

## A Submodule-Based Zero Divisor Graph for Modules

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ABSTRACT. Let  $R$  be a commutative ring with identity and  $M$  be an  $R$ -module. The zero divisor graph of  $M$  is denoted by  $\Gamma(M)$ . In this study, we are going to generalize the zero divisor graph  $\Gamma(M)$  to submodule-based zero divisor graph  $\Gamma(M, N)$  by replacing elements whose product is zero with elements whose product is in some submodule  $N$  of  $M$ . The main objective of this paper is to study the interplay of the properties of submodule  $N$  and the properties of  $\Gamma(M, N)$ .

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### 1. INTRODUCTION

Let  $R$  be a commutative ring with identity. The zero divisor graph of  $R$ , denoted  $\Gamma(R)$ , is an undirected graph whose vertices are the nonzero zero divisor of  $R$  with two distinct vertices  $x$  and  $y$  are adjacent by an edge if and only

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if  $xy = 0$ . The idea of a zero divisor graph of a commutative ring was introduced by Beck in [3] where he was mainly interested with colorings of rings. The definition above first is appeared in [2], which contains several fundamental results concerning  $\Gamma(R)$ . The zero-divisor graph of a commutative ring is further examined by Anderson, Levy and Shapiro, Mulay in [1, 9]. Also, the ideal-based zero divisor graph of  $R$  is defined by Redmond, in [12].

The zero divisor graph for modules over commutative rings has been defined by Behboodi in [4] as a generalization of zero divisor graph of rings. Let  $R$  be a commutative ring and  $M$  be an  $R$ -module, for  $x \in M$ , we denote the annihilator of the factor module  $M/Rx$  by  $I_x$ . An element  $x \in M$  is called a zero divisor, if either  $x = 0$  or  $I_x I_y M = 0$  for some  $y \neq 0$  with  $I_y \subset R$ . The set of zero divisors of  $M$  is denoted by  $Z(M)$  and the associated graph to  $M$  with vertices in  $Z^*(M) = Z(M) \setminus \{0\}$  is denoted by  $\Gamma(M)$ , such that two different vertices  $x$  and  $y$  are adjacent provided  $I_x I_y M = 0$ .

In this paper, we introduce the submodule-based zero divisor graph that is a generalization of zero divisor graph for modules. Let  $R$  be a commutative ring,  $M$  be an  $R$ -module and  $N$  be a proper submodule of  $M$ . An element  $x \in M$  is called zero divisor with respect to  $N$ , if either  $x \in N$  or  $I_x I_y M \subseteq N$  for some  $y \in M \setminus N$  with  $I_y \subset R$ . We denote  $Z(M, N)$  for the set of zero divisors of  $M$  with respect to  $N$ . Also, we denote the associated graph to  $M$  with vertices  $Z^*(M, N) = Z(M, N) \setminus N$  by  $\Gamma(M, N)$ , and two different vertices  $x$  and  $y$  are adjacent provided  $I_x I_y M \subseteq N$ .

In the second section, we define a submodule-based zero divisor graph for a module and we study basic properties of this graph. In the third section, if  $M$  is a finitely generated semisimple  $R$ -module such that its homogenous components are simple and  $N$  is a submodule of  $M$ , we determine some relations between  $\Gamma(M, N)$  and  $\Gamma(M/N)$ , where  $M/N$  is the quotient module of  $M$ , we show that the clique number and chromatic number of  $\Gamma(M, N)$  are equal. Also, we determine some submodule of  $M$  such that  $\Gamma(M, N)$  is an empty or a complete bipartite graph.

Let  $\Gamma$  be a (undirected) graph. We say that  $\Gamma$  is *connected* if there is a path between any two distinct vertices. For vertex  $x$  the number of graph edges which touch  $x$  is called the degree of  $x$  and is denoted by  $\deg(x)$ . For vertices  $x$  and  $y$  of  $\Gamma$ , we define  $d(x, y)$  to be the length of a shortest path between  $x$  and  $y$ , if there is no path, then  $d(x, y) = \infty$ . The *diameter* of  $\Gamma$  is  $\text{diam}(\Gamma) = \sup\{d(x, y) | x \text{ and } y \text{ are vertices of } \Gamma\}$ . The *girth* of  $\Gamma$ , denoted by  $\text{gr}(\Gamma)$ , is the length of a shortest cycle in  $\Gamma$  ( $\text{gr}(\Gamma) = \infty$  if  $\Gamma$  contains no cycle).

A graph  $\Gamma$  is *complete* if any two distinct vertices are adjacent. The complete graph with  $n$  vertices is denoted by  $K^n$  (we allow  $n$  to be an infinite cardinal). The *clique number*,  $\omega(\Gamma)$ , is the greatest integer  $n > 1$  such that  $K^n \subseteq \Gamma$ , and  $\omega(\Gamma) = \infty$  if  $K^n \subseteq \Gamma$  for all  $n \geq 1$ . A *complete bipartite* graph is a graph  $\Gamma$  which may be partitioned into two disjoint nonempty vertex sets  $V_1$  and  $V_2$

such that two distinct vertices are adjacent if and only if they are in different vertex sets. If one of the vertex sets is a singleton, then we call that  $\Gamma$  is a *star graph*. We denote the complete bipartite graph by  $K^{m,n}$ , where  $|V_1| = m$  and  $|V_2| = n$  (again, we allow  $m$  and  $n$  to be infinite cardinals); so a star graph is  $K^{1,n}$ , for some  $n \in \mathbb{N}$ .

The *chromatic number*,  $\chi(\Gamma)$ , of a graph  $\Gamma$  is the minimum number of colors needed to color the vertices of  $\Gamma$ , so that no two adjacent vertices share the same color. A graph  $\Gamma$  is called *planar* if it can be drawn in such a way that no two edges intersect.

Throughout this study,  $R$  is a commutative ring with nonzero identity,  $M$  is a unitary  $R$ -module and  $N$  is a proper submodule of  $M$ . Given any subset  $S$  of  $M$ , the annihilator of  $S$  is denoted by  $\text{ann}(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}$  and the cardinal number of  $S$  is denoted by  $|S|$ .

## 2. SUBMODULE-BASED ZERO DIVISOR GRAPH

Recall that  $R$  is a commutative ring,  $M$  is an  $R$ -module and  $N$  is a proper submodule of  $M$ . For  $x \in M$ , we denote  $\text{ann}(M/Rx)$  by  $I_x$ .

**Definition 2.1.** Let  $M$  be an  $R$ -module and  $N$  be a proper submodule of  $M$ . An  $x \in M$  is called a zero divisor with respect to  $N$  if  $x \in N$  or  $I_x I_y M \subseteq N$  for some  $y \in M \setminus N$  with  $I_y \subset R$ .

We denote the set of zero divisors of  $M$  with respect to  $N$  by  $Z(M, N)$  and  $Z^*(M, N) = Z(M, N) \setminus N$ . The submodule-based zero divisor graph of  $M$  with respect to  $N$ ,  $\Gamma(M, N)$ , is an undirected graph with vertices  $Z^*(M, N)$  such that distinct vertices  $x$  and  $y$  are adjacent if and only if  $I_x I_y M \subseteq N$ .

The following example shows that  $Z(M/N)$  and  $Z(M, N)$  are different from each other.

**EXAMPLE 2.2.** Let  $M = \mathbb{Z} \oplus \mathbb{Z}$  and  $N = 2\mathbb{Z} \oplus 0$ . Then  $I_{(m,n)} = 0$ , for all  $(m, n) \in \mathbb{Z} \oplus \mathbb{Z}$ . But  $I_{(m,n)+N} = 2n\mathbb{Z}$  whenever  $m \in 2\mathbb{Z}$  and  $I_{(m,n)+N} = 2\mathbb{Z}$  whenever  $m \notin 2\mathbb{Z}$ . Thus  $(1, 0), (1, 1) \in Z^*(M, N)$  are adjacent in  $\Gamma(M, N)$ , but  $(1, 0) + N, (1, 1) + N \notin Z^*(M/N)$ .

**Proposition 2.3.** If  $Z^*(M, N) = \emptyset$ , then  $\text{ann}(M/N)$  is a prime ideal of  $R$ .

*Proof.* Suppose that  $\text{ann}(M/N)$  is not prime. Then there are ideals  $I$  and  $J$  of  $R$  such that  $IJM \subset N$  but  $IM \not\subseteq N$  and  $JM \not\subseteq N$ . Let  $x \in IM \setminus N$  and  $y \in JM \setminus N$ . Then  $I_x I_y M \subseteq IJM \subseteq N$  and  $I_y \subset R$ . Thus  $x \in Z^*(M, N)$ , a contradiction. Hence,  $\text{ann}(M/N)$  is a prime ideal of  $R$ .  $\square$

**Lemma 2.4.** Let  $x, y \in Z^*(M, N)$ . If  $x - y$  is an edge in  $\Gamma(M, N)$ , then for each  $0 \neq r \in R$ , either  $ry \in N$  or  $x - ry$  is also an edge in  $\Gamma(M, N)$ .

*Proof.* Let  $x, y \in Z^*(M, N)$  and  $r \in R$ . Assume that  $x - y$  is an edge in  $\Gamma(M, N)$  and  $ry \notin N$ . Then  $I_x I_y M \subseteq N$ . It is clear that  $I_{rx} \subseteq I_x$ . So that  $I_x I_{ry} M \subseteq I_x I_y M \subseteq N$  and therefore,  $x - ry$  is an edge in  $\Gamma(M, N)$ .  $\square$

It is shown that the graphs are defined in [12] and [4], are connected with diameter less than or equal to three. Moreover, it shown that if those graphs contain a cycle, then they have the girth less than or equal to four. In the next theorems, we extend these results to a submodule-based zero divisor graph.

**Theorem 2.5.**  $\Gamma(M, N)$  is a connected graph and  $\text{diam}(\Gamma(M, N)) \leq 3$ .

*Proof.* Let  $x$  and  $y$  be distinct vertices of  $\Gamma(M, N)$ . Then, there are  $a, b \in Z^*(M, N)$  with  $I_a I_x M \subseteq N$  and  $I_b I_y M \subseteq N$  (we allow  $a, b \in \{x, y\}$ ). If  $I_a I_b M \subseteq N$ , then  $x - a - b - y$  is a path, thus  $d(x, y) \leq 3$ . If  $I_a I_b M \not\subseteq N$ , then  $Ra \cap Rb \not\subseteq N$ , and for every  $d \in (Ra \cap Rb) \setminus N$ ,  $x - d - y$  is a path of length 2,  $d(x, y) \leq 2$ , by Lemma 2.4. Hence, we conclude that  $\text{diam}(\Gamma(M, N)) \leq 3$ .  $\square$

**Theorem 2.6.** If  $\Gamma(M, N)$  contains a cycle, then  $\text{gr}(\Gamma(M, N)) \leq 4$ .

*Proof.* We have  $\text{gr}(\Gamma(M, N)) \leq 7$ , by Proposition 1.3.2 in [7] and Theorem 2.5. Assume that  $x_1 - x_2 - \cdots - x_7 - x_1$  is a cycle in  $\Gamma(M, N)$ . If  $x_1 = x_4$  then it is clear that  $\text{gr}(\Gamma(M, N)) \leq 3$ . So, suppose that  $x_1 \neq x_4$ . Then we have the following two cases:

**Case 1.** If  $x_1$  and  $x_4$  are adjacent in  $\Gamma(M, N)$ , then  $x_1 - x_2 - x_3 - x_4 - x_1$  is a cycle and  $\text{gr}(\Gamma(M, N)) \leq 4$ .

**Case 2.** Suppose that  $x_1$  and  $x_4$  are not adjacent in  $\Gamma(M, N)$ . Then  $I_{x_1} I_{x_4} M \not\subseteq N$  and so there is a  $z \in (Rx_1 \cap Rx_4) \setminus N$ . If  $z = x_1$ , then  $z \neq x_4$  and  $x_3 - x_4 - x_5 - z - x_3$  is a cycle in  $\Gamma(M, N)$ , by Lemma 2.4. If  $z \neq x_1$ , then by Lemma 2.4,  $x_1 - x_2 - z - x_7 - x_1$  is a cycle and  $\text{gr}(\Gamma(M, N)) \leq 4$ .

For cycles with length 5 or 6, by using a similar argument as above, one can shows that  $\text{gr}(\Gamma(M, N)) \leq 4$ .  $\square$

**EXAMPLE 2.7.** Assume that  $M = \mathbb{Z}$  and  $p, q$  are two prime numbers. If  $N = p\mathbb{Z}$ , then  $\Gamma(M, N) = \emptyset$ . If  $N = pq\mathbb{Z}$ , then  $\Gamma(M, N)$  is an infinite complete bipartite graph with vertex set  $V_1 \cup V_2$ , where  $V_1 = p\mathbb{Z} \setminus pq\mathbb{Z}$  and  $V_2 = q\mathbb{Z} \setminus pq\mathbb{Z}$  and so  $\text{gr}(\Gamma(M, N)) = 4$ .

**Corollary 2.8.** If  $N$  is a prime submodule of  $M$ , then  $\text{diam}(\Gamma(M, N)) \leq 2$  and  $\text{gr}(\Gamma(M, N)) = 3$ , whenever it contains a cycle.

*Proof.* Let  $x, y$  be two distinct vertices which are not adjacent in  $\Gamma(M, N)$ . Thus there is an  $a \in M \setminus N$  such that  $I_a I_x M \subseteq N$ . Since  $N$  is a prime submodule, then  $I_a M \subseteq N$ . Thus  $I_a I_y M \subseteq N$ , and then  $x - a - y$  is a path in  $\Gamma(M, N)$ . Then  $\text{diam}(\Gamma(M, N)) \leq 2$ .  $\square$

**Lemma 2.9.** Let  $|\Gamma(M, N)| \geq 3$ ,  $\text{gr}(\Gamma(M, N)) = \infty$  and  $x \in Z^*(M, N)$  with  $\text{deg}(x) > 1$ . Then  $Rx = \{0, x\}$  and  $\text{ann}(x)$  is a prime ideal of  $R$ .

*Proof.* First we show that  $Rx = \{0, x\}$ . Let  $u - x - v$  be a path in  $\Gamma(M, N)$ . Then  $u - v$  is not an edge in  $\Gamma(M, N)$  since  $\text{gr}(\Gamma(M, N)) = \infty$ . If  $x \neq rx$  for some  $r \in R$  and  $rx \notin N$ , then by Lemma 2.4,  $rx - u - x - v - rx$  is a cycle in

$\Gamma(M, N)$ , that is a contradiction. So, for every  $r \in R$  either  $rx = x$  or  $rx \in N$ . If there is an  $r \in R$  such that  $rx \in N$ , then we have either  $(1 + r)x \in N$  or  $(1 + r)x = x$ . These imply that  $x \in N$  or  $rx = 0$ . Therefore, we have shown that  $Rx = \{0, x\}$ .

Let  $a, b \in R$  and  $abx = 0$ . Then  $bx = 0$  or  $bx = x$ . Hence,  $bx = 0$  or  $ax = 0$ . So,  $\text{ann}(x)$  is a prime ideal of  $R$ .  $\square$

**Theorem 2.10.** *If  $N$  is a nonzero submodule of  $M$  and  $\text{gr}(\Gamma(M, N)) = \infty$ , then  $\Gamma(M, N)$  is a star graph.*

*Proof.* Suppose that  $\Gamma(M, N)$  is not a star graph. Then there is a path in  $\Gamma(M, N)$  such as  $u - x - y - v$ . By Lemma 2.9, we have  $Ry = \{0, y\}$  and by assumption  $u$  and  $y$  are not adjacent, thus  $I_yM \neq 0$ . So that  $I_yM = Ry$ . Also,  $x - y - v$  is a path, thus  $I_vI_yM \subseteq N$  and  $I_xI_yM \subseteq N$ . Hence,  $I_vRy \subseteq N$  and  $I_xRy \subseteq N$ . On the other hand, for every nonzero  $n \in N$ , we have

$$I_vI_{y+n}M \subseteq I_vR(y + n) \subseteq I_v(Ry + N) \subseteq N$$

and similarly  $I_xI_{y+n}M \subseteq N$ . So that  $x - y - v - (y + n) - x$  is a cycle in  $\Gamma(M, N)$ , a contradiction. Therefore,  $\Gamma(M, N)$  is a star graph.  $\square$

**Theorem 2.11.** *Let  $N$  be a nonzero submodule of  $M$ ,  $|\Gamma(M, N)| \geq 3$  and  $\Gamma(M, N)$  is a star graph. Then the following statements are true:*

- (i) *If  $x$  is the center vertex, then  $I_x = \text{ann}(M)$ .*
- (ii)  *$\Gamma(M, N)$  is a subgraph of  $\Gamma(M)$ .*

*Proof.* (i) By Lemma 2.9, we have  $Rx = \{0, x\}$ . Thus either  $I_xM = 0$  or  $I_xM = Rx$ . Assume that  $I_xM = Rx$ . If  $y$  is a vertex of  $\Gamma(M, N)$  such that  $y \neq x$ , then  $\text{deg}(y) = 1$  and  $I_xI_yM \subseteq N$ . Thus  $I_yRx \subseteq N$ . Since  $I_{x+n}I_yM \subseteq I_yR(x + n) \subseteq N$  for every nonzero element  $n \in N$  it concludes that  $y = x + n$ . In this case, every other vertices of  $\Gamma(M, N)$  are adjacent to  $y$ , a contradiction. Hence,  $I_xM = 0$  and  $I_x = \text{ann}(M)$ .

(ii) It is obvious.  $\square$

**Theorem 2.12.** *If  $|N| \geq 3$  and  $\Gamma(M, N)$  is a complete bipartite graph which is not a star graph, then  $I_x^2M \not\subseteq N$ , for every  $x \in Z^*(M, N)$ .*

*Proof.* Let  $Z^*(M, N) = V_1 \cup V_2$ , where  $V_1 \cap V_2 = \emptyset$ . Suppose that  $I_x^2M \subseteq N$  for some  $x \in Z^*(M, N)$ . Without loss of generality, we can assume that  $x \in V_1$ . By a similar argument with Lemma 2.9, either  $Rx = \{0, x\}$  or there is an  $r \in R$  such that  $x \neq rx$  and  $rx \in N$ . If  $Rx = \{0, x\}$ , then  $I_xM = Rx$ . Thus  $I_xRx \subseteq N$ . Now, for every  $y \in V_2$  and  $n \in N$  we get

$$I_yI_{x+n}M \subseteq I_yR(x + n) \subseteq I_y(Rx + N) \subseteq N$$

and  $I_xI_{x+n}M \subseteq N$ . Then,  $x + n \in V_1 \cap V_2$ , a contradiction. So, assume that  $x \neq rx$  and  $rx \in N$  for some  $r \in R$ . Since  $I_{rx+x} \subseteq I_x$ , then  $I_xI_{rx+x}M \subseteq N$  and for all  $y \in V_2$ ,  $I_yI_{rx+x}M \subseteq N$ . Thus  $rx + x \in V_1 \cap V_2$ , a contradiction.  $\square$

An  $R$ -module  $X$  is called a *multiplication-like* module if, for each nonzero submodule  $Y$  of  $X$ ,  $\text{ann}(X) \subset \text{ann}(X/Y)$ . Multiplication-like module have been studied in [8, 13].

A vertex  $x$  of a connected graph  $G$  is a *cut-point*, if there are vertices  $u, v$  of  $G$  such that  $x$  is in every path from  $u$  to  $v$  and  $x \neq u, x \neq v$ . For a connected graph  $G$ , an edge  $E$  of  $G$  is defined to be a *bridge* if  $G - \{E\}$  is disconnected, see [6].

**Theorem 2.13.** *Let  $M$  be a multiplication-like module and  $N$  be a nonzero submodule of  $M$ . Then  $\Gamma(M, N)$  has no cut-points.*

*Proof.* Suppose that  $x$  is a cut-point of  $\Gamma(M, N)$ . Then there exist vertices  $u, v \in M \setminus N$  such that  $x$  lies on every path from  $u$  to  $v$ . By Theorem 2.5, the shortest path from  $u$  to  $v$  has length 2 or 3.

**Case 1.** Suppose that  $u-x-v$  is a path of shortest length from  $u$  to  $v$ . Since  $x$  is a cut point  $x, u, v$  aren't in a cycle. By a similar argument to that of Lemma 2.9, we have  $Rx = \{0, x\}$ . On the other hand,  $I_x M \subseteq Rx$  and  $M$  is a multiplication-like module, so we have  $I_x M = Rx$ . Hence  $I_u Rx \subseteq N$  and  $I_v Rx \subseteq N$ . Also, for every nonzero  $n \in N$ , we have  $I_u I_{x+n} M \subseteq I_u (Rx + N) \subseteq N$  and  $I_v I_{x+n} M \subseteq N$ . Therefore,  $u - (x + n) - v$  is a path from  $u$  to  $v$ , a contradiction.

**Case 2.** Suppose that  $u-x-y-v$  is a path in  $\Gamma(M, N)$ . Then, we have  $I_x M = Rx$  and for every nonzero  $n \in N$ , we have  $I_y I_{x+n} M \subseteq N$  and  $I_u I_{x+n} M \subseteq N$ . Thus  $u - (x + n) - y - v$  is a path from  $u$  to  $v$ , a contradiction.  $\square$

**Theorem 2.14.** *Let  $M$  be a multiplication-like module and  $N$  be a nonzero submodule of  $M$ . Then  $\Gamma(M, N)$  has a bridge if and only if  $\Gamma(M, N)$  is a graph on two vertices.*

*Proof.* If  $|\Gamma(M, N)| = 3$ , then  $\Gamma(M, N) = K^3$ , by Theorem 2.11, and it has no bridge. Assume that  $|\Gamma(M, N)| \geq 4$  and  $x - y$  is a bridge. Thus there is not a cycle containing  $x - y$ . Without loss of generality, we can assume that  $\deg(x) > 1$ . Thus, there exists a vertex  $z \neq y$  such that  $z - x$  is an edge of  $\Gamma(M, N)$ . Then  $Rx = \{0, x\}$  and  $I_x M = Rx$ . Hence, for every  $n \in N$ ,  $I_z I_{x+n} M \subseteq N$  and  $I_y I_{x+n} M \subseteq N$ , a contradiction. Therefore,  $\Gamma(M, N)$  has not a bridge. The converse is clear.  $\square$

### 3. SUBMODULE-BASED ZERO DIVISOR GRAPH OF SEMISIMPLE MODULES

A nonzero  $R$ -module  $X$  is called simple if its only submodules are  $(0)$  and  $X$ . An  $R$ -module  $X$  is called semisimple if it is a direct sum of simple modules. Also,  $X$  is called homogenous semisimple if it is a direct sum of isomorphic simple modules.

In this section,  $R$  is a commutative ring and  $M$  is a finitely generated semisimple  $R$ -module such that its homogenous components are simple and

$N$  is a submodule of  $M$ . The following theorem has a crucial role in this section.

**Theorem 3.1.** *Let  $x, y \in M \setminus N$ . Then  $x, y$  are adjacent in  $\Gamma(M, N)$  if and only if  $Rx \cap Ry \subseteq N$ .*

*Proof.* Let  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of  $M$ . By assumption  $N$  is a submodule of  $M$ , so there exists a subset  $A$  of  $I$  such that  $M = N \oplus (\bigoplus_{i \in A} M_i)$  and so  $\text{ann}(M/N) = \text{ann}(\bigoplus_{i \in A} M_i) = \bigcap_{i \in A} \text{ann}(M_i)$ . Assume that  $x, y \in M \setminus N$  are adjacent in  $\Gamma(M, N)$  and  $Rx \cap Ry \not\subseteq N$ . Thus there exists  $\alpha \in I$  such that  $M_\alpha \subseteq (Rx \cap Ry) \setminus N$ . Also, there exist subsets  $B \subset I$  and  $C \subset I$  such that  $M = Rx \oplus (\bigoplus_{i \in B} M_i)$  and  $M = Ry \oplus (\bigoplus_{i \in C} M_i)$ . Therefore,  $I_x = \bigcap_{i \in B} \text{ann}(M_i)$  and  $I_y = \bigcap_{i \in C} \text{ann}(M_i)$ . Since  $I_x I_y M \subseteq N$ , we have  $I_x I_y \subseteq \text{ann}(M/N)$ . For every  $i, j \in I$ ,  $\text{ann}(M_i)$  and  $\text{ann}(M_j)$  are coprime, then

$$\begin{aligned} I_x I_y = \left[ \bigcap_{i \in B} \text{ann}(M_i) \right] \left[ \bigcap_{i \in C} \text{ann}(M_i) \right] &= \prod_{i \in B \cup C} \text{ann}(M_i) \\ &\subseteq \bigcap_{i \in A} \text{ann}(M_i) \subseteq \text{ann}(M_r), \end{aligned}$$

for all  $r \in A$ . Thus for any  $r \in A$  there exists  $j_r \in B \cup C$  such that  $\text{ann}(M_{j_r}) \subseteq \text{ann}(M_r)$ . So that  $\text{ann}(M_{j_r}) = \text{ann}(M_r)$  implies that  $M_{j_r} \cong M_r$  and by hypothesis  $M_{j_r} = M_r$ . Hence,

$$M_\alpha \subseteq \bigoplus_{i \in A} M_i \subseteq \bigoplus_{j \in B \cup C} M_j.$$

Thus there exists  $\gamma \in B \cup C$  such that  $M_\alpha = M_\gamma$ , also

$$M_\alpha \subseteq Rx \cap Ry = \left( \bigoplus_{i \in I \setminus B} M_i \right) \cap \left( \bigoplus_{i \in I \setminus C} M_i \right).$$

Therefore,  $\alpha \in I \setminus (B \cup C)$ , a contradiction. The converse is obvious.  $\square$

**Corollary 3.2.** *Let  $x, y \in M \setminus N$  be such that  $x + N \neq y + N$ . Then*

- (i)  *$x$  and  $y$  are adjacent in  $\Gamma(M, N)$  if and only if  $x + N$  and  $y + N$  are adjacent in  $\Gamma(M/N)$ .*
- (ii) *if  $x$  and  $y$  are adjacent in  $\Gamma(M, N)$ , then all distinct elements of  $x + N$  and  $y + N$  are adjacent in  $\Gamma(M, N)$ .*

*Proof.* (i) Let  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of  $M$ . Suppose that  $x$  and  $y$  are adjacent in  $\Gamma(M, N)$ ,  $Rx = \bigoplus_{i \in A} M_i$ ,  $Ry = \bigoplus_{i \in B} M_i$  and  $N = \bigoplus_{i \in C} M_i$ . Then  $Rx + N = \bigoplus_{i \in A \cup C} M_i$  and  $Ry + N = \bigoplus_{i \in B \cup C} M_i$ . Thus,

$$(Rx + N) \cap (Ry + N) = \bigoplus_{i \in (A \cup C) \cap (B \cup C)} M_i = \bigoplus_{i \in (A \cap B) \cup C} M_i = (Rx \cap Ry) + N.$$

By Theorem 3.1, we have  $Rx \cap Ry \subseteq N$  hence,

$$I_{x+N} I_{y+N} M \subseteq (Rx + N) \cap (Ry + N) = (Rx \cap Ry) + N = N.$$

Therefore,  $x + N$  and  $y + N$  are adjacent in  $\Gamma(M/N)$ . The converse is obvious.

(ii) Let  $x, y \in Z^*(M, N)$  be adjacent in  $\Gamma(M, N)$ . Then  $Rx \cap Ry \subseteq N$  by Theorem 3.1. So for every  $n, n' \in N$  we have

$$I_{x+n}I_{y+n'}M \subseteq R(x+n) \cap R(y+n') \subseteq (Rx+N) \cap (Ry+N) = N.$$

Hence,  $x+n$  and  $y+n'$  are adjacent in  $\Gamma(M, N)$ .  $\square$

In the following theorem, we prove that the clique number of graphs  $\Gamma(M, N)$  and  $\Gamma(M/N)$  are equal.

**Theorem 3.3.** *If  $N$  is a nonzero submodule of  $M$ , then  $\omega(\Gamma(M/N)) = \omega(\Gamma(M, N))$ .*

*Proof.* First we show that  $I_{m+N}^2M \not\subseteq N$  for each  $0 \neq m+N \in M/N$ . Assume that  $N = \bigoplus_{i \in A} M_i$  and  $m = (m_i)_{i \in I} \in M \setminus N$ . Then  $I_{m+N} = \bigcap_{i \notin A, m_i=0} \text{ann}(M_i)$ . Hence,  $I_{m+N} = I_{m+N}^2$ . Thus  $I_{m+N}^2M \not\subseteq N$  since there is at least one  $j \in I \setminus A$  such that  $m_j \neq 0$ .

Now, Corollary 3.2 implies that  $\omega(\Gamma(M/N)) \leq \omega(\Gamma(M, N))$ . Thus, it is enough to consider the case where  $\omega(\Gamma(M/N)) = d < \infty$ . Assume that  $G$  is a complete subgraph of  $\Gamma(M, N)$  with vertices  $m_1, m_2, \dots, m_{d+1}$ , we provide a contradiction. Consider the subgraph  $G_*$  of  $\Gamma(M/N)$  with vertices  $m_1 + N, \dots, m_{d+1} + N$ . By Corollary 3.2,  $G_*$  is a complete subgraph of  $\Gamma(M, N)$ . Thus  $m_j + N = m_k + N$  for some  $1 \leq j, k \leq d+1$  with  $j \neq k$  since  $\omega(\Gamma(M/N)) = d$ . We have  $I_{m_j}I_{m_k}M \subseteq N$ . Therefore,  $Rm_j \cap Rm_k \subseteq N$  and so  $I_{m_j+N}I_{m_k+N}M \subseteq N$ . Hence,  $I_{m_j+N}^2M \subseteq N$ , that is a contradiction.  $\square$

In the following theorem, we show that there is a relation between  $\omega(\Gamma(M, N))$  and  $\chi(\Gamma(M, N))$ .

**Theorem 3.4.** *Assume that  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of  $M$  and  $N = \bigoplus_{i \in A} M_i$  is a submodule of  $M$  for some  $A \subset I$ . Then  $\omega(\Gamma(M, N)) = \chi(\Gamma(M, N)) = |I| - |A|$ .*

*Proof.* Suppose that  $I \setminus A = \{1, \dots, n\}$  so  $M_1, \dots, M_n \not\subseteq N$ . Let for  $1 \leq k \leq n-1$

$$L^k = \{m \in M : m \text{ has } k \text{ nonzero components}\}$$

and let for  $1 \leq s \leq n$

$$L_s^1 = \{m \in L^1 : \text{the } s^{\text{th}} \text{ component of } m \text{ is nonzero}\}.$$

If  $m \in L_s^1$  and  $m' \in L_t^1$  for some  $1 \leq s, t \leq n$  with  $s \neq t$ , then  $m$  and  $m'$  are adjacent and so  $K^n$  is a subgraph of  $\Gamma(M, N)$ . Thus  $\omega(\Gamma(M, N)) \geq n$ . If  $m, m' \in L_s^1$  for some  $1 \leq s \leq n$ , then  $m, m'$  are not adjacent because  $\text{ann}(M_s) \not\subseteq I_m I_{m'}$  and so the elements of  $L_s^1$  have same color. On the other hand, if  $x \in L^t$  with  $t > 1$ , then there is not a complete subgraph  $K^h$  of  $\Gamma(M, N)$  containing  $x$ , such that  $h \geq n$ . Thus  $\omega(\Gamma(M, N)) = n \leq \chi(\Gamma(M, N))$ . Also, if  $x \in L^t$  with  $t > 1$ , then there is an  $s$  with  $1 \leq s \leq n$  such that  $x$  is not



adjacent to each element of  $L_s^1$ . Thus the color of  $x$  is same as the elements of  $L_s^1$ . Thus  $\chi(\Gamma(M, N)) = n$ .  $\square$

The Kuartowski's Theorem states: A graph  $G$  is planar if and only if it contains no subgraph homeomorphic to  $K^5$  or  $K^{3,3}$ .

**Theorem 3.5.** *Let  $N$  be a nonzero proper submodule of  $M$  such that  $N$  is not prime. Then  $\Gamma(M, N)$  is not planar.*

*Proof.* Assume that  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of  $M$  and  $N = \bigoplus_{i \in A} M_i$  for some  $A \subset I$ . Let  $I \setminus A = \{i, j\}$ . Then  $\Gamma(M, N)$  is a complete bipartite graph  $K^{n,m}$ , where  $n = (|M_i| - 1)(\prod_{k \in I - \{i,j\}} |M_k|)$  and  $m = (|M_j| - 1)(\prod_{k \in I - \{i,j\}} |M_k|)$ . By hypotheses  $N$  is a nonzero and  $M_i$ 's are non-isomorphic, so we have  $n, m \geq 3$ . Hence  $\Gamma(M, N)$  has a subgraph homeomorphic to  $K^{3,3}$ . The cases  $|I \setminus A| \geq 3$  are similar to that of the case  $|I \setminus A| = 2$ .  $\square$

**Theorem 3.6.** *A nonzero submodule  $N$  of  $M$  is prime if and only if  $Z^*(M, N) = \emptyset$ .*

*Proof.* Let  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of  $M$  and  $N$  is prime. Then  $N = \bigoplus_{i \in I \setminus \{k\}} M_i$ , for some  $k \in I$ . If  $x \in Z^*(M, N)$ , then there exists a  $y \in M \setminus N$  such that  $I_x I_y M \subseteq N$ . If  $x \neq y$ , then  $Rx \cap Ry \subseteq N$ , by Theorem 3.1. Thus either  $M_k \not\subseteq Rx$  or  $M_k \not\subseteq Ry$ . Hence, either  $Rx \subseteq N$  or  $Ry \subseteq N$ , a contradiction. Now, suppose that  $x = y$  so by  $I_x^2 M \subseteq N$  and hypotheses  $I_x M \subseteq N$ . Thus  $I_{x+n} I_x M \subseteq N$  for every  $0 \neq n \in N$ . By a similar argument, we have either  $x \in N$  or  $x + n \in N$ , a contradiction. Hence,  $Z^*(M, N) = \emptyset$ .

Conversely, assume that  $Z^*(M, N) = \emptyset$ . Then  $\text{ann}(M/N)$  is prime ideal of  $R$  by Proposition 2.3 and there exists a  $k \in I$  such that  $\text{ann}(M/N) = \text{ann}(M_k)$ . Hence,  $N = \bigoplus_{i \in I \setminus \{k\}} M_i$  is a prime submodule of  $M$ .  $\square$

A proper submodule  $N$  of  $M$  is called 2-absorbing if whenever  $a, b \in R$ ,  $m \in M$  and  $abm \in N$ , then  $am \in N$  or  $bm \in N$  or  $ab \in \text{ann}(M/N)$ , see [10, 11]. In the following results, we study the behavior of  $\Gamma(M, N)$  whenever  $N$  is a 2-absorbing submodule of  $M$ .

**Theorem 3.7.** *A submodule  $N$  of  $M$  is 2-absorbing if and only if at most two components of  $M$  are zero in  $N$ .*

*Proof.* Let  $M = \bigoplus_{i \in I} M_i$ , where  $M_i$ 's are non-isomorphic simple submodules of  $M$ . Suppose that  $N$  is a 2-absorbing submodule of  $M$  and  $N = \bigoplus_{i \in A} M_i$ , where  $A = I \setminus \{s, t, k\}$ . Since for all  $i \in I$ ,  $\text{ann}(M_i)$  is prime, there are  $a \in \text{ann}(M_s) \setminus (\text{ann}(M_t) \cup \text{ann}(M_k))$ ,  $b \in \text{ann}(M_t) \setminus (\text{ann}(M_s) \cup \text{ann}(M_k))$  and  $c \in \bigcap_{j \in I \setminus (A - \{s,t\})} \text{ann}(M_j) \setminus (\text{ann}(M_s) \cup \text{ann}(M_t))$ . Now,  $abc \in \text{ann}(M/N)$  but  $ab \notin \text{ann}(M/N)$ ,  $ac \notin \text{ann}(M/N)$  and  $bc \notin \text{ann}(M/N)$ . This contradict with

Theorem 2.3 in [10]. Thus  $|A| \geq |I| - 2$  and at most two components of  $M$  are zero in  $N$ .

Conversely, if one component of  $M$  is zero in  $N$ , then  $N$  is a prime submodule of  $M$ . Suppose that  $N = \bigoplus_{i \in A} M_i$ , where  $A = I \setminus \{i, j\}$ . Thus  $M_i, M_j \not\subseteq N$ . Suppose that  $a, b \in R$ ,  $(m_i)_{i \in I} = m \in M \setminus N$  and  $abm \in N$ . Then either  $m_i \neq 0$  or  $m_j \neq 0$ . If  $m_i \neq 0$  and  $m_j \neq 0$ , then  $ab \in \text{ann}(M_i) \cap \text{ann}(M_j) = \text{ann}(M/N)$ . If  $m_i \neq 0$  and  $m_j = 0$ , then  $ab \in \text{ann}(M_i)$  and so either  $a \in \text{ann}(M_i)$  or  $b \in \text{ann}(M_i)$ . Hence,  $am \in N$  or  $bm \in N$ . The case  $m_i = 0$  and  $m_j \neq 0$ , is similar to the previous case. Therefore,  $N$  is a 2-absorbing submodule of  $M$ .  $\square$

**Theorem 3.8.**  *$N$  is a 2-absorbing submodule of  $M$  if and only if  $Z^*(M, N) = \emptyset$  or  $\Gamma(M, N)$  is a complete bipartite graph.*

*Proof.* Let  $N$  be a 2-absorbing submodule of  $M$ . If  $N$  is prime, then  $Z^*(M, N) = \emptyset$ , by Theorem 3.6. Now, assume that  $N = \bigoplus_{i \in I \setminus \{j, k\}} M_i$  for some  $j, k \in I$  and  $(m_i)_{i \in I} = m \in M \setminus N$ . Thus  $I_m = \bigcap_{\{i \in I : m_i = 0\}} \text{ann}(M_i)$ . If  $m_j \neq 0$  and  $m_k \neq 0$ , then  $m \notin Z(M, N)$ . Let  $V_1 = \{(m_i)_{i \in I} \in M \setminus N : m_j = 0\}$  and  $V_2 = \{(m_i)_{i \in I} \in M \setminus N : m_k = 0\}$ . Thus  $m - m'$  is an edge of  $\Gamma(M, N)$  for every  $m \in V_1$  and  $m' \in V_2$ . Also, every vertices in  $V_1$  and  $V_2$  are not adjacent. Hence,  $\Gamma(M, N)$  is a complete bipartite graph.

Now, suppose that  $\Gamma(M, N)$  is a complete bipartite graph and  $N$  is not 2-absorbing. By Theorem 3.7, there are at least three components  $M_s, M_t, M_k$  such that  $M_s, M_t, M_k \not\subseteq N$ . For  $i = s, t, k$  let  $v_i = (m_i)_{i \in I}$ , where  $m_i \neq 0$  and  $m_j = 0$  for all  $j \neq i$ . Then  $v_s - v_t - v_k - v_s$  is a cycle in  $\Gamma(M, N)$ . Thus  $\text{gr}(\Gamma(M, N)) = 3$  and so  $\Gamma(M, N)$  is not bipartite graph, by Theorem 1 of Sec. 1.2 in [5]. Hence,  $N$  is a 2-absorbing submodule of  $M$ .  $\square$

**EXAMPLE 3.9.** Let  $M = \mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7$ . Then every nonzero submodule  $N$  of  $M$  is 2-absorbing. Thus either  $Z^*(M, N) = \emptyset$  or  $\Gamma(M, N)$  is a complete bipartite graph. In particular, if  $N = \mathbb{Z}_7$ , then  $\Gamma(M, N) = K^{7,28}$ .

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