On the 2-Adjointable Operators and Superstability of them between 2-Pre Hilbert $C^*$-Module Spaces

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Abstract. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a $C^*$-algebra. Next, we present the concept of 2-linear operators over a $C^*$-algebra. Our result improve the main result of the paper [Z. Lewandowska, On 2-normed sets, Glasnik Mat., 38(58) (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert $C^*$-modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

Keywords: $C^*$-Algebra, 2-Adjointable mapping, Superstability.

2000 Mathematics subject classification: 46L08, 46L09.

1. Introduction

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion $\phi$-perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert $C^*$-modules(see [3]).
In this paper, first, we introduce the definition 2-pre Hilbert $C^\ast$-module spaces and give several important properties. Next, we present the concept of 2-linear operators over a $C^\ast$-algebra which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjointable mappings between 2-pre Hilbert $C^\ast$-modules and prove an analogue of $\phi$-perturbation of adjoitable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

2. 2-Pre Hilbert Modules

Let $X$ be a left module over a $C^\ast$-algebra $A$. An action of $a \in A$ on $X$ is denoted by $a.x \in X$, $x \in X$.

**Definition 2.1.** A 2-pre Hilbert $A$-module is a left $A$-module $X$ equipped with $A$-valued function defined on $X \times X \times X$ satisfying the following conditions:

1. $(x, x|z)$ is a positive element in $A$ for any $x, z \in X$ and $(x, x|z) = 0$ if and only if $x$ and $z$ are linearly dependent;
2. $(x, x|z) = (z, z|x)$ for any $x, z \in X$;
3. $(y, x|z) = (x, y|z)^\ast$ for any $x, y, z \in X$;
4. $(ax + x', y|z) = \alpha(x, y|z) + (x', y|z)$ for any $\alpha \in C$ and $x, x', y, z \in X$;
5. $(ax, y|z) = a(x, y|z)$ for any $a \in A$.

The map $(., .|.)$ is called $A$-valued 2-inner product and $(X, (., .|.) )$ is called 2-pre Hilbert $C^\ast$-module space.

**Example 2.2.** Every 2-inner product space is a 2-pre Hilbert $C$-module.

**Example 2.3.** Let $A$ be a $C^\ast$-algebra and $J \subseteq A$ a left ideal. Then $J$ can be equipped with the structure of 2-pre Hilbert $A$-module with $A$-valued inner product $(x, y|z) := xy^\ast zz^\ast - xz^\ast yz^\ast$ for any $x, y, z \in A$.

**Definition 2.4.** Let $X$ be a 2-pre Hilbert $A$-module. We can define a function \[ |||., .|., .|||_X \] on $X \times X$ by \[ |||x|z||_X = ||(x, x|z)||^{1/2} \] for all $x, z \in X$.

**Lemma 2.5.** \[ |||., .|., .|||_X \] satisfies the following conditions:

1. $||ax|z||_X \leq ||a|| \ ||x|z||_X$ for any $x, z \in X$ and $a \in A$;
2. $(x, y|z) (y, x|z) \leq ||y|z||_X^2 (x, x|z)$ for any $x, y, z \in X$;
3. \[ ||(x, y|z)||^2 \leq ||(x, x|z)|| ||(y, y|z)|| \]

**Proof.** $N1$ is obvious; $N3$ follows from $N2$, so let us prove $N2$. Let $\phi$ be a positive linear functional on $A$. Then $\phi((., .|.) )$ is usual 2-inner product on $X$. Applying the Schwartz inequality for 2-inner product (see [2],
On the 2-adjointable operators and superstability of · · ·

page 3) we obtain for all \(x, y, z \in X\),
\[
\phi((x, y|z) (y, x|z)) = \phi((x, y|z)y, x|z))
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} \phi(((x, y|z)y, (x, y|z)y|z))^{\frac{1}{2}}
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} \phi((x, y|z) (y, y|z) (x, y|z) x|z)^{\frac{1}{2}}
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} \|y, y|z\|^{\frac{1}{2}} \phi((x, y|z) (y, x|z))^{\frac{1}{2}}.
\]

Thus, for any positive linear functional \(\phi\), we have
\[
\phi((x, y|z) (y, z|z)) \leq \|y, y|z\|^{\frac{1}{2}} \phi((x, x|z))
\]
hence
\[
(x, y|z) (y, z|z) \leq \|y, y|z\|^{\frac{1}{2}} (x, x|z).
\]

\[\square\]

**Theorem 2.6.** The function \(\|.|\|_X\) is a 2-norm on \(X\).

**Proof.** Now, we verify that \(\|.|\|_X\) satisfies the following properties of 2-norms:

1) \(I_3\) and \(I_4\) show that \(\|\alpha x, y\|_X = \|\alpha x, \alpha y\|_X = \|\alpha\| \|x, y\|_X\) for all \(x, y \in X\) and \(\alpha \in \mathbb{C}\).

2) \(I_1\) follows that \(\|x, y\|_X = 0\) if and only if \(x\) and \(y\) are linearly dependent for all \(x, y \in X\).

3) it follows from \(I_2\) that \(\|x, y\|_X = \|(x, x|y)\|^{\frac{1}{2}} = \|y, x|_X\) for all \(x, y \in X\).

4) By proposition 2.5 \((N3)\), we have
\[
\|x + x', y\|_X^2 = \|(x + x', x + x'|y)\| = \|(x, x|y) + (x', x|y) + (x', x'|y))\|
\]
\[
\leq \|(x, x|y)\| + 2\|(x, x'|y)\| + \|(x', x'|y)\|
\]
\[
\leq (\|(x, x|y)\|^{\frac{1}{2}} + \|(x', x'|y)\|^{\frac{1}{2}})^2 = (\|x, y|_X + \|x', y|_X)^2
\]
for all \(x, x', y \in X\). This show that \((X, \|.|\|_X)\) is a 2-normed space. \[\square\]

3. 2-ADJOINTABLE MAPPINGS

In continue, we let \(A\) be a \(C^*\)-algebra. Now, we start with following definition.

**Definition 3.1.** Let \(X\) and \(Y\) be two 2-pre Hilbert \(A\)-modules. An operator \(f : X \times X \to Y\) is said to be \(A\)-2 linear if it satisfies the following conditions:

1) \(f(x + y, z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w)\) for all \(x, y, z, w \in X\);

2) \(f(\alpha x, \beta y) = \alpha \beta f(x, y)\) for all \(\alpha, \beta \in \mathbb{C}\) and \(x, y \in X\);

3) \(f(ax, by) = a, b^* f(x, y)\) for all \(x, y \in X\) and \(a, b \in A\).

**Example 3.2.** Let \(X\) be a 2-pre Hilbert \(A\)-module and \(z \in X\). Define \(f : X \times X \to A\) by \(f(x, y) = (x, y|z)\). Then \(f\) is a \(A\)-2 linear operator.
**Definition 3.3.** Let $X$ and $Y$ be two 2-pre Hilbert $A$-modules. A mapping $f : X \times X \rightarrow Y$ is called 2-adjointable if there exists a mapping $g : Y \times Y \rightarrow X$ such that

$$(f(x, y), s | t) = (x, y | g(s, t))$$

(3.1)

for all $x, y \in X$ and $s, t \in Y$. The mapping $g$ is denoted by $f^*$ and is called the 2-adjoint of $f$.

**Lemma 3.4.** Let $X$ be a 2-pre Hilbert $A$-module and $\dim(X) > 1$. If $(x, y | z) = 0$ for all $y, z \in X$, then $x = 0$.

**Proof.** Suppose $x \neq 0$. Let $x$ and $y$ be linearly independent. Then by hypothesis $(x, x, | y) = 0$ and this is contradiction. $\square$

**Lemma 3.5.** Every 2-adjointable mapping is $A$-2 linear.

**Proof.** Let $f : X \times X \rightarrow Y$ be a 2-adjointable mapping. Then there exists a mapping $g : Y \times Y \rightarrow X$ such that (3.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $a, b \in A$, we have

$$(f(\alpha ax + \beta bz + w), s | t) = (\alpha ax + \beta bz + w | g(s, t))$$

$$= \alpha \overline{a}ab^* (x, z | g(s, t)) + \alpha a (x, w | g(s, t)) + \beta \overline{b}b^* (y, z | g(s, t)) + (y, w | g(s, t))$$

$$= (\alpha \overline{a}ab^* f(x, z) + \alpha a f(x, w) + \beta \overline{b}b^* f(y, z) + f(y, w), s | t).$$

It follows from lemma 3.4 that $f$ is $A$-2 linear. $\square$

4. Superstability of 2-adjointable mappings

In this section, $X$ and $Y$ denote 2-pre Hilbert $A$-modules and $\dim(X) > 1$, $\dim(Y) > 1$ and $\phi : X^2 \times Y^2 \rightarrow [0, \infty)$ is a function. We start our work with following definition.

**Definition 4.1.** A (not necessarily $A$-2 linear) mapping $f : X \times X \rightarrow Y$ is called $\phi$-perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily $A$-2 linear) $g : Y \times Y \rightarrow X$ such that

$$||(f(x, y), s | t) - (x, y | g(s, t))|| \leq \phi(x, y, s, t)$$

(4.1)

for all $x, y \in X$ and $s, t \in Y$.

**Theorem 4.2.** Let $f : X \times X \rightarrow Y$ be a $\phi$-perturbation of a 2-adjointable mapping with corresponding mapping $g : Y \times Y \rightarrow X$. Suppose for some sequence $c_n$ of non-zero complex numbers the following conditions hold:

$$\lim_{n \rightarrow \infty} |c_n|^{-1} \phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y)$$

(4.2)
\[
\lim_{n \to \infty} |c_n|^{-1} \phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y)
\]

(4.3)

Then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

**Proof.** Let \( \lambda \in \mathbb{C} \) be an arbitrary number. Putting \( \lambda x \) instead \( x \) in (4.1), we get

\[
\| (f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t) \| \leq \phi(\lambda x, y, s, t)
\]

multiplication of (4.1) by \( |\lambda| \), we have

\[
\| (\|f(\lambda x, y), s \mid t) - \lambda (x, y \mid g(s, t)) \| \leq |\lambda| \phi(x, y, s, t)
\]

Thus,

\[
\| (f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t) \| \leq \phi(\lambda x, y, s, t) + |\lambda| \phi(x, y, s, t)
\]

(4.4)

Replacing \( c_n s \) by \( s \) in (4.4), we get

\[
\| (f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t) \| \leq |c_n|^{-1} \phi(\lambda x, y, c_n s, t) + |\lambda| |c_n|^{-1} \phi(x, y, c_n s, t)
\]

hence, as \( n \to \infty \), applying (4.3) we obtain

\[
(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t) = 0 \quad (\lambda \in \mathbb{C}, x, y \in X, s, t \in Y).
\]

It follows from proposition 3.4 that

\[
f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x, y \in X)
\]

(4.5)

Now, we take \( c_n x \) instead \( x \) in (4.1) to get

\[
\| (f(c_n x, y), s \mid t) - (c_n x, y \mid g(t, s)) \| \leq \phi(c_n x, y, s, t).
\]

It follows from (4.5) that

\[
\| (f(x, y), s \mid t) - (x, y \mid g(s, t)) \| \leq |c_n|^{-1} \phi(c_n x, y, s, t)
\]

hence, as \( n \to \infty \), applying (4.2) we get

\[
(f(x, y), s \mid t) = (x, y \mid g(s, t)) \quad (x, y \in X, s, t \in Y).
\]

Therefore \( f \) is 2-adjointable and by Lemma 3.5, \( f \) is \( A \)-2 linear. \( \square \)

In the following, we let \( c_n = a^n \) that \( a > 1 \). We get the following results.

**Corollary 4.3.** If \( f : X \times X \to Y \) is a \( \phi \)-perturbation of a 2-adjointable mapping, where

\[
\phi(x, y, s, t) = \epsilon \| x \|_{X}^{p} \| y \|_{X}^{r} \| s \| \| t \|_{Y}^{q} \quad (\epsilon \geq 0, \ 0 < p < 1, \ 0 < q < 1),
\]

then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2-linear.

**Corollary 4.4.** If \( f : X \times X \to Y \) is a \( \phi \)-perturbation of a 2-adjointable mapping, where

\[
\phi(x, y, s, t) = \epsilon_1 \| x \|_{X}^{p} \| y \|_{X}^{r} + \epsilon_2 \| s \| \| t \|_{Y}^{q} \quad (\epsilon_1 \geq 0, \ \epsilon_2 \geq 0, \ 0 < p < 1, \ 0 < q < 1).
\]

Then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

**Acknowledgments**

We would like to thank the referee for his/her careful reading of the paper.
REFERENCES