On the 2-Adjointable Operators and Superstability of them between 2-Pre Hilbert $C^*$-Module Spaces

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Abstract. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a $C^*$-algebra. Next, we present the concept of 2-linear operators over a $C^*$-algebra. Our result improve the main result of the paper [Z. Lewandowska, On 2-normed sets, Glasnik Mat., 38(58) (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert $C^*$-modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

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1. Introduction

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and et. defined the notion $\phi$-perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert $C^*$-modules(see [3]).

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In this paper, first, we introduce the definition 2-pre Hilbert $C^*$-module spaces and give several important properties. Next, we present the concept of 2-linear operators over a $C^*$-algebra which coincides with Lewandowska’s definition (see [4, 5]). Also, we define 2-adjointable mappings between 2-pre Hilbert $C^*$-modules and prove an analogue of $\phi$-perturbation of adjoitable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

2. 2-Pre Hilbert Modules

Let $X$ be a left module over a $C^*$-algebra $A$. An action of $a \in A$ on $X$ is denoted by $a.x \in X$, $x \in X$.

**Definition 2.1.** A 2-pre Hilbert $A$-module is a left $A$-module $X$ equipped with $A$-valued function defined on $X \times X \times X$ satisfying the following conditions:

1. $I_1$ $(x, z|x)_IL_2$) $(x, z|x) = (z, z|x)$ for any $x, z \in X$;

2. $I_2$ $(y, x|z)$ is a positive element in $A$ for any $x, z \in X$;

3. $I_3$ $(x, y|z) = \alpha (x, y|z) + \alpha^* (x, y|z)$ for any $\alpha \in \mathbb{C}$ and $x, y, z \in X$;

4. $I_4$ $(ax + x', y|z)$ = $a(x, y|z)$ for any $a \in A$.

The map $(., .|.)$ is called $A$-valued 2-inner product and $(X, (., .|.)$) is called 2-pre Hilbert $C^*$-module space.

**Example 2.2.** Every 2-inner product space is a 2-pre Hilbert $C^*$-module.

**Example 2.3.** Let $A$ be a $C^*$-algebra and $J \subseteq A$ be a left ideal. Then $J$ can be equipped with the structure of 2-pre Hilbert $A$-module with $A$-valued inner product $(x, y|z) := x^* y^* z^* - x^* z^* y^*$ for any $x, y, z \in A$.

**Definition 2.4.** Let $X$ be a 2-pre Hilbert $A$-module. we can define a function $|||.|.|.|_X$ on $X \times X$ by $|||x|z||_X = ||(x, x|z)||^2$ for all $x, z \in X$.

**Lemma 2.5.** $|||.|.|.|_X$ satisfies the following conditions:

N1) $||ax|z||_X \leq ||a|| \ ||x|z||_X$ for any $x, z \in X$ and $a \in A$;

N2) $(x, y|z) \leq \|y|z\|_X^2 (x, x|z)$ for any $x, y, z \in X$;

N3) $(||x, y|z||^2 \leq ||(x, x|z)|| \ ||(y, y|z)||$

**Proof.** N1 is obvious; N3 follows from N2. Let $\phi$ be a positive linear functional on $A$. Then $\phi((., .|.)$) is usual 2-inner product on $X$. Applying the Schwartz inequality for 2-inner product (see [2],
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page 3) we obtain for all \( x, y, z \in X \),
\[
\phi((x, y|z) (y, x|z)) = \phi((x, y|z)y, x|z))
\]
\[
\leq \phi((x, x|z))^{\frac{1}{2}} \phi(((x, y|z)y, (x, y|z)y)z) z) z)
\]
\[
\leq \phi((x, x|z))^{\frac{1}{2}} \phi((x, y|z) (y, y|z) (x, y|z) z) z)
\]
\[
\leq \phi((x, x|z))^{\frac{1}{2}} ||(y, y|z)||^{\frac{1}{2}} \phi((x, y|z) (y, x|z) z).
\]
Thus, for any positive linear functional \( \phi \), we have
\[
\phi((x, y|z) (y, x|z)) \leq ||y|z||_{X}^\frac{1}{2} \phi((x, x|z))
\]

hence
\[
(x, y|z) (y, x|z) \leq ||y|z||_{X}^\frac{1}{2} (x, x|z).
\]

\( \square \)

**Theorem 2.6.** The function \( ||.|.|_{X} \) is a 2-norm on \( X \).

**Proof.** Now, we verify that \( ||.|.|_{X} \) satisfies the following properties of 2-norms:
1) \( I_{2} \) and \( I_{4} \) show that \( ||\alpha x||y||_{X} = ||(\alpha x, \alpha y)||_{X}^\frac{1}{2} = ||\alpha| ||y||_{X} \) for all \( x, y \in X \) and \( \alpha \in C \).
2) \( I_{1} \) follows that \( ||x||y||_{X} = 0 \) if and only if \( x \) and \( y \) are linearly dependent for all \( x, y \in X \).
3) it follows from \( I_{2} \) that \( ||x||y||_{X} = ||(x, x)|y||_{X}^\frac{1}{2} = ||y||x||_{X} \) for all \( x, y \in X \).
4) By proposition 2.5 \((N3)\), we have
\[
||x + x'||y||_{X}^\frac{1}{2} = ||(x + x', x + x'|y)|| = ||(x, x)|y| + (x', x)|y| + (x, x')|y)||
\]
\[
\leq ||(x, x)|y|| + 2||x, x'|y|y|| + ||(x', x'|y)||
\]
\[
\leq (||x, x|y||^{\frac{1}{2}} + ||(x', x'|y)||^{\frac{1}{2}})^{2} = (||x|y||x + ||x'|y||x)^{2}
\]
for all \( x, x', y \in X \). This show that \( (X, ||.|.|_{X}) \) is a 2-normed space. \( \square \)

3. 2-ADJOINTABLE MAPPINGS

In continue, we let \( A \) be a \( C^{*}\)-algebra. Now, we start with following definition.

**Definition 3.1.** Let \( X \) and \( Y \) be two 2-pre Hilbert \( A \)-modules. An operator \( f : X \times X \rightarrow Y \) is said to be \( A\)-2 linear if it satisfies the following conditions:
1) \( f(x + y, z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w) \) for all \( x, y, z, w \in X \);
2) \( f(\alpha x, \beta y) = \alpha \beta f(x, y) \) for all \( \alpha, \beta \in C \) and \( x, y \in X \);
3) \( f(\alpha x, by) = a \cdot b^{*}f(x, y) \) for all \( x, y \in X \) and \( a, b \in A \).

**Example 3.2.** Let \( X \) be a 2-pre Hilbert \( A \)-module and \( z \in X \). Define \( f : X \times X \rightarrow A \) by \( f(x, y) = (x, y|z) \). Then \( f \) is a \( A\)-2 linear operator.
Definition 3.3. Let $X$ and $Y$ be two 2-pre Hilbert $A$-modules. A mapping $f : X \times X \to Y$ is called 2-adjointable if there exists a mapping $g : Y \times Y \to X$ such that
\[
(f(x, y), s \mid t) = (x, y \mid g(s, t))
\]
for all $x, y \in X$ and $s, t \in Y$. The mapping $g$ is denoted by $f^*$ and is called the 2-adjointable of $f$.

Lemma 3.4. Let $X$ be a 2-pre Hilbert $A$-module and $\dim(X) > 1$. If $(x, y \mid z) = 0$ for all $y, z \in X$, then $x = 0$.

Proof. Suppose $x \neq 0$. Let $x$ and $y$ be linearly independent. Then by hypothesis $(x, x, |y) = 0$ and this is contradiction. \[\square\]

Lemma 3.5. Every 2-adjointable mapping is $A$-2 linear.

Proof. Let $f : X \times X \to Y$ be a 2-adjointable mapping. Then there exists a mapping $g : Y \times Y \to X$ such that (3.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $a, b \in A$, we have
\[
(f(ax + y, bz + w), s \mid t) = (ax + y, bz + w \mid g(s, t))
\]
\[\begin{align*}
&= a\alpha b^* (x, z \mid g(s, t)) + \alpha a(x, w \mid g(s, t)) + \beta b^* (y, z \mid g(s, t)) + (y, w \mid g(s, t)) \\
&= \alpha a\beta b^* (f(x, z), s \mid t) + \alpha a(f(x, w), s \mid t) + \beta b^* (f(y, z), s \mid t) + (f(y, w), s \mid t) \\
&= (\alpha a\beta b^* f(x, z) + \alpha a f(x, w) + \beta b^* f(y, z) + f(y, w), s \mid t).
\end{align*}
\]
It follows from lemma 3.4 that $f$ is $A$-2 linear. \[\square\]

4. Superstability of 2-adjointable mappings

In this section, $X$ and $Y$ denote 2-pre Hilbert $A$-modules and $\dim(X) > 1$, $\dim(Y) > 1$ and $\phi : X^2 \times Y^2 \to [0, \infty)$ is a function. We start our work with following definition.

Definition 4.1. A (not necessarily $A$-2 linear) mapping $f : X \times X \to Y$ is called $\phi$-perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily $A - 2$ linear) $g : Y \times Y \to X$ such that
\[
\|(f(x, y), s \mid t) - (x, y \mid g(s, t))\| \leq \phi(x, y, s, t)
\]
for all $x, y \in X$ and $s, t \in Y$.

Theorem 4.2. Let $f : X \times X \to Y$ be a $\phi$-perturbation of a 2-adjointable mapping with corresponding mapping $g : Y \times Y \to X$. Suppose for some sequence $c_n$ of non-zero complex numbers the following conditions hold:
\[
\lim_{n \to \infty} |c_n|^{-1} \phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y)
\]
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\[
\lim_{n \to \infty} |c_n|^{-1} \phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y) \quad (4.3)
\]

Then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

**Proof.** Let \( \lambda \in \mathbb{C} \) be an arbitrary number. Putting \( \lambda x \) instead \( x \) in (4.1), we get

\[
\|(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t)\| \leq \phi(\lambda x, y, s, t)
\]

multiplication of (4.1) by \( |\lambda| \), we have

\[
\|(\lambda f(x, y), s \mid t) - \lambda(x, y \mid g(s, t))\| \leq |\lambda| \phi(x,y,s,t)
\]

Thus,

\[
\|(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t)\| \leq \phi(\lambda x, y, s, t) + |\lambda| \phi(x,y,s,t) \quad (4.4)
\]

Replacing \( c_n s \) by \( s \) in (4.4), we get

\[
\|(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t)\| \leq |c_n|^{-1} \phi(\lambda x, y, c_n s, t) + |\lambda| |c_n|^{-1} \phi(x, y, c_n s, t)
\]

hence, as \( n \to \infty \), applying (4.3) we obtain

\[
(f(\lambda x, y), s \mid t) - (\lambda f(x, y), s \mid t) = 0 \quad (\lambda \in \mathbb{C}, x,y \in X, s,t \in Y).
\]

It follows from proposition 3.4 that

\[
f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x,y \in X) \quad (4.5)
\]

Now, we take \( c_n x \) instead \( x \) in (4.1) to get

\[
\|(f(c_n x, y), s \mid t) - (c_n x, y \mid g(s,t))\| \leq \phi(c_n x, y, s, t).
\]

It follows from (4.5) that

\[
\|(f(x, y), s \mid t) - (x, y \mid g(s,t))\| \leq |c_n|^{-1} \phi(c_n x, y, s, t)
\]

hence, as \( n \to \infty \), applying (4.2) we get

\[
(f(x, y), s \mid t) = (x, y \mid g(s,t)) \quad (x,y \in X, s,t \in Y).
\]

Therefore \( f \) is 2-adjointable and by Lemma 3.5, \( f \) is \( A \)-2 linear. \( \square \)

In the following, we let \( c_n = a^n \) that \( a > 1 \). We get the following results.

**Corollary 4.3.** If \( f : X \times X \to Y \) is a \( \phi \)-perturbation of a 2-adjointable mapping, where

\[
\phi(x, y, s, t) = \epsilon \|x\|_X^{p} \|y\|_X^{q} \|s\|_Y^{r} \|t\|_Y^{s} \quad (\epsilon > 0, 0 < p < 1, 0 < q < 1),
\]

then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2-linear.

**Corollary 4.4.** If \( f : X \times X \to Y \) is a \( \phi \)-perturbation of a 2-adjointable mapping, where

\[
\phi(x, y, s, t) = \epsilon_1 \|x\|_X^{p} + \epsilon_2 \|s\|_Y^{q} \quad (\epsilon_1 \geq 0, \epsilon_2 \geq 0, 0 < p < 1, 0 < q < 1).
\]

Then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

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REFERENCES


