# On the 2-Adjointable Operators and Superstability of them between 2-Pre Hilbert $C^{*}$-Module Spaces 

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#### Abstract

In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a $C^{*}$-algebra. Next, we present the concept of 2 -linear operators over a $C^{*}$-algebra. Our result improve the main result of the paper [Z. Lewandowska, On 2-normed sets, Glasnik Mat., 38(58) (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert $\mathrm{C}^{*}$-modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.


Keywords: $C^{*}$-Algebra, 2-Adjointable mapping, Superstability.

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## 1. Introduction

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2 -inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion $\phi$-perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert $C^{*}$-modules(see [3]).

[^0]In this paper, first, we introduce the definition 2-pre Hibert $C^{*}$-module spaces and give several important properties. Next, we present the concept of 2-linear operators over a $C^{*}$-algebra which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjoinable mappings between 2-pre Hilbert $C^{*}$-modules and prove an analogue of $\phi$-perturbation of adjoitable mappings in $\operatorname{paper}([3])$.

We refer the interested reader to monographs $[6,7,8,9]$ and references therein for more information.

## 2. 2-Pre Hilbert Modules

Let $X$ be a left module over a $C^{*}$-algebra $A$. An action of $a \in A$ on $X$ is denoted by $a . x \in X, x \in X$.

Definition 2.1. A 2-pre Hilbert $A$-module is a left $A$-module $X$ equipped with $A$-valued function defined on $X \times X \times X$ satisfing the following conditions:
$\left.I_{1}\right)(x, x \mid z)$ is a positive element in $A$ for any $x, z \in X$ and $(x, x \mid z)=0$ if and only if $x$ and $z$ are linearly dependent;
$\left.I_{2}\right)(x, x \mid z)=(z, z \mid x)$ for any $x, z \in X$;
$\left.I_{3}\right)(y, x \mid z)=(x, y \mid z)^{*}$ for any $x, y, z \in X$;
$\left.I_{4}\right)\left(\alpha x+x^{\prime}, y \mid z\right)=\alpha(x, y \mid z)+\left(x^{\prime}, y \mid z\right)$ for any $\alpha \in \mathbb{C}$ and $x, x^{\prime}, y, z \in X$;
$\left.I_{5}\right)(a x, y \mid z)=a(x, y \mid z)$ for any $x, y, z \in X$ and any $a \in A$.
The map $(., . \mid$.$) is called A$-valued 2-inner product and $(X,(., . \mid)$.$) is called 2-pre$ Hilbert $C^{*}$-module space.

Example 2.2. Every 2-inner product space is a 2 -pre Hilbert $\mathbb{C}$-module.
Example 2.3. Let $A$ be a $C^{*}$-algebra and $J \subseteq A$ be a left ideal. Then $J$ can be equipped with the structure of 2 -pre Hilbert $A$-module with $A$-valued inner product $(x, y \mid z):=x y^{*} z z^{*}-x z^{*} z y^{*}$ for any $x, y, z \in A$.

Definition 2.4. Let $X$ be a 2 -pre Hilbert $A$-module. we can define a function $\|\cdot|\cdot|\|_{X}$ on $X \times X$ by $\|x \mid z\|_{X}=\|(x, x \mid z)\|^{\frac{1}{2}}$ for all $x, z \in X$.

Lemma 2.5. ||.|.|| $\|_{X}$ satisfies the following conditions:
N1) $\left\|a x\left|z\left\|_{X} \leq\right\| a\left\|\|x \mid z\|_{X}\right.\right.\right.$ for any $x, z \in X$ and $a \in A$;
N2) $(x, y \mid z)(y, x \mid z) \leq\|y \mid z\|_{X}^{2}(x, x \mid z)$ for any $x, y, z \in X$;
N3) $\|(x, y \mid z)\|^{2} \leq\|(x, x \mid z)\|\|(y, y \mid z)\|$
Proof. $N 1$ is obvious; $N 3$ follows from $N 2$, so let us prove $N 2$.
Let $\phi$ be a positive linear functional on $A$. Then $\phi((., \mid)$.$) is usual 2-inner$ product on $X$. Applying the Schwartz inequality for 2-inner product (see [2],
page 3) we obtain for all $x, y, z \in X$,

$$
\begin{aligned}
\phi((x, y \mid z)(y, x \mid z)) & =\phi((x, y \mid z) y, x \mid z)) \\
& \leq \phi((x, x \mid z))^{\frac{1}{2}} \phi(((x, y \mid z) y,(x, y \mid z) y \mid z))^{\frac{1}{2}} \\
& \leq \phi((x, x \mid z))^{\frac{1}{2}} \phi\left((x, y \mid z)(y, y \mid z)(x, y \mid z)^{*}\right)^{\frac{1}{2}} \\
& \leq \phi((x, x \mid z))^{\frac{1}{2}}\|(y, y \mid z)\|^{\frac{1}{2}} \phi((x, y \mid z)(y, x \mid z))^{\frac{1}{2}}
\end{aligned}
$$

Thus, for any positive linear functional $\phi$, we have

$$
\phi((x, y \mid z)(y, x \mid z)) \leq\|y \mid z\|_{X}^{2} \phi((x, x \mid z))
$$

hence

$$
(x, y \mid z)(y, x \mid z) \leq\|y \mid z\|_{X}^{2}(x, x \mid z)
$$

Theorem 2.6. The function $\|. \mid \cdot\|_{X}$ is a 2 -norm on $X$.
Proof. Now, we verify that $\|.|\cdot|\|_{X}$ satisfies the following properties of 2-norms:

1) $I_{3}$ and $I_{4}$ show that $\left\|\alpha x\left|y\left\|_{X}=\right\|(\alpha x, \alpha x \mid y)\left\|^{\frac{1}{2}}=|\alpha|\right\| x\right| y\right\|_{X}$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$.
2) $I_{1}$ follows that $\|x \mid y\|_{X}=0$ if and only if $x$ and $y$ are linearly dependent for all $x, y \in X$.
3) it follows from $I_{2}$ that $\left\|x\left|y\left\|_{X}=\right\|(x, x \mid y)\left\|^{\frac{1}{2}}=\right\| y\right| x\right\|_{X}$ for all $x, y \in X$.
4) By proposition $2.5(N 3)$, we have

$$
\begin{aligned}
\left\|x+x^{\prime} \mid y\right\|_{X}^{2} & =\left\|\left(x+x^{\prime}, x+x^{\prime} \mid y\right)\right\|=\left\|(x, x \mid y)+\left(x^{\prime}, x \mid y+\left(x, x^{\prime} \mid y\right)+\left(x^{\prime}, x^{\prime} \mid y\right)\right)\right\| \\
& \leq\|(x, x \mid y)\|+2\left\|\left(x, x^{\prime} \mid y\right)\right\|+\left\|\left(x^{\prime}, x^{\prime} \mid y\right)\right\| \\
& \leq\left(\|(x, x \mid y)\|^{\frac{1}{2}}+\left\|\left(x^{\prime}, x^{\prime} \mid y\right)\right\|^{\frac{1}{2}}\right)^{2}=\left(\left\|x\left|y\left\|_{X}+\right\| x^{\prime}\right| y\right\|_{X}\right)^{2}
\end{aligned}
$$

for all $x, x^{\prime}, y \in X$. This show that $\left(X,\|\cdot|\cdot|\|_{X}\right)$ is a 2 -normed space.

## 3. 2-ADJOINTABLE MAPPINGS

In continue, we let $A$ be a $C^{*}$-algebra. Now, we start with following definition. Definition 3.1. Let $X$ and $Y$ be two 2-pre Hilbert $A$-modules. An operator $f: X \times X \rightarrow Y$ is said to be $A-2$ linear if it satisfies the following conditions:

1) $f(x+y, z+w)=f(x, z)+f(x, w)+f(y, z)+f(y, w)$ for all $x, y, z, w \in X$;
2) $f(\alpha x, \beta y)=\alpha \bar{\beta} f(x, y)$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$;
3) $f(a x, b y)=a . b^{*} . f(x, y)$ for all $x, y \in X$ and $a, b \in A$.

Example 3.2. Let $X$ be a 2-pre Hilbert $A$-module and $z \in X$. Define $f$ : $X \times X \rightarrow A$ by $f(x, y)=(x, y \mid z)$. Then $f$ is a $A$ - 2 linear operator.

Definition 3.3. Let $X$ and $Y$ be two 2-pre Hilbert $A$-modules. A mapping $f: X \times X \rightarrow Y$ is called 2-adjointable if there exists a mapping $g: Y \times Y \rightarrow X$ such that

$$
\begin{equation*}
(f(x, y), s \mid t)=(x, y \mid g(s, t)) \tag{3.1}
\end{equation*}
$$

for all $x, y \in X$ and $s, t \in Y$. The mapping $g$ is denoted by $f^{*}$ and is called the 2-adjointable of $f$.

Lemma 3.4. Let $X$ be a 2-pre Hilbert A-module and $\operatorname{dim}(X)>1$. If $(x, y \mid z)=$ 0 for all $y, z \in X$, then $x=0$.

Proof. Suppose $x \neq 0$. Let $x$ and $y$ be linearly independent. Then by hypothesis $(x, x, \mid y)=0$ and this is contradiction.

Lemma 3.5. Every 2-adjonable mapping is $A$ - 2 linear.
Proof. Let $f: X \times X \rightarrow Y$ be a 2-adjoinable mapping. Then there exists a mapping $g: Y \times Y \rightarrow X$ such that (3.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $\alpha, \beta \in \mathbb{C}$, every $a, b \in A$, we have

$$
\begin{aligned}
& (f(\alpha a x+y, \beta b z+w), s \mid t)=(\alpha a x+y, \beta b z+w \mid g(s, t)) \\
& \quad=\alpha \bar{\beta} a b^{*}(x, z \mid g(s, t))+\alpha a(x, w \mid g(s, t))+\bar{\beta} b^{*}(y, z \mid g(s, t))+(y, w \mid g(s, t)) \\
& \quad=\alpha \bar{\beta} a b^{*}(f(x, z), s \mid t)+\alpha a(f(x, w), s \mid t)+\bar{\beta} b^{*}(f(y, z), s \mid t)+(f(y, w), s \mid t) \\
& \quad=\left(\alpha \bar{\beta} a b^{*} f(x, z)+\alpha a f(x, w)+\bar{\beta} b^{*} f(y, z)+f(y, w), s \mid t\right) .
\end{aligned}
$$

It follows from lemma 3.4 that $f$ is $A-2$ linear.

## 4. SUPERSTABILITY OF 2-ADJOINABLE MAPPINGS

In this section, $X$ and $Y$ denote 2-pre Hilbert $A$-modules and $\operatorname{dim}(X)>1$, $\operatorname{dim}(Y)>1$ and $\phi: X^{2} \times Y^{2} \rightarrow[0, \infty)$ is a function. We start our work with following definition.
Definition 4.1. $A$ (not necessarily $A$-2 linear) mapping $f: X \times X \rightarrow Y$ is called
$\phi$-perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily $A-2$ - linear) $g: Y \times Y \rightarrow X$ such that

$$
\begin{equation*}
\|(f(x, y), s \mid t)-(x, y \mid g(s, t))\| \leq \phi(x, y, s, t) \tag{4.1}
\end{equation*}
$$

for all $x, y \in X$ and $s, t \in Y$.
Theorem 4.2. Let $f: X \times X \rightarrow Y$ be a $\phi$-perturbation of a 2-adjointable mapping with corresponding mapping $g: Y \times Y \rightarrow X$. Suppose for some sequence $c_{n}$ of non-zero complex numbers the following conditions hold:

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|c_{n}\right|^{-1} \phi\left(c_{n} x, y, s, t\right)=0 \quad(x, y \in X, s, t \in Y) \tag{4.2}
\end{equation*}
$$

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left|c_{n}\right|^{-1} \phi\left(x, y, c_{n} s, t\right)=0 \quad(x, y \in X, s, t \in Y) \tag{4.3}
\end{equation*}
$$

Then $f$ is 2-adjointable and hence $f$ is $A$-2 linear.
Proof. Let $\lambda \in \mathbb{C}$ be an arbitrary number. Putting $\lambda x$ instead $x$ in (4.1), we get

$$
\|(f(\lambda x, y), s \mid t)-(\lambda x, y \mid g(s, t))\| \leq \phi(\lambda x, y, s, t)
$$

multiplication of (4.1) by $|\lambda|$, we have

$$
\|(\lambda f(x, y), s \mid t)-\lambda(x, y \mid g(s, t))\| \leq|\lambda| \phi(x, y, s, t)
$$

Thus,

$$
\begin{equation*}
\|(f(\lambda x, y), s \mid t)-(\lambda f(x, y), s \mid t)\| \leq \phi(\lambda x, y, s, t)+|\lambda| \phi(x, y, s, t) \tag{4.4}
\end{equation*}
$$

Replacing $c_{n} s$ by $s$ in (4.4), we get

$$
\| f(\lambda x, y), s \mid t)-(\lambda f(x, y), s \mid t) \| \leq\left|c_{n}\right|^{-1} \phi\left(\lambda x, y, c_{n} s, t\right)+|\lambda|\left|c_{n}\right|^{-1} \phi\left(x, y, c_{n} s, t\right)
$$

hence, as $n \rightarrow \infty$, applying (4.3) we obtain

$$
(f(\lambda x, y), s \mid t)-(\lambda f(x, y), s \mid t)=0 \quad(\lambda \in \mathbb{C}, x, y \in X, s, t \in Y)
$$

It follows from proposition 3.4 that

$$
\begin{equation*}
f(\lambda x, y)=\lambda f(x, y) \quad(\lambda \in \mathbb{C}, x, y \in X) \tag{4.5}
\end{equation*}
$$

Now, we take $c_{n} x$ instead $x$ in (4.1) to get

$$
\left\|\left(f\left(c_{n} x, y\right), s \mid t\right)-\left(c_{n} x, y \mid g(t, s)\right)\right\| \leq \phi\left(c_{n} x, y, s, t\right)
$$

It follows from (4.5) that

$$
\|(f(x, y), s \mid t)-(x, y \mid g(s, t))\| \leq\left|c_{n}\right|^{-1} \phi\left(c_{n} x, y, s, t\right)
$$

hence, as $n \rightarrow \infty$, applying (4.2) we get

$$
(f(x, y), s \mid t)=(x, y \mid g(s, t) \quad(x, y \in X, s, t \in Y)
$$

Therefore $f$ is 2 -adjointable and by Lemma $3.5, f$ is $A-2$ linear.
In the following, we let $c_{n}=a^{n}$ that $a>1$. we get the following results.
Corollary 4.3. If $f: X \times X \rightarrow Y$ is a $\phi$-perturbation of a 2-adjointable mapping, where
$\phi(x, y, s, t)=\epsilon\left\|x\left|y\left\|_{X}^{p}\right\| s\right| t\right\|_{Y}^{q}(\epsilon \geq 0,0<p<1,0<q<1)$, then $f$ is 2-adjointable and hence $f$ is A-2-linear.

Corollary 4.4. If $f: X \times X \rightarrow Y$ is a $\phi$-perturbation of a 2-adjointable mapping, where
$\phi(x, y, s, t)=\epsilon_{1}\left\|x\left|y\left\|_{X}^{p}+\epsilon_{2}\right\| s\right| t\right\|_{Y}^{q}\left(\epsilon_{1} \geq 0, \epsilon_{2} \geq 0,0<p<1,0<q<1\right)$. Then $f$ is 2-adjointable and hence $f$ is $A$-2 linear.

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## References

1. Y.J. Cho, P.C.S. Lin, S.S. Kim, A. Misiak, Theory of 2-Inner Product Spaces, Nova Science Publishers, Inc., New York, 2001.
2. S.S. Dragomir, Y.J. Cho, S.S. Kim, A. Sofo, Some Boas-Bellman Type Inequalitys in 2-Inner Product Space, JIPAM, 6(2),(2005), article 55.
3. M. Frank, P. Găvruta, M.S. Moslehian, Superstability of Adjointable Mappings on Hilbert $C^{*}$ - Modules, Appl. Anal. Discrete Math., No3, (2009), 39-45.
4. Z. Lewandowska, On 2-Normed Sets, Glasnik Mat., 38(58), (2003), 99-110.
5. Z. Lewandowska, Bounded 2-Linear Operators on 2-Normed Sets, Glas. Mat. Ser. III, 39(59) (2), (2004), 301-312.
6. A, Ashyani, H, Mohammadinejad, O, RabieiMotlagh, Stability Analysis of Mathematical Model of Virus Therapy for Cancer, Iranian Journal of Mathematical Sciences and Informatics, 11( 2), (2016), 97-110.
7. H, Sadeghi, Generalized Approximate Amenability of Direct Sum of Banach Algebras, Iranian Journal of Mathematical Sciences and Informatics, 13(1), (2018), 75-87.
8. M.E. Gordji, M. Ramezani, Approximate inner products on Hilbert C*-modules; A Fixed Point Approach, Operators and Matrices, 6(4), (2012), 757-766.
9. M.E. Gordji, M. Ramezani, Y.J. Cho, H. Baghani, Approximate Lie brackets: A Fixed Point Approach, Journal of Inequalities and Applications, (2012), doi:10.1186/1029-242X-2012-125.

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