A Submodule-Based Zero Divisor Graph for Modules

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Abstract. Let \( R \) be a commutative ring with identity and \( M \) be an \( R \)-module. The zero divisor graph of \( M \) is denoted by \( \Gamma(M) \). In this study, we are going to generalize the zero divisor graph \( \Gamma(M) \) to submodule-based zero divisor graph \( \Gamma(M, N) \) by replacing elements whose product is zero with elements whose product is in some submodule \( N \) of \( M \). The main objective of this paper is to study the interplay of the properties of submodule \( N \) and the properties of \( \Gamma(M, N) \).

Keywords: Zero divisor graph, Submodule-based zero divisor graph, Semisimple module.


1. Introduction

Let \( R \) be a commutative ring with identity. The zero divisor graph of \( R \), denoted \( \Gamma(R) \), is an undirected graph whose vertices are the nonzero zero divisor of \( R \) with two distinct vertices \( x \) and \( y \) are adjacent by an edge if and only if

\[ xy = 0 \]
if $xy = 0$. The idea of a zero divisor graph of a commutative ring was introduced by Beck in [3] where he was mainly interested with colorings of rings. The definition above first is appeared in [2], which contains several fundamental results concerning $\Gamma(R)$. The zero-divisor graph of a commutative ring is further examined by Anderson, Levy and Shapiro, Mulay in [1, 9]. Also, the ideal-based zero divisor graph of $R$ is defined by Redmond, in [12].

The zero divisor graph for modules over commutative rings has been defined by Behboodi in [4] as a generalization of zero divisor graph of rings. Let $R$ be a commutative ring and $M$ be an $R$-module, for $x \in M$, we denote the annihilator of the factor module $M/Rx$ by $I_x$. An element $x \in M$ is called a zero divisor, if either $x = 0$ or $I_xI_yM = 0$ for some $y \neq 0$ with $I_y \subset R$. The set of zero divisors of $M$ is denoted by $Z(M)$ and the associated graph to $M$ with vertices in $Z^*(M) = Z(M) \setminus \{0\}$ is denoted by $\Gamma(M)$, such that two different vertices $x$ and $y$ are adjacent provided $I_xI_yM = 0$.

In this paper, we introduce the submodule-based zero divisor graph that is a generalization of zero divisor graph for modules. Let $R$ be a commutative ring, $M$ be an $R$-module and $N$ be a proper submodule of $M$. An element $x \in M$ is called zero divisor with respect to $N$, if either $x \in N$ or $I_xI_yM \subseteq N$ for some $y \in M \setminus N$ with $I_y \subset R$. We denote $Z(M, N)$ for the set of zero divisors of $M$ with respect to $N$. Also, we denote the associated graph to $M$ with vertices $Z^*(M, N) = Z(M, N) \setminus N$ by $\Gamma(M, N)$, and two different vertices $x$ and $y$ are adjacent provided $I_xI_yM \subseteq N$.

In the second section, we define a submodule-based zero divisor graph for a module and we study basic properties of this graph. In the third section, if $M$ is a finitely generated semisimple $R$-module such that its homogenous components are simple and $N$ is a submodule of $M$, we determine some relations between $\Gamma(M, N)$ and $\Gamma(M/N)$, where $M/N$ is the quotient module of $M$, we show that the clique number and chromatic number of $\Gamma(M, N)$ are equal. Also, we determine some submodule of $M$ such that $\Gamma(M, N)$ is an empty or a complete bipartite graph.

Let $\Gamma$ be a (undirected) graph. We say that $\Gamma$ is \textit{connected} if there is a path between any two distinct vertices. For vertex $x$ the number of graph edges which touch $x$ is called the degree of $x$ and is denoted by $\deg(x)$. For vertices $x$ and $y$ of $\Gamma$, we define $d(x, y)$ to be the length of a shortest path between $x$ and $y$, if there is no path, then $d(x, y) = \infty$. The \textit{diameter} of $\Gamma$ is $\text{diam}(\Gamma) = \sup\{d(x, y) | x$ and $y$ are vertices of $\Gamma\}$. The \textit{girth} of $\Gamma$, denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$ ($\text{gr}(\Gamma) = \infty$ if $\Gamma$ contains no cycle).

A graph $\Gamma$ is \textit{complete} if any two distinct vertices are adjacent. The complete graph with $n$ vertices is denoted by $K^n$ (we allow $n$ to be an infinite cardinal). The \textit{clique number}, $\omega(\Gamma)$, is the greatest integer $n > 1$ such that $K^n \subseteq \Gamma$, and $\omega(\Gamma) = \infty$ if $K^n \subseteq \Gamma$ for all $n \geq 1$. A \textit{complete bipartite} graph is a graph $\Gamma$ which may be partitioned into two disjoint nonempty vertex sets $V_1$ and $V_2$.
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such that two distinct vertices are adjacent if and only if they are in different
vertex sets. If one of the vertex sets is a singleton, then we call that $\Gamma$ is a star
graph. We denote the complete bipartite graph by $K^{m,n}$, where $|V_1| = m$ and
$|V_2| = n$ (again, we allow $m$ and $n$ to be infinite cardinals); so a star graph is
$K^{1,n}$, for some $n \in \mathbb{N}$.

The chromatic number, $\chi(\Gamma)$, of a graph $\Gamma$ is the minimum number of colors
needed to color the vertices of $\Gamma$, so that no two adjacent vertices share the
same color. A graph $\Gamma$ is called planar if it can be drawn in such a way that
no two edges intersect.

Throughout this study, $R$ is a commutative ring with nonzero identity, $M$ is
a unitary $R$-module and $N$ is a proper submodule of $M$. Given any subset $S$
of $M$, the annihilator of $S$ is denoted by $\text{ann}(S) = \{r \in R | rs = 0$ for all $s \in S\}$
and the cardinal number of $S$ is denoted by $|S|$.

2. Submodule-based Zero Divisor Graph

Recall that $R$ is a commutative ring, $M$ is an $R$-module and $N$ is a proper
submodule of $M$. For $x \in M$, we denote $\text{ann}(M/Rx)$ by $I_x$.

Definition 2.1. Let $M$ be an $R$-module and $N$ be a proper submodule of $M$.
An $x \in M$ is called a zero divisor with respect to $N$ if $x \in N$ or $I_x \neq 0$
for some $y \in M \setminus N$ with $I_y \subset R$.

We denote the set of zero divisors of $M$ with respect to $N$ by $Z(M,N)$
and $Z^*(M, N) = Z(M,N) \setminus N$. The submodule-based zero divisor graph of $M$
with respect to $N$, $\Gamma(M,N)$, is an undirected graph with vertices $Z^*(M,N)$
such that distinct vertices $x$ and $y$ are adjacent if and only if $I_x I_y M \subseteq N$.

The following example shows that $Z(M/N)$ and $Z(M,N)$ are different from each other.

Example 2.2. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and $N = 2\mathbb{Z} \oplus 0$. Then $I_{(m,n)} = 0$, for all
$(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$. But $I_{(m,n)+N} = 2n\mathbb{Z}$ whenever $m \in 2\mathbb{Z}$ and $I_{(m,n)+N} = 2\mathbb{Z}$
whenever $m \notin 2\mathbb{Z}$. Thus $(1,0), (1,1) \in Z^*(M,N)$ are adjacent in $\Gamma(M,N)$, but
$(1,0) + N, (1,1) + N \notin Z^*(M/N)$.

Proposition 2.3. If $Z^*(M,N) = \emptyset$, then $\text{ann}(M/N)$ is a prime ideal of $R$.

Proof. Suppose that $\text{ann}(M/N)$ is not prime. Then there are ideals $I$ and $J$
of $R$ such that $IJM \subset N$ but $IM \nsubseteq N$ and $JM \nsubseteq N$. Let $x \in IM \setminus N$
and $y \in JM \setminus N$. Then $I_x I_y M \subseteq IJM \subseteq N$ and $I_y \subset R$. Thus $x \in Z^*(M,N)$, a
contradiction. Hence, $\text{ann}(M/N)$ is a prime ideal of $R$. □

Lemma 2.4. Let $x, y \in Z^*(M,N)$. If $x - y$ is an edge in $\Gamma(M,N)$, then for
each $0 \neq r \in R$, either $ry \in N$ or $x - ry$ is also an edge in $\Gamma(M,N)$.

Proof. Let $x, y \in Z^*(M,N)$ and $r \in R$. Assume that $x - y$ is an edge in $\Gamma(M,N)$
and $ry \notin N$. Then $I_x I_y M \subseteq N$. It is clear that $I_{rx} \subseteq I_x$. So that
$I_x I_y M \subseteq I_x I_y M \subseteq N$ and therefore, $x - ry$ is an edge in $\Gamma(M,N)$. □
It is shown that the graphs are defined in [12] and [4], are connected with diameter less than or equal to three. Moreover, it shown that if those graphs contain a cycle, then they have the girth less than or equal to four. In the next theorems, we extend these results to a submodule-based zero divisor graph.

**Theorem 2.5.** \( \Gamma(M, N) \) is a connected graph and \( \text{diam}(\Gamma(M, N)) \leq 3 \).

**Proof.** Let \( x \) and \( y \) be distinct vertices of \( \Gamma(M, N) \). Then, there are \( a, b \in Z^+(M, N) \) with \( I_aI_xM \subseteq N \) and \( I_yI_bM \subseteq N \) (we allow \( a, b \in \{x, y\} \)). If \( I_aI_bM \subseteq N \), then \( x - a - b - y \) is a path, thus \( d(x, y) \leq 3 \). If \( I_aI_bM \nsubseteq N \), then \( Ra \cap Rb \nsubseteq N \), and for every \( d \in (Ra \cap Rb) \setminus N \), \( x - d - y \) is a path of length 2, \( d(x, y) \leq 2 \), by Lemma 2.4. Hence, we conclude that \( \text{diam}(\Gamma(M, N)) \leq 3 \). \( \square \)

**Theorem 2.6.** If \( \Gamma(M, N) \) contains a cycle, then \( \text{gr}(\Gamma(M, N)) \leq 4 \).

**Proof.** We have \( \text{gr}(\Gamma(M, N)) = 7 \), by Proposition 1.3.2 in [7] and Theorem 2.5. Assume that \( x_1 - x_2 - \cdots - x_7 - x_1 \) is a cycle in \( \Gamma(M, N) \). If \( x_1 = x_4 \) then it is clear that \( \text{gr}(\Gamma(M, N)) \leq 3 \). So, suppose that \( x_1 \neq x_4 \). Then we have the following two cases:

**Case 1.** If \( x_1 \) and \( x_4 \) are adjacent in \( \Gamma(M, N) \), then \( x_1 - x_2 - x_3 - x_4 - x_1 \) is a cycle and \( \text{gr}(\Gamma(M, N)) \leq 4 \).

**Case 2.** Suppose that \( x_1 \) and \( x_4 \) are not adjacent in \( \Gamma(M, N) \). Then \( I_{x_1}I_{x_4}M \nsubseteq N \) and so there is a \( z \in (Rx_1 \cap Rx_4) \setminus N \). If \( z = x_1 \), then \( z \neq x_4 \) and \( x_3 - x_4 - x_3 - z - x_3 \) is a cycle in \( \Gamma(M, N) \), by Lemma 2.4. If \( z \neq x_1 \), then by Lemma 2.4, \( x_1 - x_2 - z - x_7 - x_1 \) is a cycle and \( \text{gr}(\Gamma(M, N)) \leq 4 \).

For cycles with length 5 or 6, by using a similar argument as above, one can shows that \( \text{gr}(\Gamma(M, N)) \leq 4 \). \( \square \)

**Example 2.7.** Assume that \( M = \mathbb{Z} \) and \( p, q \) are two prime numbers. If \( N = p\mathbb{Z} \), then \( \Gamma(M, N) = \emptyset \). If \( N = pq\mathbb{Z} \), then \( \Gamma(M, N) \) is an infinite complete bipartite graph with vertex set \( V_1 \cup V_2 \), where \( V_1 = p\mathbb{Z} \setminus pq\mathbb{Z} \) and \( V_2 = q\mathbb{Z} \setminus pq\mathbb{Z} \) and so \( \text{gr}(\Gamma(M, N)) = 4 \).

**Corollary 2.8.** If \( N \) is a prime submodule of \( M \), then \( \text{diam}(\Gamma(M, N)) \leq 2 \) and \( \text{gr}(\Gamma(M, N)) = 3 \), whenever it contains a cycle.

**Proof.** Let \( x, y \) be two distinct vertices which are not adjacent in \( \Gamma(M, N) \). Thus there is an \( a \in M \setminus N \) such that \( I_aI_xM \subseteq N \). Since \( N \) is a prime submoduule, then \( I_aM \subseteq N \). Thus \( I_aI_yM \subseteq N \), and then \( x - a - y \) is a path in \( \Gamma(M, N) \). Then \( \text{diam}(\Gamma(M, N)) \leq 2 \). \( \square \)

**Lemma 2.9.** Let \( |\Gamma(M, N)| \geq 3 \), \( \text{gr}(\Gamma(M, N)) = \infty \) and \( x \in Z^+(M, N) \) with \( \text{deg}(x) > 1 \). Then \( Rx = \{0, x\} \) and \( \text{ann}(x) \) is a prime ideal of \( R \).

**Proof.** First we show that \( Rx = \{0, x\} \). Let \( u - x - v \) be a path in \( \Gamma(M, N) \). Then \( u - v \) is not an edge in \( \Gamma(M, N) \) since \( \text{gr}(\Gamma(M, N)) = \infty \). If \( x \neq rx \) for some \( r \in R \) and \( rx \notin N \), then by Lemma 2.4, \( rx - u - x - v - rx \) is a cycle in
If there is an $r \in R$ such that $rx \in N$, then we have either $(1+r)x \in N$ or $(1+r)x = x$. These imply that $x \in N$ or $rx = 0$. Therefore, we have shown that $Rx = \{0,x\}$.

Let $a, b \in R$ and $abx = 0$. Then $bx = 0$ or $bx = x$. Hence, $bx = 0$ or $ax = 0$.

So, $\text{ann}(x)$ is a prime ideal of $R$. □

**Theorem 2.10.** If $N$ is a nonzero submodule of $M$ and $\text{gr}(\Gamma(M, N)) = \infty$, then $\Gamma(M, N)$ is a star graph.

**Proof.** Suppose that $\Gamma(M, N)$ is not a star graph. Then there is a path in $\Gamma(M, N)$ such as $u - x - y - v$. By Lemma 2.9, we have $Ry = \{0, y\}$ and by assumption $u$ and $y$ are not adjacent, thus $I_yM \neq 0$. So that $I_yM = Ry$. Also, $x - y - v$ is a path, thus $I_xI_yM \subseteq N$ and $I_xI_yM \subseteq N$. Hence, $I_xRy \subseteq N$ and $I_xRy \subseteq N$. On the other hand, for every nonzero $n \in N$, we have

$$I_x I_y + M \subseteq I_x (R(y+n) \subseteq I_x (Ry + N) \subseteq N$$

and similarly $I_x I_y + M \subseteq N$. So that $x - y - v - (y + n) - x$ is a cycle in $\Gamma(M, N)$, a contradiction. Therefore, $\Gamma(M, N)$ is a star graph. □

**Theorem 2.11.** Let $N$ be a nonzero submodule of $M$, $|\Gamma(M, N)| \geq 3$ and $\Gamma(M, N)$ is a star graph. Then the following statements are true:

(i) If $x$ is the center vertex, then $I_x = \text{ann}(M)$.

(ii) $\Gamma(M, N)$ is a subgraph of $\Gamma(M)$.

**Proof.** (i) By Lemma 2.9, we have $Rx = \{0,x\}$. Thus either $I_xM = 0$ or $I_xM = Rx$. Assume that $I_xM = Rx$. If $y$ is a vertex of $\Gamma(M, N)$ such that $y \neq x$, then $\text{deg}(y) = 1$ and $I_xI_yM \subseteq N$. Thus $I_xRx \subseteq N$. Since $I_xI_y + M \subseteq I_xR(x+n) \subseteq N$ for every nonzero element $n \in N$ it concludes that $y = x + n$. In this case, every other vertices of $\Gamma(M, N)$ are adjacent to $y$, a contradiction. Hence, $I_xM = 0$ and $I_x = \text{ann}(M)$.

(ii) It is obvious. □

**Theorem 2.12.** If $|N| \geq 3$ and $\Gamma(M, N)$ is a complete bipartite graph which is not a star graph, then $I_x^2M \not\subseteq N$, for every $x \in Z^2(M, N)$.

**Proof.** Let $Z^2(M, N) = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$. Suppose that $I_x^2M \subseteq N$ for some $x \in Z^2(M, N)$. Without loss of generality, we can assume that $x \in V_1$.

By a similar argument with Lemma 2.9, either $Rx = \{0,x\}$ or there is an $r \in R$ such that $x \neq rx$ and $rx \in N$. If $Rx = \{0,x\}$, then $I_xM = Rx$. Thus $I_xRx \subseteq N$. Now, for every $y \in V_2$ and $n \in N$ we get

$$I_y I_x + M \subseteq I_y R(x+n) \subseteq I_y (Rx + N) \subseteq N$$

and $I_x I_y + M \subseteq N$. Then, $x + n \in V_1 \cap V_2$, a contradiction. So, assume that $x \neq rx$ and $rx \in N$ for some $r \in R$. Since $I_{rx+x} \subseteq I_x$, then $I_y I_{rx+x}M \subseteq N$ and for all $y \in V_2$, $I_y I_{rx+x}M \subseteq N$. Thus $rx+x \in V_1 \cap V_2$, a contradiction. □
An $R$-module $X$ is called a multiplication-like module if, for each nonzero submodule $Y$ of $X$, $\text{ann}(X) \subset \text{ann}(X/Y)$. Multiplication-like module have been studied in [8, 13].

A vertex $x$ of a connected graph $G$ is a cut-point, if there are vertices $u, v$ of $G$ such that $x$ is in every path from $u$ to $v$ and $x \neq u, x \neq v$. For a connected graph $G$, an edge $E$ of $G$ is defined to be a bridge if $G - \{E\}$ is disconnected, see [6].

**Theorem 2.13.** Let $M$ be a multiplication-like module and $N$ be a nonzero submodule of $M$. Then $\Gamma(M, N)$ has no cut-points.

**Proof.** Suppose that $x$ is a cut-point of $\Gamma(M, N)$. Then there exist vertices $u, v \in M \setminus N$ such that $x$ lies on every path from $u$ to $v$. By Theorem 2.5, the shortest path from $u$ to $v$ has length 2 or 3.

**Case 1.** Suppose that $u - x - v$ is a path of shortest length from $u$ to $v$. Since $x$ is a cut point, $u, v$ aren’t in a cycle. By a similar argument to that of Lemma 2.9, we have $Rx = \{0, x\}$. On the other hand, $I_x M \subseteq Rx$ and $M$ is a multiplication-like module, so we have $I_x M = Rx$. Hence $I_u Rx \subseteq N$ and $I_v Rx \subseteq N$. Also, for every nonzero $n \in N$, we have $I_u I_{x+n} M \subseteq I_u (Rx + N) \subseteq N$ and $I_v I_{x+n} M \subseteq N$. Therefore, $u - (x + n) - v$ is a path from $u$ to $v$, a contradiction.

**Case 2.** Suppose that $u - x - y - v$ is a path in $\Gamma(M, N)$. Then, we have $I_x M = Rx$ and for every nonzero $n \in N$, we have $I_y I_{x+n} M \subseteq N$ and $I_u I_{x+n} M \subseteq N$. Thus $u - (x + n) - y - v$ is a path from $u$ to $v$, a contradiction. □

**Theorem 2.14.** Let $M$ be a multiplication-like module and $N$ be a nonzero submodule of $M$. Then $\Gamma(M, N)$ has a bridge if and only if $\Gamma(M, N)$ is a graph on two vertices.

**Proof.** If $|\Gamma(M, N)| = 3$, then $\Gamma(M, N) = K^3$, by Theorem 2.11, and it has no bridge. Assume that $|\Gamma(M, N)| \geq 4$ and $x - y$ is a bridge. Thus there is not a cycle containing $x - y$. Without loss of generality, we can assume that $\text{deg}(x) > 1$. Thus, there exists a vertex $z \neq y$ such that $z - x$ is an edge of $\Gamma(M, N)$. Then $Rx = \{0, x\}$ and $I_x M = Rx$. Hence, for every $n \in N$, $I_z I_{x+n} M \subseteq N$ and $I_y I_{x+n} M \subseteq N$, a contradiction. Therefore, $\Gamma(M, N)$ has not a bridge. The converse is clear. □

3. **Submodule-based Zero Divisor Graph of Semisimple Modules**

A nonzero $R$-module $X$ is called simple if its only submodules are $(0)$ and $X$. An $R$-module $X$ is called semisimple if it is a direct sum of simple modules. Also, $X$ is called homogenous semisimple if it is a direct sum of isomorphic simple modules.

In this section, $R$ is a commutative ring and $M$ is a finitely generated semisimple $R$-module such that its homogenous components are simple and
N is a submodule of M. The following theorem has a crucial role in this section.

**Theorem 3.1.** Let \( x, y \in M \setminus N \). Then \( x, y \) are adjacent in \( \Gamma(M, N) \) if and only if \( Rx \cap Ry \subseteq N \).

**Proof.** Let \( M = \bigoplus_{i \in I} M_i \), where \( M_i \)'s are non-isomorphic simple submodules of \( M \). By assumption \( N \) is a submodule of \( M \), so there exists a subset \( A \) of \( I \) such that \( M = N \oplus (\bigoplus_{i \in A} M_i) \) and so \( \text{ann}(M/N) = \bigoplus_{i \in A} \text{ann}(M_i) = \bigcap_{i \in A} \text{ann}(M_i) \). Assume that \( x, y \in M \setminus N \) are adjacent in \( \Gamma(M, N) \) and \( Rx \cap Ry \not\subseteq N \). Thus there exists \( \alpha \in I \) such that \( M_{\alpha} \subseteq (Rx \cap Ry) \setminus N \). Also, there exist subsets \( B \subset I \) and \( C \subset I \) such that \( M = Rx \oplus (\bigoplus_{i \in B} M_i) \) and \( M = Ry \oplus (\bigoplus_{i \in C} M_i) \). Therefore, \( I_x = \bigcap_{i \in B} \text{ann}(M_i) \) and \( I_y = \bigcap_{i \in C} \text{ann}(M_i) \). Since \( I_xI_yM \not\subseteq N \), we have \( I_xI_y \subseteq \text{ann}(M/N) \). For every \( i, j \in I \), \( \text{ann}(M_i) \) and \( \text{ann}(M_j) \) are coprime, then

\[
I_xI_y = \left[ \bigcap_{i \in B} \text{ann}(M_i) \right] \left[ \bigcap_{i \in C} \text{ann}(M_i) \right] = \prod_{i \in B \cup C} \text{ann}(M_i) \\
\subseteq \bigcap_{i \in A} \text{ann}(M_i) \subseteq \text{ann}(M_r),
\]

for all \( r \in A \). Thus for any \( r \in A \) there exists \( j_r \in B \cup C \) such that \( \text{ann}(M_{j_r}) \subseteq \text{ann}(M_r) \). So that \( \text{ann}(M_{j_r}) = \text{ann}(M_r) \) implies that \( M_{j_r} \cong M_r \) and by hypothesis \( M_{j_r} = M_r \). Hence,

\[
M_{\alpha} \subseteq \bigoplus_{i \in A} M_i \subseteq \bigoplus_{i \in B \cup C} M_j.
\]

Thus there exists \( \gamma \in B \cup C \) such that \( M_{\alpha} = M_{\gamma} \), also

\[
M_{\alpha} \subseteq Rx \cap Ry = (\bigoplus_{i \in I \setminus B} M_i) \cap (\bigoplus_{i \in I \setminus C} M_i).
\]

Therefore, \( \alpha \in I \setminus (B \cup C) \), a contradiction. The converse is obvious. \( \square \)

**Corollary 3.2.** Let \( x, y \in M \setminus N \) be such that \( x + N \neq y + N \). Then

(i) \( x \) and \( y \) are adjacent in \( \Gamma(M, N) \) if and only if \( x + N \) and \( y + N \) are adjacent in \( \Gamma(M/N) \).

(ii) if \( x \) and \( y \) are adjacent in \( \Gamma(M, N) \), then all distinct elements of \( x + N \) and \( y + N \) are adjacent in \( \Gamma(M, N) \).

**Proof.** (i) Let \( M = \bigoplus_{i \in I} M_i \), where \( M_i \)'s are non-isomorphic simple submodules of \( M \). Suppose that \( x \) and \( y \) are adjacent in \( \Gamma(M, N) \), \( Rx = \bigoplus_{i \in A} M_i \), \( Ry = \bigoplus_{i \in B} M_i \) and \( N = \bigoplus_{i \in C} M_i \). Then \( Rx + N = \bigoplus_{i \in A \cup C} M_i \) and \( Ry + N = \bigoplus_{i \in B \cup C} M_i \). Thus,

\[
(Rx + N) \cap (Ry + N) = \bigoplus_{i \in (A \cup C) \cap (B \cup C)} M_i = \bigoplus_{i \in (A \cap B) \cup C} M_i = (Rx \cap Ry) + N.
\]

By Theorem 3.1, we have \( Rx \cap Ry \not\subseteq N \) hence,

\[
I_x + N I_y + N M \subseteq (Rx + N) \cap (Ry + N) = (Rx \cap Ry) + N = N.
\]
Therefore, \( x + N \) and \( y + N \) are adjacent in \( \Gamma(M/N) \). The converse is obvious.

(ii) Let \( x, y \in Z^*(M, N) \) be adjacent in \( \Gamma(M, N) \). Then \( Rx \cap Ry \subseteq N \) by Theorem 3.1. So for every \( n, n' \in N \) we have

\[
I_{x+n}I_{y+n'}M \subseteq R(x+n) \cap R(y+n') \subseteq (Rx+N) \cap (Ry+N) = N.
\]

Hence, \( x + n \) and \( y + n' \) are adjacent in \( \Gamma(M, N) \). \( \square \)

In the following theorem, we prove that the clique number of graphs \( \Gamma(M, N) \) and \( \Gamma(M/N) \) are equal.

**Theorem 3.3.** If \( N \) is a nonzero submodule of \( M \), then \( \omega(\Gamma(M/N)) = \omega(\Gamma(M, N)) \).

**Proof.** First we show that \( I^2_{m+N}M \nsubseteq N \) for each \( 0 \neq m + N \in M/N \).

Assume that \( N = \bigoplus_{i \in A} M_i \) and \( m = (m_i)_{i \in I} \in M \setminus N \). Then \( I_{m+N} = \bigcap_{i \in A, m_i = 0} \text{ann}(M_i) \). Hence, \( I_{m+N} = I^2_{m+N} \). Thus \( I^2_{m+N}M \nsubseteq N \) since there is at least one \( j \in I \setminus A \) such that \( m_j \neq 0 \).

Now, Corollary 3.2 implies that \( \omega(\Gamma(M/N)) \leq \omega(\Gamma(M, N)) \). Thus, it is enough to consider the case where \( \omega(\Gamma(M/N)) = d < \infty \). Assume that \( G \) is a complete subgraph of \( \Gamma(M, N) \) with vertices \( m_1, m_2, \cdots, m_{d+1} \), we provide a contradiction. Consider the subgraph \( G_* \) of \( \Gamma(M/N) \) with vertices \( m_1, m_2, \cdots, m_{d+1} + N \). By Corollary 3.2, \( G_* \) is a complete subgraph of \( \Gamma(M, N) \). Thus \( m_j + N = m_k + N \) for some \( 1 \leq j, k \leq d + 1 \) with \( j \neq k \) since \( \omega(\Gamma(M/N)) = d \). We have \( I_{m_j} I_{m_k} M \subseteq N \). Therefore, \( Rm_j \cap Rm_k \subseteq N \) and so \( I_{m_j+N} I_{m_k+N} M \subseteq N \). Hence, \( I^2_{m_j+N}M \subseteq N \), that is a contradiction. \( \square \)

In the following theorem, we show that there is a relation between \( \omega(\Gamma(M, N)) \) and \( \chi(\Gamma(M/N)) \).

**Theorem 3.4.** Assume that \( M = \bigoplus_{i \in I} M_i \), where \( M_i \)'s are non-isomorphic simple submodules of \( M \) and \( N = \bigoplus_{i \in A} M_i \) is a submodule of \( M \) for some \( A \subset I \). Then \( \omega(\Gamma(M/N)) = \chi(\Gamma(M, N)) = |I| - |A| \).

**Proof.** Suppose that \( I \setminus A = \{1, \cdots, n\} \) so \( M_1, \cdots, M_n \nsubseteq N \). Let for \( 1 \leq k \leq n - 1 \)

\[
L^k = \{ m \in M : m \text{ has } k \text{ nonzero components} \}
\]

and let for \( 1 \leq s \leq n \)

\[
L^1_s = \{ m \in L^1 : \text{the } s^{th} \text{ component of } m \text{ is nonzero} \}.
\]

If \( m \in L^1_s \) and \( m' \in L^1_t \) for some \( 1 \leq s, t \leq n \) with \( s \neq t \), then \( m \) and \( m' \) are adjacent and so \( K^n \) is a subgraph of \( \Gamma(M, N) \). Thus \( \omega(\Gamma(M/N)) \geq n \).

If \( m, m' \in L^1_s \) for some \( 1 \leq s \leq n \), then \( m, m' \) are not adjacent because \( \text{ann}(M_s) \nsubseteq I_m I_{m'} \) and so the elements of \( L^1_s \) have same color. On the other hand, if \( x \in L^t \) with \( t > 1 \), then there is not a complete subgraph \( K^b \) of \( \Gamma(M, N) \) containing \( x \), such that \( b \geq n \). Thus \( \omega(\Gamma(M, N)) = n \leq \chi(\Gamma(M/N)) \).

Also, if \( x \in L^t \) with \( t > 1 \), then there is an \( s \) with \( 1 \leq s \leq n \) such that \( x \) is not
adjacent to each element of $L_1^s$. Thus the color of $x$ is same as the elements of $L_1^s$. Thus $\chi(\Gamma(M, N)) = n$. □

The Kwartowski’s Theorem states: A graph $G$ is planar if and only if it contains no subgraph homeomorphic to $K^5$ or $K^{3,3}$.

**Theorem 3.5.** Let $N$ be a nonzero proper submodule of $M$ such that $N$ is not prime. Then $\Gamma(M, N)$ is not planar.

**Proof.** Assume that $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$ and $N = \bigoplus_{i \in I} M_i$ for some $A \subseteq I$. Let $I \setminus A = \{i, j\}$. Then $\Gamma(M, N)$ is a complete bipartite graph $K^{n,m}$, where $n = (|\mathcal{I}| - 1)(\prod_{k \in I \setminus \{i, j\}} |M_k|)$ and $m = (|\mathcal{I}| - 1)(\prod_{k \in I \setminus \{i, j\}} |M_k|)$. By hypotheses $N$ is a nonzero and $M_i$’s are non-isomorphic, so we have $n, m \geq 3$. Hence $\Gamma(M, N)$ has a subgraph homeomorphic to $K^{3,3}$. The cases $|I \setminus A| \geq 3$ are similar to that of the case $|I \setminus A| = 2$. □

**Theorem 3.6.** A nonzero submodule $N$ of $M$ is prime if and only if $Z^*(M, N) = \emptyset$.

**Proof.** Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$ and $N$ is prime. Then $N = \bigoplus_{i \in I \setminus \{k\}} M_i$, for some $k \in I$. If $x \in Z^*(M, N)$, then there exists a $y \in M \setminus N$ such that $I_x I_y M \subseteq N$. If $x \neq y$, then $Rx \cap Ry \subseteq N$, by Theorem 3.1. Thus either $M_k \not\subseteq Rx$ or $M_k \not\subseteq Ry$. Hence, either $Rx \not\subseteq N$ or $Ry \not\subseteq N$, a contradiction. Now, suppose that $x = y$ so by $I_x^2 M \subseteq N$ and hypotheses $I_x M \subseteq N$. Thus $I_x I_y M \subseteq N$ for every $0 \neq n \in N$. By a similar argument, we have either $x \in N$ or $x + n \in N$, a contradiction. Hence, $Z^*(M, N) = \emptyset$.

Conversely, assume that $Z^*(M, N) = \emptyset$. Then $\text{ann}(M/N)$ is prime ideal of $R$ by Proposition 2.3 and there exists a $k \in I$ such that $\text{ann}(M/N) = \text{ann}(M_k)$. Hence, $N = \bigoplus_{i \in I \setminus \{k\}} M_i$ is a prime submodule of $M$. □

A proper submodule $N$ of $M$ is called 2-absorbing if whenever $a, b \in R$, $m \in M$ and $am \in N$, then $am \in N$ or $bm \in N$, ab $\in \text{ann}(M/N)$, see [10, 11]. In the following results, we study the behavior of $\Gamma(M, N)$ whenever $N$ is a 2-absorbing submodule of $M$.

**Theorem 3.7.** A submodule $N$ of $M$ is 2-absorbing if and only if at most two components of $M$ are zero in $N$.

**Proof.** Let $M = \bigoplus_{i \in I} M_i$, where $M_i$’s are non-isomorphic simple submodules of $M$. Suppose that $N$ is a 2-absorbing submodule of $M$ and $N = \bigoplus_{i \in A} M_i$, where $A = I \setminus \{s, t, k\}$. Since for all $i \in I$, $\text{ann}(M_i)$ is prime, there are $a \in \text{ann}(M_k) \setminus (\text{ann}(M_l) \cup \text{ann}(M_k))$, $b \in \text{ann}(M_s) \setminus (\text{ann}(M_l) \cup \text{ann}(M_k))$ and $c \in \bigcap_{j \in I \setminus \{s, t, l\}} \text{ann}(M_j) \setminus (\text{ann}(M_i) \cup \text{ann}(M_k))$. Now, $abc \in \text{ann}(M/N)$ but $ab \not\in \text{ann}(M/N)$, $ac \not\in \text{ann}(M/N)$ and $bc \not\in \text{ann}(M/N)$. This contradict with
Theorem 2.3 in [10]. Thus \( |A| \geq |I| - 2 \) and at most two components of \( M \) are zero in \( N \).

Conversely, if one component of \( M \) is zero in \( N \), then \( N \) is a prime submodule of \( M \). Suppose that \( N = \bigoplus_{i \in A} M_i \), where \( A = I \setminus \{i, j\} \). Thus \( M_i, M_j \not\subseteq N \). Suppose that \( a, b \in R, (m_i)_{i \in I} = m \in M \setminus N \) and \( abm \in N \). Then either \( m_i \neq 0 \) or \( m_j \neq 0 \). If \( m_i \neq 0 \) and \( m_j \neq 0 \), then \( ab \in \text{ann}(M_i) \cap \text{ann}(M_j) = \text{ann}(M/N) \).

If \( m_i \neq 0 \) and \( m_j = 0 \), then \( ab \in \text{ann}(M_i) \) and so either \( a \in \text{ann}(M_i) \) or \( b \in \text{ann}(M_i) \). Hence, \( am \in N \) or \( bm \in N \). The case \( m_i = 0 \) and \( m_j \neq 0 \), is similar to the previous case. Therefore, \( N \) is a 2-absorbing submodule of \( M \).

\[ \square \]

**Theorem 3.8.** \( N \) is a 2-absorbing submodule of \( M \) if and only if \( Z^*(M, N) = \emptyset \) or \( \Gamma(M, N) \) is a complete bipartite graph.

**Proof.** Let \( N \) be a 2-absorbing submodule of \( M \). If \( N \) is prime, then \( Z^*(M, N) = \emptyset \), by Theorem 3.6. Now, assume that \( N = \bigoplus_{i \in I \setminus \{j, k\}} M_i \) for some \( j, k \in I \) and \( (m_i)_{i \in I} = m \in M \setminus N \). Thus \( I_m = \bigcap_{i \in I : m_i = 0} \text{ann}(M_i) \). If \( m_j \neq 0 \) and \( m_k \neq 0 \), then \( m \notin Z(M, N) \).

Let \( V_1 = \{(m_i)_{i \in I} \in M \setminus N : m_j = 0\} \) and \( V_2 = \{(m_i)_{i \in I} \in M \setminus N : m_k = 0\} \). Thus \( m - m' \) is an edge in \( \Gamma(M, N) \) for every \( m \in V_1 \) and \( m' \in V_2 \). Also, every vertices in \( V_1 \) and \( V_2 \) are not adjacent. Hence, \( \Gamma(M, N) \) is a complete bipartite graph.

Now, suppose that \( \Gamma(M, N) \) is a complete bipartite graph and \( N \) is not 2-absorbing. By Theorem 3.7, there are at least three components \( M_s, M_t, M_k \) such that \( M_s, M_t, M_k \not\subseteq N \). For \( i = s, t, k \) let \( v_i = (m_i)_{i \in I} \), where \( m_i \neq 0 \) and \( m_j = 0 \) for all \( j \neq i \). Then \( v_s - v_t - v_k - v_s \) is a cycle in \( \Gamma(M, N) \). Thus \( \text{gr}(\Gamma(M, N)) = 3 \) and so \( \Gamma(M, N) \) is not bipartite graph, by Theorem 1 of Sec. 1.2 in [5]. Hence, \( N \) is a 2-absorbing submodule of \( M \).

\[ \square \]

**Example 3.9.** Let \( M = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7 \). Then every nonzero submodule \( N \) of \( M \) is 2-absorbing. Thus either \( Z^*(M, N) = \emptyset \) or \( \Gamma(M, N) \) is a complete bipartite graph. In particular, if \( N = \mathbb{Z}_7 \), then \( \Gamma(M, N) = K^7_{28} \).

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**References**


