On the 2-Adjointable Operators and Superstability of them between 2-Pre Hilbert $C^*$-Module Spaces

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ABSTRACT. In this paper, first, we introduce the new concept of 2-inner product on Banach modules over a $C^*$-algebra. Next, we present the concept of 2-linear operators over a $C^*$-algebra. Our result improve the main result of the paper [Z. Lewandowska, On 2-normed sets, Glasnik Mat., 38(58) (2003), 99-110]. In the end of this paper, we define the notions 2-adjointable mappings between 2-pre Hilbert $C^*$-modules and prove superstability of them in the spirit of Hyers-Ulam-Rassias.

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1. Introduction

The concept of 2-inner product has been intensively studied by many authors in the last three decades. The basic definitions and elementary properties of 2-inner product spaces can be found in [1] and [2].

Recently, M.Frank and e.t. defined the notion $\phi$-perturbation of an adjointable mapping and proved the superstability of an adjointable mapping on Hilbert $C^*$-modules(see [3]).

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In this paper, first, we introduce the definition 2-pre Hilbert $C^*$-module spaces and give several important properties. Next, we present the concept of 2-linear operators over a $C^*$-algebra which coincides with Lewandowska's definition (see [4, 5]). Also, we define 2-adjointable mappings between 2-pre Hilbert $C^*$-modules and prove an analogue of $\phi$-perturbation of adjoitable mappings in paper([3]).

We refer the interested reader to monographs [6, 7, 8, 9] and references therein for more information.

2. 2-Pre Hilbert Modules

Let $X$ be a left module over a $C^*$-algebra $A$. An action of $a \in A$ on $X$ is denoted by $a.x \in X$, $x \in X$.

**Definition 2.1.** A 2-pre Hilbert $A$-module is a left $A$-module $X$ equipped with $A$-valued function defined on $X \times X \times X$ satisfying the following conditions:

1. $(x, x|z)$ is a positive element in $A$ for any $x, z \in X$ and $(x, x|z) = 0$ if and only if $x$ and $z$ are linearly dependent;
2. $(x, x|z) = (z, z|x)$ for any $x, z \in X$;
3. $(y, x|z) = (x, y|z)^*$ for any $x, y, z \in X$;
4. $(xy + x', y|z) = \alpha(x, y|z) + (x', y|z)$ for any $\alpha \in \mathbb{C}$ and $x, x', y, z \in X$;
5. $(ax, y|z) = a(x, y|z)$ for any $x, y, z \in X$ and any $a \in A$.

The map $(., .|.)$ is called $A$-valued 2-inner product and $(X, (., .|.)$) is called 2-pre Hilbert $C^*$-module space.

**Example 2.2.** Every 2-inner product space is a 2-pre Hilbert $C$-module.

**Example 2.3.** Let $A$ be a $C^*$-algebra and $J \subseteq A$ be a left ideal. Then $J$ can be equipped with the structure of 2-pre Hilbert $A$-module with $A$-valued inner product $(x, y|z) := xy^*zz^* - xz^*zy^*$ for any $x, y, z \in A$.

**Definition 2.4.** Let $X$ be a 2-pre Hilbert $A$-module, we can define a function $\|\cdot, .|\|_X$ on $X \times X$ by $\|x|z\|_X = \|(x, x|z)\|^{1/2}$ for all $x, z \in X$.

**Lemma 2.5.** $\|\cdot, .|\|_X$ satisfies the following conditions:

1. $\|\alpha x|z\|_X \leq ||\alpha|| \|x|z\|_X$ for any $x, z \in X$ and $\alpha \in A$;
2. $(x, y|z) (y, x|z) \leq \|y|z\|_X^2 (x, x|z)$ for any $x, y, z \in X$;
3. $||(x, y|z)||^2 \leq \|(x, x|z)|| \|(y, y|z)||$

**Proof.** $N1$ is obvious; $N3$ follows from $N2$, so let us prove $N2$.

Let $\phi$ be a positive linear functional on $A$. Then $\phi((., .|.)$) is usual 2-inner product on $X$. Applying the Schwartz inequality for 2-inner product (see [2],
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page 3) we obtain for all \( x, y, z \in X \),
\[
\phi((x, y|z) (y, z|x)) = \phi((x, y|z)y, x|z))
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} \phi(((x, y|z)y, (x, y|z)y|z))^\frac{1}{2}
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} \phi((x, y|z) (y, y|z) (x, y|z)^*)^\frac{1}{2}
\]
\[
\leq \phi((x, x|z))^\frac{1}{2} \|(y, y|z)\|^\frac{1}{2} \phi((x, y|z) (y, x|z))^\frac{1}{2}.
\]
Thus, for any positive linear functional \( \phi \), we have
\[
\phi((x, y|z) (y, z|x)) \leq \|(y, y|z)\|^\frac{1}{2} \phi((x, x|z))
\]
hence
\[
(x, y|z) (y, z|x) \leq \|(y, y|z)\|^\frac{1}{2} (x, x|z).
\]

\[\square\]

**Theorem 2.6.** The function \( \|\cdot, \cdot\|_X \) is a 2-norm on \( X \).

**Proof.** Now, we verify that \( \|\cdot, \cdot\|_X \) satisfies the following properties of 2-norms:

1) \( I_3 \) and \( I_4 \) show that \( \|\alpha x|y\|_X = \|(\alpha x, \alpha x|y)\|_X = \|\alpha\| \|x|y\|_X \) for all \( x, y \in X \) and \( \alpha \in \mathbb{C} \).

2) \( I_1 \) follows that \( |x|y|_X = 0 \) if and only if \( x \) and \( y \) are linearly dependent for all \( x, y \in X \).

3) it follows from \( I_2 \) that \( \|x|y|_X = \|(x, x|y)\|_X = \|y|x|_X \) for all \( x, y \in X \).

4) By proposition 2.5 (\( N3 \)), we have
\[
\|(x + x'|y|_X^2 = \|(x + x', x + x'|y)\| = \|(x, y) + (x', x|y) + (x', x'|y)\| \leq \|(x, x|y)\| + 2\|(x, x'|y)\| + \|(x', x'|y)\|
\]
\[
\leq (\|(x, x|y)\| + 2\|(x, x'|y)\| + 1\|(x', x'|y)\|)^2 = (\|(x|y|_X + \|x'|y|_X)^2
\]
for all \( x, x', y \in X \). This show that \( (X, \|\cdot, \cdot\|_X) \) is a 2-normed space. \( \square \)

3. 2-ADJOINTABLE MAPPINGS

In continue, we let \( A \) be a \( C^* \)-algebra. Now, we start with following definition.

**Definition 3.1.** Let \( X \) and \( Y \) be two 2-pre Hilbert \( A \)-modules. An operator \( f : X \times X \to Y \) is said to be \( A \)-2 linear if it satisfies the following conditions:

1) \( f(x + + z + w) = f(x, z) + f(x, w) + f(y, z) + f(y, w) \) for all \( x, y, z, w \in X \);

2) \( f(\alpha x, \beta y) = \alpha \beta f(x, y) \) for all \( \alpha, \beta \in \mathbb{C} \) and \( x, y \in X \);

3) \( f(ax, by) = a \cdot b^* f(x, y) \) for all \( x, y \in X \) and \( a, b \in A \).

**Example 3.2.** Let \( X \) be a 2-pre Hilbert \( A \)-module and \( z \in X \). Define \( f : X \times X \to A \) by \( f(x, y) = (x, y|z) \). Then \( f \) is a \( A \)-2 linear operator.
Definition 3.3. Let $X$ and $Y$ be two 2-pre Hilbert $A$-modules. A mapping $f : X \times X \to Y$ is called 2-adjointable if there exists a mapping $g : Y \times Y \to X$ such that
\[
(f(x, y), s \mid t) = (x, y \mid g(s, t))
\]
for all $x, y \in X$ and $s, t \in Y$. The mapping $g$ is denoted by $f^*$ and is called the 2-adjoint of $f$.

Lemma 3.4. Let $X$ be a 2-pre Hilbert $A$-module and $\dim(X) > 1$. If $(x, y \mid z) = 0$ for all $y, z \in X$, then $x = 0$.

Proof. Suppose $x \neq 0$. Let $x$ and $y$ be linearly independent. Then by hypothesis $(x, x, \mid y) = 0$ and this is contradiction. \hfill $\square$

Lemma 3.5. Every 2-adjointable mapping is $A$-2 linear.

Proof. Let $f : X \times X \to Y$ be a 2-adjointable mapping. Then there exists a mapping $g : Y \times Y \to X$ such that (3.1) holds. For every $x, y, z, w \in X$, every $s, t \in Y$, every $a, b \in A$, we have
\[
(f(a\alpha x + y, \beta b z + w), s \mid t) = (a\alpha x + y, \beta b z + w \mid g(s, t))
\]
\[
= a\alpha \beta b^* (x, z \mid g(s, t)) + a\alpha (x, w \mid g(s, t)) + \beta b^* (y, z \mid g(s, t)) + (y, w \mid g(s, t))
\]
\[
= a\alpha \beta b^* (f(x, z), s \mid t) + a\alpha (f(x, w), s \mid t) + \beta b^* (f(y, z), s \mid t) + (f(y, w), s \mid t)
\]
\[
= (a\alpha \beta b^* f(x, z) + a\alpha f(x, w) + \beta b^* f(y, z) + f(y, w), s \mid t).
\]
It follows from lemma 3.4 that $f$ is $A$-2 linear. \hfill $\square$

4. Superstability of 2-adjointable mappings

In this section, $X$ and $Y$ denote 2-pre Hilbert $A$-modules and $\dim(X) > 1$, $\dim(Y) > 1$ and $\phi : X^2 \times Y^2 \to [0, \infty)$ is a function. We start our work with following definition.

Definition 4.1. A (not necessarily $A$-2 linear) mapping $f : X \times X \to Y$ is called $\phi$-perturbation of an 2-adjointable mapping if there exists a mapping (not necessarily $A$-2 linear) $g : Y \times Y \to X$ such that
\[
\| (f(x, y), s \mid t) - (x, y \mid g(s, t)) \| \leq \phi(x, y, s, t)
\]
for all $x, y \in X$ and $s, t \in Y$.

Theorem 4.2. Let $f : X \times X \to Y$ be a $\phi$-perturbation of a 2-adjointable mapping with corresponding mapping $g : Y \times Y \to X$. Suppose for some sequence $c_n$ of non-zero complex numbers the following conditions hold:
\[
\lim_{n \to \infty} |c_n|^{-1} \phi(c_n x, y, s, t) = 0 \quad (x, y \in X, s, t \in Y)
\]
\[
\lim_{n \to \infty} |c_n|^{-1} \phi(x, y, c_n s, t) = 0 \quad (x, y \in X, s, t \in Y)
\] (4.3)

Then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

**Proof.** Let \( \lambda \in \mathbb{C} \) be an arbitrary number. Putting \( \lambda x \) instead \( x \) in (4.1), we get

\[
\|f(\lambda x, y, s | t) - (\lambda x, y | g(s, t))\| \leq \phi(\lambda x, y, s, t)
\]

and hence, as \( n \to \infty \), applying (4.3) we obtain

\[
\|f(\lambda f(x, y), s | t) - \lambda f(x, y | g(s, t))\| \leq |\lambda| \phi(x, y, s, t)
\]

Thus,

\[
\|f(\lambda x, y, s | t) - (\lambda f(x, y), s | t)\| \leq \phi(\lambda x, y, s, t) + |\lambda| \phi(x, y, s, t) \quad (4.4)
\]

Replacing \( c_n s \) by \( s \) in (4.4), we get

\[
\|f(\lambda x, y, s | t) - (\lambda f(x, y), s | t)\| \leq |c_n|^{-1} \phi(\lambda x, y, c_n s, t) + |\lambda| |c_n|^{-1} \phi(x, y, c_n s, t)
\]

Replacing \( c_n s \) by \( s \) in (4.4), we get

\[
\|f(\lambda (x, y), s | t) - (\lambda f(x, y), s | t)\| = 0 \quad (\lambda \in \mathbb{C}, x, y \in X, s, t \in Y).
\]

It follows from proposition 3.4 that

\[
f(\lambda x, y) = \lambda f(x, y) \quad (\lambda \in \mathbb{C}, x, y \in X)
\] (4.5)

Now, we take \( c_n x \) instead \( x \) in (4.1) to get

\[
\|f(c_n x, y, s | t) - (c_n x, y | g(s, t))\| \leq \phi(c_n x, y, s, t).
\]

It follows from (4.5) that

\[
\|f(x, y, s | t) - (x, y | g(s, t))\| \leq |c_n|^{-1} \phi(c_n x, y, s, t)
\]

hence, as \( n \to \infty \), applying (4.2) we get

\[
f(x, y, s | t) = (x, y | g(s, t)) \quad (x, y \in X, s, t \in Y).
\]

Therefore \( f \) is 2-adjointable and by Lemma 3.5, \( f \) is \( A \)-2 linear. \( \square \)

In the following, we let \( c_n = a^n \) that \( a > 1 \). We get the following results.

**Corollary 4.3.** If \( f : X \times X \to Y \) is a \( \phi \)-perturbation of a 2-adjointable mapping, where

\[
\phi(x, y, s, t) = \epsilon \|x\|_X^p \|y\|_Y^q \|s\|_Y^r \|t\|_Y^s \quad (\epsilon \geq 0, 0 < p < 1, 0 < q < 1),
\]

then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2-linear.

**Corollary 4.4.** If \( f : X \times X \to Y \) is a \( \phi \)-perturbation of a 2-adjointable mapping, where

\[
\phi(x, y, s, t) = \epsilon_1 \|x\|_X^p \|y\|_Y^q + \epsilon_2 \|s\|_Y^r \|t\|_Y^s \quad (\epsilon_1 \geq 0, \epsilon_2 \geq 0, 0 < p < 1, 0 < q < 1).
\]

Then \( f \) is 2-adjointable and hence \( f \) is \( A \)-2 linear.

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