by Theorem 3.1. Thus either $M_k \not\subseteq Rx$ or $M_k \not\subseteq Ry$. Hence, either $Rx \subseteq N$ or $Ry \subseteq N$, a contradiction. Now, suppose that $x = y$ so by $I_x^2 M \subseteq N$ and hypotheses $I_x M \subseteq N$. Thus $I_{x+n} I_x M \subseteq N$ for every $0 \neq n \in N$. By a similar argument, we have either $x \in N$ or $x + n \in N$, a contradiction. Hence, $Z^*(M,N) = \emptyset$.

Conversely, assume that $Z^*(M,N) = \emptyset$. Then $\mathrm{ann}(M/N)$ is prime ideal of R by Proposition 2.3 and there exists a $k \in I$ such that $\mathrm{ann}(M/N) = \mathrm{ann}(M_k)$. Hence, $N = \bigoplus_{i\in I\setminus\{k\}} M_i$ is a prime submodule of M. □

A proper submodule N of M is called 2-absorbing if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in \mathrm{ann}(M/N)$, see [10, 11]. In the following results, we study the behavior of $\Gamma(M,N)$ whenever N is a 2-absorbing submodule of M .

Theorem 3.7. *A submodule N of M is 2-absorbing if and only if at most two components of M are zero in N.*

Proof. Let $M = \bigoplus_{i\in I} M_i$, where M_i's are non-isomorphic simple submodules of M. Suppose that N is a 2-absorbing submodule of M and $N = \bigoplus_{i\in A} M_i$, where $A = I \setminus \{s,t,k\}$. Since for all $i \in I$, $\mathrm{ann}(M_i)$ is prime, there are $a \in \mathrm{ann}(M_s) \setminus (\mathrm{ann}(M_t) \cup \mathrm{ann}(M_k))$, $b \in \mathrm{ann}(M_t) \setminus (\mathrm{ann}(M_s) \cup \mathrm{ann}(M_k))$ and $c \in \bigcap_{j\in I\setminus(A-\{s,t\})} \mathrm{ann}(M_j) \setminus (\mathrm{ann}(M_s) \cup \mathrm{ann}(M_t))$. Now, $abc \in \mathrm{ann}(M/N)$ but $ab \notin \mathrm{ann}(M/N)$, $ac \notin \mathrm{ann}(M/N)$ and $bc \notin \mathrm{ann}(M/N)$. This contradict with

Theorem 2.3 in [10]. Thus $|A| \geq |I| - 2$ and at most two components of M are zero in N.

Conversely, if one component of M is zero in N, then N is a prime submodule of M. Suppose that $N = \bigoplus_{i \in A} M_i$, where $A = I \setminus \{i, j\}$. Thus $M_i, M_j \not\subseteq N$. Suppose that $a, b \in R$, $(m_i)_{i \in I} = m \in M \setminus N$ and $abm \in N$. Then either $m_i \neq 0$ or $m_j \neq 0$. If $m_i \neq 0$ and $m_j \neq 0$, then $ab \in \mathrm{ann}(M_i) \cap \mathrm{ann}(M_j) = \mathrm{ann}(M/N)$. If $m_i \neq 0$ and $m_j = 0$, then $ab \in \mathrm{ann}(M_i)$ and so either $a \in \mathrm{ann}(M_i)$ or $b \in \mathrm{ann}(M_i)$. Hence, $am \in N$ or $bm \in N$. The case $m_i = 0$ and $m_j \neq 0$, is similar to the previous case. Therefore, N is a 2-absorbing submodule of M. □

Theorem 3.8. *N is a 2-absorbing submodule of M if and only if $Z^*(M, N) = \emptyset$ or $\Gamma(M, N)$ is a complete bipartite graph.*

Proof. Let N be a 2-absorbing submodule of M. If N is prime, then $Z^*(M, N) = \emptyset$, by Theorem 3.6. Now, assume that $N = \bigoplus_{i \in I \setminus \{j,k\}} M_i$ for some $j, k \in I$ and $(m_i)_{i \in I} = m \in M \setminus N$. Thus $I_m = \bigcap_{\{i \in I : m_i = 0\}} \mathrm{ann(M_i)}$. If $m_j \neq 0$ and $m_k \neq 0$, then $m \notin Z(M, N)$. Let $V_1 = \{(m_i)_{i \in I} \in M \setminus N : m_j = 0\}$ and $V_2 = \{(m_i)_{i \in I} \in M \setminus N : m_k = 0\}$. Thus $m - m'$ is an edge of $\Gamma(M, N)$ for every $m \in V_1$ and $m' \in V_2$. Also, every vertices in V_1 and V_2 are not adjacent. Hence, $\Gamma(M, N)$ is a complete bipartite graph.

Now, suppose that $\Gamma(M, N)$ is a complete bipartite graph and N is not 2-absorbing. By Theorem 3.7, there are at least three components M_s, M_t, M_k such that $M_s, M_t, M_k \not\subseteq N$. For $i = s, t, k$ let $v_i = (m_i)_{i \in I}$, where $m_i \neq 0$ and $m_j = 0$ for all $j \neq i$. Then $v_s - v_t - v_k - v_s$ is a cycle in $\Gamma(M, N)$. Thus $\mathrm{gr(\Gamma(M,N))} = 3$ and so $\Gamma(M, N)$ is not bipartite graph, by Theorem 1 of Sec. 1.2 in [5]. Hence, N is a 2-absorbing submodule of M. □

Example 3.9. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$. Then every nonzero submodule N of M is 2-absorbing. Thus either $Z^*(M, N) = \emptyset$ or $\Gamma(M, N)$ is a complete bipartite graph. In particular, if $N = \mathbb{Z}_7$, then $\Gamma(M, N) = K^{7,28}$.

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