Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces

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Abstract. The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems

\[ -\text{div} \left( a(x, u, \nabla u) + \Phi(u) \right) + g(x, u, \nabla u) = f - \text{div} F, \]

in a bounded open set \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), in the framework of Orlicz-Sobolev spaces without any restriction on the \( M \)-function of the Orlicz spaces, where \( -\text{div} \left( a(x, u, \nabla u) \right) \) is a Leray-Lions operator defined from \( W^{1,0}_{M}(\Omega) \) into its dual, \( \Phi \in C^0(\mathbb{R}, \mathbb{R}^N) \). The function \( g(x, u, \nabla u) \) is a nonlinear lower order term with natural growth with respect to \( |\nabla u| \), satisfying the sign condition and the datum \( \mu \) is assumed to belong to \( L^1(\Omega) + W^{-1}E_M(\Omega) \).

Keywords: Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.


1. Introduction

Let \( \Omega \) be a bounded open set of \( \mathbb{R}^N , N \geq 2 \), and let \( M \) be an \( N \)-function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem

\[
\begin{cases}
A(u) - \text{div} \Phi(u) + g(x, u, \nabla u) = f - \text{div} F & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Received 30 July 2016; Accepted 12 August 2017
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Here, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$, while the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The non everywhere defined nonlinear operator $A(u) = -\text{div}(a(x, u, \nabla u))$ acts from its domain $D(A) \subset W^1_0 L_M(\Omega)$ into $W^{-1} L_N(\Omega)$. The function $a(x, u, \nabla u)$ is assumed to satisfy, among others, $a(x, u, \nabla u)$ nonstandard growth condition governed by the $N$-function $M$, and the source term $f \in L^1(\Omega)$ and $|F| \in E_N(\Omega)$. We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with $\Phi \equiv 0$, $g \equiv 0$ and $L^1(\Omega)$-data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when $f$ is a bounded Radon measure datum and $g$ grows at most like $|\nabla u|^{p-1}$ by Beta et al. in [9, 10, 11] with $\Phi \equiv 0$ and by Guibé and Mercaldo in [23, 24] when $\Phi(u)$ behaves at most like $|u|^{p-1}$. Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in $L^1(\Omega)$. Recall that to get both existence and uniqueness of a solution to problems with $L^1$-data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall’Aglio [14].

The authors in [5] have dealt with the equation (1.1) with $g = g(x, u)$ and $\mu \in W^{-1} E_N(\Omega)$, under the restriction that the $N$-function $M$ satisfies the $\Delta_2$-condition. This work was then extended in [2] for $N$-functions not satisfying necessarily the $\Delta_2$-condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the $N$-function $M$. Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $\mu \in W^{-1} E_N(\Omega)$, an existence result has been proved in [3]. Specific examples to which our results apply include the following:

\[
-\text{div} \left( |\nabla u|^{p-2} \nabla u + |u|^s u \right) + u|\nabla u|^p = \mu \quad \text{in} \quad \Omega,
\]

\[
-\text{div} \left( |\nabla u|^{p-2} \nabla u \log^\beta (1 + |\nabla u|) + |u|^s u \right) = \mu \quad \text{in} \quad \Omega,
\]

\[
-\text{div} \left( \frac{M(|\nabla u|) \nabla u}{|\nabla u|^2} + |u|^s u \right) + M(|\nabla u|) = \mu \quad \text{in} \quad \Omega,
\]

where $p > 1$, $s > 0$, $\beta > 0$ and $\mu$ is a given Radon measure on $\Omega$.

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form $f - \text{div} F$ with $f \in L^1(\Omega)$ and $|F| \in E_N(\Omega)$, in the setting of Orlicz spaces without any restriction on the $N$-functions $M$. The approximate equations provide a $W^1_0 L_M(\Omega)$ bound for the corresponding solution $u_n$. This allows us to obtain
a function $u$ as a limit of the sequence $u_n$. Hence, appear two difficulties. The first one is how to give a sense to $\Phi(u)$, the second difficulty lies in the need of the convergence almost everywhere of the gradients of $u_n$ in $\Omega$. This is done by using suitable test functions built upon $u_n$ which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an $N$-function, i.e., $M$ is continuous, increasing, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$, and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$. Equivalently, $M$ admits the representation:

$$M(t) = \int_0^t a(s) \, ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to $+\infty$ as $t \to +\infty$.

The conjugate of $M$ is also an $N$-function and it is defined by $\overline{M} = \int_0^t \bar{a}(s) \, ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is the function $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1]).

An $N$-function $M$ is said to satisfy the $\Delta_2$-condition if, for some $k$,

$$M(2t) \leq kM(t) \quad \forall t \geq 0, \quad (2.1)$$

when (2.1) holds only for $t \geq t_0 > 0$ then $M$ is said to satisfy the $\Delta_2$-condition near infinity. Moreover, we have the following Young’s inequality

$$st \leq M(t) + M(s), \quad \forall s, t \geq 0.$$

Given two $N$-functions, we write $P \ll Q$ to indicate $P$ grows essentially less rapidly than $Q$; i.e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if

$$\lim_{t \to \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $k_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$ is defined as the set of (equivalence classes of) real valued measurable functions $u$ on $\Omega$ such that

$$\int_{\Omega} M(|u(x)|) \, dx < +\infty \quad (\text{resp.} \quad \int_{\Omega} M \left( \frac{|u(x)|}{\lambda} \right) \, dx < +\infty \text{ for some } \lambda > 0).$$
The set $L_M(\Omega)$ is a Banach space under the norm
\[ \|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\}, \]
and $k_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\text{int}}(\Omega)$ by means of the pairing $\int_{\Omega} uv \, dx$, and the dual norm of $L_{\text{int}}(\Omega)$ is equivalent to $\| \cdot \|_{M,\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm
\[ \|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \| D^\alpha u \|_{M,\Omega}. \]

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod L_{\text{int}})$ and $\sigma(\prod L_M, \prod L_{\text{int}})$.

The space $W^1_1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W^1_1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod L_{\text{int}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that $u_n$ converges to $u$ for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_{\Omega} M \left( \frac{\lambda}{\lambda} u_n - D^\alpha u \right) \, dx \to 0$ for all $|\alpha| \leq 1$.

This implies convergence for $\sigma(\prod L_M, \prod L_{\text{int}})$. If $M$ satisfies the $\Delta_2$ condition on $\mathbb{R}^n$ (near infinity only when $\Omega$ has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\text{int}}(\Omega)$ [resp. $W^{-1}E_{\text{int}}(\Omega)$] denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\text{int}}(\Omega)$ [resp. $E_{\text{int}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain $\Omega$ has the segment property if for every $x \in \partial \Omega$ there exists an open set $G_x$ and a nonzero vector $y_x$ such that $x \in G_x$ and if $z \in \Omega \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$. The following lemmas can be found in [6].

**Lemma 2.1.** Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $M$ be an $N$-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then
\[ \frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{ x \in \Omega : u(x) \notin D \}, \\ 0 & \text{a.e. in } \{ x \in \Omega : u(x) \in D \}. \end{cases} \]
Lemma 2.2. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of $F'$ is finite. Let $M$ be an $N$-function, then the mapping $F : W^1 L_M(\Omega) \to W^1 L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\mathcal{T}})$.

Lemma 2.3. ([21]) Let $\Omega$ have the segment property. Then for each $\nu \in W^1_0 L_M(\Omega)$, there exists a sequence $\nu_n \in D(\Omega)$ such that $\nu_n$ converges to $\nu$ for the modular convergence in $W^1_0 L_M(\Omega)$. Furthermore, if $\nu \in W^1_0 L_M(\Omega) \cap L^\infty(\Omega)$, then

$$||\nu_n||_{L^\infty(\Omega)} \leq (N + 1)||\nu||_{L^\infty(\Omega)}.$$

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.4. Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure. Let $M, P, Q$ be $N$-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:

$$|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

where $k_1, k_2$ are real constants and $c(x) \in E_Q(\Omega)$.

Then the Nemytskii operator $N_f$ defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

We will also use the following technical lemma.

Lemma 2.5. ([26]) If $\{f_n\} \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in $\Omega$, $f_n, f \geq 0$ a.e. in $\Omega$ and $\int_\Omega f_n(x) \, dx \to \int_\Omega f(x) \, dx$, then $f_n \to f$ in $L^1(\Omega)$.

3. Structural Assumptions and Main Result

Throughout the paper $\Omega$ will be a bounded subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property. Let $M$ and $P$ be two $N$-functions such that $P \ll M$. Let $A$ be the non everywhere defined operator defined from its domain $\mathcal{D}(\Omega) \subset W^1_0 L_M(\Omega)$ into $W^{-1} L_{\mathcal{T}}(\Omega)$ given by

$$A(u) := - \text{div} a(\cdot, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. We assume that there exist a nonnegative function $c(x)$ in $E_{\mathcal{T}}(\Omega)$, $\alpha > 0$ and positive real constants $k_1, k_2, k_3$ and $k_4$, such that for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\xi' \in \mathbb{R}^N$ ($\xi \neq \xi'$) and for almost every $x \in \Omega$

$$|a(x, s, \xi)| \leq c(x) + k_1 P^{-1} M(k_2|s|) + k_3 M^{-1} M(k_4|\xi|),$$

(3.1)
Here, \( g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is a Carathéodory function satisfying for almost every \( x \in \Omega \) and for all \( s \in \mathbb{R} \), \( \xi \in \mathbb{R}^N \),
\[
|g(x, s, \xi)| \leq b(|s|) \left( d(x) + M(|\xi|) \right),
\]
and
\[
g(x, s, \xi)s \geq 0,
\]
where \( b : \mathbb{R} \to \mathbb{R}^+ \) is a continuous and increasing function while \( d \) is a given nonnegative function in \( L^1(\Omega) \).

The right-hand side of (1.1) and \( \Phi : \mathbb{R} \to \mathbb{R}^N \), are assumed to satisfy
\[
f \in L^1(\Omega) \text{ and } |F| \in L^1(\Omega),
\]
\[
\Phi \in C^0(\mathbb{R}, \mathbb{R}^N).
\]

Our aim in this paper is to give a meaning to a possible solution of (1.1). In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution \( u \) of problem (1.1) is the Orlicz-Sobolev space \( W^1_{0}\mathcal{M}(\Omega) \). But when \( u \) is only in \( W^1_{0}\mathcal{M}(\Omega) \) there is no reason for \( \Phi(u) \) to be in \( (L^1(\Omega))^N \) since no growth hypothesis is assumed on the function \( \Phi \). Thus, the term \( \text{div} (\Phi(u)) \) may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally through a pointwise multiplication of equation (1.1) by \( h(u) \) where \( h \) belongs to \( C^1_c(\mathbb{R}) \), the class of \( C^1(\mathbb{R}) \) functions with compact support.

**Definition 3.1.** A measurable function \( u : \Omega \to \mathbb{R} \) is called a renormalized solution of (1.1) if \( u \in W^1_{0}\mathcal{M}(\Omega) \), \( a(x, u, \nabla u) \in (L^1(\Omega))^N \), \( g(x, u, \nabla u) \in L^1(\Omega) \), \( g(x, u, \nabla u)u \in L^1(\Omega) \),
\[
\lim_{m \to +\infty} \int_{\{x \in \Omega : m \leq |u(x)| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx = 0,
\]
and
\[
\begin{cases}
- \text{div} a(x, u, \nabla u)h(u) = \text{div} (\Phi(u)h(u)) + h'(u)\Phi(u)\nabla u \\
+ g(x, u, \nabla u)h(u) = fh(u) - \text{div} (Fh(u)) + h'(u)F \nabla u \text{ in } \mathcal{D}'(\Omega),
\end{cases}
\]
for every \( h \in C^1_c(\mathbb{R}) \).

**Remark 3.2.** Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for \( h \) in \( C^1_c(\mathbb{R}) \) and \( u \) in \( W^1_{0}\mathcal{M}(\Omega) \), \( h(u) \) belongs to \( W^1_{0}\mathcal{M}(\Omega) \) and for \( \varphi \) in \( \mathcal{D}(\Omega) \) the function \( \varphi h(u) \) belongs to \( W^1_{0}\mathcal{M}(\Omega) \). Since \( (\text{div} a(x, u, \nabla u)) \in W^{-1}\mathcal{M}(\Omega) \), we also have
\[
\langle - \text{div} a(x, u, \nabla u)h(u), \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \langle - \text{div} a(x, u, \nabla u), \varphi h(u) \rangle_{W^{-1}\mathcal{M}(\Omega), W^1_{0}\mathcal{M}(\Omega)} 
\quad \forall \varphi \in \mathcal{D}(\Omega).
\]
Finally, since $\Phi h$ and $\Phi h'$ belong to $C^0_0(\mathbb{R})^N$, for any measurable function $u$ we have $\Phi(u)h(u)$ and $\Phi(u)h'(u) \in (L^\infty(\Omega))^N$ and then $\text{div} (\Phi(u)h(u)) \in W^{-1,\infty}(\Omega)$ and $\Phi(u)h'(u) \in L_M(\Omega)$.

Our main result is the following

**Theorem 3.3.** Suppose that assumptions (3.1)-(3.7) are fulfilled. Then, problem (1.1) has at least one renormalized solution.

**Remark 3.4.** The condition (3.4) can be replaced by the weaker one

$$|g(x, s, \xi)| \leq d(x) + b(|s|)M(|\xi|),$$

with $b : \mathbb{R} \to \mathbb{R}^+$ a continuous function belonging to $L^1(\mathbb{R})$ and $d(x) \in L^1(\Omega)$.

Actually the original equation (1.1) will be recovered whenever $h(u) \equiv 1$, but unfortunately this cannot happen in general strong additional requirements on $u$. Therefore, (3.8) is to be viewed as a weaker form of (1.1).

4. **Proof of the Main Result**

From now on, we will use the standard truncation function $T_k$, $k > 0$, defined for all $s \in \mathbb{R}$ by $T_k(s) = \max\{-k, \min\{k, s\}\}$.

**Step 1: Approximate problems.** Let $f_n$ be a sequence of $L^\infty(\Omega)$ functions that converge strongly to $f$ in $L^1(\Omega)$. For $n \in \mathbb{N}$, $n \geq 1$, let us consider the following sequence of approximate equations

$$-\text{div} a(x, u_n, \nabla u_n) + \text{div} \Phi_n(u_n) + g_n(x, u_n, \nabla u_n) = f_n - F \text{ in } D'(\Omega),$$

(4.1)

where we have set $\Phi_n(s) = \Phi(T_n(s))$ and $g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{2} |g(x, s, \xi)|}$. For fixed $n > 0$, it’s obvious to observe that

$$g_n(x, s, \xi) \geq 0, \quad |g_n(x, s, \xi)| \leq |g(x, s, \xi)| \quad \text{and} \quad |g_n(x, s, \xi)| \leq n.$$ 

Moreover, since $\Phi$ is continuous one has $|\Phi_n(t)| \leq \max_{|t| \leq n} |\Phi(t)|$. Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduce that there exists at least one solution $u_n$ of the approximate Dirichlet problem (4.1) in the sense

$$\begin{cases}
u_n \in W^1_0L_M(\Omega), \ a(x, u_n, \nabla u_n) \in (L^\infty(\Omega))^N \\
\int_\Omega a(x, u_n, \nabla u_n) \nabla v dx + \int_\Omega \Phi_n(u_n) \nabla v dx \\
+ \int_\Omega g_n(x, u_n, \nabla u_n) v dx = (f_n, v) + \int_\Omega F \nabla v dx, \text{ for every } v \in W^1_0L_M(\Omega).
\end{cases}$$

(4.2)
Step 2: Estimation in $W^{1}_{0}L_{M}(\Omega)$. Taking $u_{n}$ as function test in problem (4.2), we obtain
\[
\int_{\Omega} a(x,u_{n},\nabla u_{n})\nabla u_{n}dx + \int_{\Omega} \Phi_{n}(u_{n})\nabla u_{n}dx \\
+ \int_{\Omega} g_{n}(x,u_{n},\nabla u_{n})u_{n}dx = (f_{n},u_{n}) + \int_{\Omega} F\nabla u_{n}dx.
\] (4.3)

Define $\tilde{\Phi}_{n} \in (C^{1}(\mathbb{R}))^{N}$ as $\tilde{\Phi}_{n}(t) = \int_{0}^{t} \Phi_{n}(\tau)d\tau$. Then formally \[
\text{div}(\tilde{\Phi}_{n}(u_{n})) = \Phi_{n}(u_{n})\nabla u_{n}, \quad u_{n} = 0 \text{ on } \partial \Omega, \quad \tilde{\Phi}_{n}(0) = 0 \text{ and by the Divergence theorem}
\]
\[
\int_{\Omega} \Phi_{n}(u_{n})\nabla u_{n}dx = \int_{\Omega} \text{div}(\tilde{\Phi}_{n}(u_{n}))dx = \int_{\partial \Omega} \tilde{\Phi}_{n}(u_{n})\,n\,ds = 0,
\]
where $\vec{n}$ is the outward pointing unit normal field of the boundary $\partial \Omega$ ($ds$ may be used as a shorthand for $\vec{n}\,ds$). Thus, by virtue of (3.5) and Young's inequality, we get \[
\int_{\Omega} a(x,u_{n},\nabla u_{n})\nabla u_{n}dx \leq C_{1} + \frac{\alpha}{2} \int_{\Omega} M(|\nabla u_{n}|)dx,
\] (4.4)
which, together with (3.3) give \[
\int_{\Omega} M(|\nabla u_{n}|)dx \leq C_{2}.
\] (4.5)
Moreover, we also have \[
\int_{\Omega} g_{n}(x,u_{n},\nabla u_{n})u_{n}dx \leq C_{3}.
\] (4.6)

As a consequence of (4.5) there exist a subsequence of $\{u_{n}\}_{n}$, still indexed by $n$, and a function $u \in W^{1}_{0}L_{M}(\Omega)$ such that \[
\begin{align*}
u_{n} &\rightharpoonup u \text{ weakly in } W^{1}_{0}L_{M}(\Omega) \text{ for } \sigma(\Pi L_{M}(\Omega),\Pi E_{M}(\Omega)), \\
u_{n} &\to u \text{ strongly in } E_{M}(\Omega) \text{ and a.e. in } \Omega.
\end{align*}
\] (4.7)

Step 3: Boundedness of $(a(x,u_{n},\nabla u_{n}))_{n}$ in $(L_{M}(\Omega))^{N}$. Let $w \in (E_{M}(\Omega))^{N}$ with $\|w\|_{M} \leq 1$. Thanks to (3.2), we can write \[
(a(x,u_{n},\nabla u_{n}) - (a(x,u_{n},\frac{w}{k_{4}})))(\nabla u_{n} - \frac{w}{k_{4}}) \geq 0,
\]
which implies \[
\frac{1}{k_{4}} \int_{\Omega} a(x,u_{n},\nabla u_{n})wdx \leq \int_{\Omega} a(x,u_{n},\nabla u_{n})\nabla u_{n}dx \\
+ \int_{\Omega} a(x,u_{n},\frac{w}{k_{4}})(\frac{w}{k_{4}} - \nabla u_{n})dx.
\]

Thanks to (4.4) and (4.5), one has \[
\int_{\Omega} a(x,u_{n},\nabla u_{n})\nabla u_{n}dx \leq C_{5}.
\]
Define $\lambda = 1 + k_1 + k_3$. By the growth condition (3.1) and Young’s inequality, one can write
\[
\left| \int_\Omega a\left(x, u_n, \frac{w}{k_4} \right) \left| \frac{w}{k_4} - \nabla u_n \right| dx \right| \\
\leq \left( 1 + \frac{1}{k_4} \right) \left( \int_\Omega \mathcal{M}(c(x)) dx + k_1 \int_\Omega \mathcal{M}^{-1}(M(k_2|u_n|)) dx \\
+ k_3 \int_\Omega M(|w|) dx \right) + \frac{\lambda}{k_4} \int_\Omega M(|w|) dx + \lambda \int_\Omega M(|\nabla u_n|) dx.
\]

By virtue of [18] and Lemma 4.14 of [20], there exists an $N$-function $Q$ such that $M \ll Q$ and the space $W^1_0 L_M(\Omega)$ is continuously embedded into $L_Q(\Omega)$. Thus, by (4.5) there exists a constant $c_0 > 0$, not depending on $n$, satisfying $\|u_n\|_Q \leq c_0$. Since $M \ll Q$, we can write $M(k_2 t) \leq Q(\frac{t}{c_0})$, for $t > 0$ large enough. As $P \ll M$, we can find a constant $c_1$, not depending on $n$, such that $\int_\Omega \mathcal{M}^{-1}(M(k_2|u_n|)) dx \leq \int_\Omega Q\left( \frac{|u_n|}{c_0} \right) + c_1$. Hence, we conclude that the quantity $\int a(x, u_n, \nabla u_n) w dx$ is bounded from above for all $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Using the Orlicz norm we deduce that
\[
\left( a(x, u_n, \nabla u_n) \right)_{n} \text{ is bounded in } (L_{\mathcal{M}}(\Omega))^N. \tag{4.8}
\]

**Step 4: Renormalization identity for the approximate solutions.** For any $m \geq 1$, define $\theta_m(r) = T_{m+1}(r) - T_m(r)$. Observe that by [19, Lemma2] one has $\theta_m(u_n) \in W^1_0 L_M(\Omega)$. The use of $\theta_m(u_n)$ as test function in (4.2) yields
\[
\int_{\{|m| \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle + \int_{\{|m| \leq |u_n| \leq m+1\}} F \nabla u_n dx,
\]

By Hölder’s inequality and 4.5 we have
\[
\int_{\{|m| \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle \\
+ C_6 \int_{\{|m| \leq |u_n| \leq m+1\}} \mathcal{M}(|F|) dx.
\]

It’s not hard to see that
\[
\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.
\]

So that by (4.5) and (4.7) one can deduce that
\[
\theta_m(u_n) \rightharpoonup \theta_m(u) \text{ weakly in } W^1_0 L_M(\Omega) \text{ for } \sigma(\mathcal{M}(\Omega), \mathcal{M}(\Omega)).
\]

Note that as $m$ goes to $\infty$, $\theta_m(u) \rightarrow 0$ weakly in $W^1_0 L_M(\Omega)$ for $\sigma(\mathcal{M}(\Omega), \mathcal{M}(\Omega))$, and since $f_n$ converges strongly in $L^1(\Omega)$, by Lebesgue’s theorem we have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{|m| \leq |u_n| \leq m+1\}} \mathcal{M}(|F|) dx = \lim_{m \to \infty} \lim_{n \to \infty} \langle f_n, \theta_m(u_n) \rangle = 0.
\]
By (3.3) we finally have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx = 0. \quad (4.9)
\]

**Step 5:** Almost everywhere convergence of the gradients. Define
\[
\phi(s) = se^{\lambda s^2} \text{ with } \lambda = \left(\frac{b(k)}{2\alpha}\right)^2. \quad \text{One can easily verify that for all } s \in \mathbb{R} \quad \phi'(s) - \frac{b(k)}{\alpha} |\phi(s)| \geq \frac{1}{2}. \quad (4.10)
\]

For \( m \geq k \), we define the function \( \psi_m \) by
\[
\begin{cases}
\psi_m(s) = 1 & \text{if } |s| \leq m, \\
\psi_m(s) = m + 1 - |s| & \text{if } m \leq |s| \leq m + 1, \\
\psi_m(s) = 0 & \text{if } |s| \geq m + 1.
\end{cases}
\]

By virtue of [21, Theorem 4] there exists a sequence \( \{v_j\}_j \subset D(\Omega) \) such that \( v_j \to u \) in \( W^1_0 L_1(\Omega) \) for the modular convergence and a.e. in \( \Omega \). Let us define the following functions \( \theta^j_n = T_k(u_n) - T_k(v_j) \), \( \theta^j = T_k(u) - T_k(v_j) \) and \( z_{n,m}^j = \phi(\theta^j_n) \psi_m(u_n) \). Using \( z_{n,m}^j \in W^1_0 L_1(\Omega) \) as test function in (4.2) we get
\[
\begin{align*}
\int_\Omega a(x, u_n, \nabla u_n) \nabla z_{n,m}^j \, dx &+ \int_\Omega \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) \, dx \\
&+ \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \nabla \psi_m'(u_n) \phi(T_k(u_n) - T_k(v_j)) \, dx \\
&+ \int \Phi(u_n) \nabla u_n \nabla \psi_m(u_n) \phi(T_k(u_n) - T_k(v_j)) \, dx \\
&+ \int \nabla g_n(x, u_n, \nabla u_n) z_{n,m}^j \, dx = \int \nabla f_n \nabla z_{n,m}^j \, dx + \int F \nabla z_{n,m}^j \, dx.
\end{align*}
\]

From now on we denote by \( \epsilon_i(n, j), \quad i = 0, 1, 2, \ldots \), various sequences of real numbers which tend to zero, when \( n \) and \( j \to +\infty \), i.e.
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \epsilon_i(n, j) = 0.
\]

In view of (4.7), we have \( z_{n,m}^j \to \phi(\theta^j) \psi_m(u) \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) as \( n \to +\infty \), which yields
\[
\lim_{n \to +\infty} \int_\Omega f_n \nabla z_{n,m}^j \, dx = \int_\Omega f \phi(\theta^j) \psi_m(u) \, dx,
\]
and since \( \phi(\theta^j) \to 0 \) weakly in \( L^\infty(\Omega) \) for \( \sigma(L^\infty, L^1) \) as \( j \to +\infty \), we have
\[
\lim_{j \to +\infty} \int_\Omega f \phi(\theta^j) \psi_m(u) \, dx = 0.
\]

Thus, we write
\[
\int_\Omega f_n \nabla z_{n,m}^j \, dx = \epsilon_0(n, j).
\]

Thanks to (4.5) and (4.7), we have as \( n \to +\infty \),
\[
z_{n,m}^j \to \phi(\theta^j) \psi_m(u) \text{ in } W^1_0 L_1(\Omega) \text{ for } \sigma(\Pi L_1(\Omega), \Pi E^M(\Omega)).
which implies that
\[
\lim_{n \to +\infty} \int_{\Omega} F \nabla z_{n,m}^j \, dx = \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_{m}(u) \, dx + \int_{\Omega} F \nabla u \phi(\theta) \psi_{m}(u) \, dx
\]

On the one hand, by Lebesgue’s theorem we get
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_{m}(u) \, dx = 0,
\]
on the other hand, we write
\[
\int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_{m}(u) \, dx = \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_{m}(u) \, dx - \int_{\Omega} F \nabla T_k(v_j) \phi'(\theta^j) \psi_{m}(u) \, dx,
\]
so that, by Lebesgue’s theorem one has
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_{m}(u) \, dx = \int_{\Omega} F \nabla T_k(u) \psi_{m}(u) \, dx.
\]
Let \( \lambda > 0 \) such that \( M \left( \frac{\nabla v_j - \nabla u}{\lambda} \right) \to 0 \) strongly in \( L^1(\Omega) \) as \( j \to +\infty \) and \( M \left( \frac{\nabla u}{\lambda} \right) \in L^1(\Omega) \), the convexity of the \( N \)-function \( M \) allows us to have
\[
M \left( \frac{\nabla T_k(v_j) \phi' \phi(\theta^j) \psi_{m}(u) - \nabla T_k(u) \psi_{m}(u)}{\lambda} \right) = \frac{1}{4} M \left( \frac{\nabla v_j - \nabla u}{\lambda} \right) + \frac{1}{4} \left( 1 + \frac{1}{\sigma(2k)} \right) M \left( \frac{\nabla u}{\lambda} \right).
\]
Then, by using the modular convergence of \( \{ \nabla v_j \} \) in \( (L_M(\Omega))^N \) and Vitali’s theorem, we obtain
\[
\nabla T_k(v_j) \phi' \phi(\theta^j) \psi_{m}(u) \to \nabla T_k(u) \psi_{m}(u) \text{ in } (L_M(\Omega))^N,
\]
as \( j \) tends to \( +\infty \), for the modular convergence, and then
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla T_k(u) \phi' \phi(\theta^j) \psi_{m}(u) \, dx = \int_{\Omega} F \nabla T_k(u) \psi_{m}(u) \, dx.
\]
We have proved that
\[
\int_{\Omega} F \nabla z_{n,m}^j \, dx = \epsilon_1 \langle n, j \rangle.
\]
It’s easy to see that by the modular convergence of the sequence \( \{ v_j \} \), one has
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\{ m \leq |u_n| \leq m+1 \}} \Phi_n(u_n) \nabla u_n \psi_{m}'(u_n) \phi \left( T_k(u_n) - T_k(v_j) \right) \, dx = 0,
\]
while for the third term in the left-hand side of (4.11) we can write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi \left( T_k(u_n) - T_k(v_j) \right) \psi_{m}(u_n) \, dx
\]
\[
= \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi' \phi(\theta^j_n) \psi_{m}(u_n) \, dx - \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi' \phi(\theta^j_n) \psi_{m}(u_n) \, dx.
\]
Firstly, we have
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'((\theta_n^j) \psi_m(u_n) dx = 0.
\]
In view of (4.7), one has
\[
\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n) \to \Phi(u) \phi'(\theta^j) \psi_m(u),
\]
almost everywhere in \( \Omega \) as \( n \) tends to \(+\infty\). Furthermore, we can check that
\[
\|\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n)\|_{\mathcal{M}} \leq M(c_m \phi'(2k))|\Omega| + 1,
\]
where \( c_m = \max_{|t| \leq m+1} \Phi(t) \). Applying [27, Theorem 14.6] we get
\[
\lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'((\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx.
\]
Using the modular convergence of the sequence \( \{v_j\} \), we obtain
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'((\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx.
\]
Then, using again the Divergence theorem we get
\[
\int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx = 0.
\]
Therefore, we write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx = \epsilon_2(n, j).
\]
Since \( g_n(x, u_n, \nabla u_n)z_{n,m}^j \geq 0 \) on the set \( \{ |u_n| > k \} \) and \( \psi_m(u_n) = 1 \) on the set \( \{ |u_n| \leq k \} \), from (4.11) we obtain
\[
\int \Omega a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx + \int \{|u_n| \leq k\} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \leq \epsilon_3(n, j),
\]
We now evaluate the first term of the left-hand side of (4.12) by writing
\[
\int \Omega a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx
\]

\[
= \int \Omega a(x, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) \psi_m(u_n) dx
\]

\[
+ \int \Omega a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) dx
\]

\[
= \int \Omega a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) dx
\]

\[
- \int \{|u_n| > k\} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx
\]

\[
+ \int \Omega a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) dx.
\]
and then
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j \, dx \\
= \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^\theta) \right) \\
\qquad \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^\theta \right) \phi'(\theta_n^j) \, dx \\
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^\theta) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^\theta \right) \phi'(\theta_n^j) \, dx \\
- \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \nabla T_k(v_j) \phi'(\theta_n^j) \, dx \\
- \int_{\{u_n > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx \\
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) \, dx,
\] (4.13)

where by \( \chi_s^\theta, s > 0 \), we denote the characteristic function of the subset
\[\Omega^j_s = \{ x \in \Omega : |\nabla T_k(v_j)| \leq s \} .\]

For fixed \( m \) and \( s \), we will pass to the limit in \( n \) and then in \( j \) in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^\theta) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^\theta \right) \phi'(\theta_n^j) \, dx \to +\infty.
\]
Since by lemma (2.4) one has
\[a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^\theta) \phi'(\theta_n^j) \to a(x, T_k(u), \nabla T_k(v_j) \chi_s^\theta) \phi'(\theta^j),\]
strongly in \((E_{M^j}(\Omega))^N\) as \( n \to \infty \), while by (4.5)
\[\nabla T_k(u_n) \to \nabla T_k(u),\]
weakly in \((L_M(\Omega))^N\). Let \( \chi^s \) denote the characteristic function of the subset
\[\Omega^s = \{ x \in \Omega : |\nabla T_k(u)| \leq s \} .\]
As \( \nabla T_k(v_j) \chi^s \to \nabla T_k(u) \chi_s \) strongly in \((E_M(\Omega))^N\) as \( j \to +\infty \), one has
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_s^\theta) \cdot \left( \nabla T_k(u) - \nabla T_k(v_j) \chi_s^\theta \right) \phi'(\theta^j) \, dx \to 0,
\]
as \( j \to \infty \). Then
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s^\theta) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_s^\theta \right) \phi'(\theta_n^j) \, dx = c_4(n, j). \quad (4.14)
\]
We now estimate the third term of (4.13). It’s easy to see that by (3.3), \( a(x, s, 0) = 0 \) for almost everywhere \( x \in \Omega \) and for all \( s \in \mathbb{R} \). Thus, from (4.8) we have that \( \left( a(x, T_k(u_n), \nabla T_k(u_n)) \right)_n \) is bounded in \((L_{M^j}(\Omega))^N\) for all \( k \geq 0 \).
Therefore, there exist a subsequence still indexed by $n$ and a function $l_k$ in $(L^\infty(\Omega))^N$ such that
\[
  a(x, T_k(u_n), \nabla T_k(u_n)) \to l_k \text{ weakly in } (L^\infty(\Omega))^N \text{ for } \sigma(\Pi L^\infty, HE_M). \quad (4.15)
\]

Then, since $\nabla T_k(v_j)\chi_{\Omega \setminus \Omega'_j} \in (E^\infty(\Omega))^N$, we obtain
\[
  \int_{\Omega \setminus \Omega'_j} a(x, T_k(u_n)) \nabla T_k(v_j) \phi'(\theta^j_n) dx \to \int_{\Omega \setminus \Omega'_j} l_k \nabla T_k(v_j) \phi' \psi_j dx,
\]
as $n \to +\infty$. The modular convergence of $\{v_j\}$ allows us to get
\[
  - \int_{\Omega \setminus \Omega'_j} l_k \nabla T_k(v_j) \phi' \psi_j dx \to - \int_{\Omega \setminus \Omega'} l_k \nabla T_k(u) dx,
\]
as $j \to +\infty$. This proves
\[
  - \int_{\Omega \setminus \Omega'_j} a(x, T_k(u_n)) \nabla T_k(v_j) \phi' \psi_j dx = - \int_{\Omega \setminus \Omega'} l_k \nabla T_k(u) dx + \epsilon_5(n, j). \quad (4.16)
\]

As regards the fourth term, observe that $\psi_m(u_n) = 0$ on the subset $\{|u_n| \geq m + 1\}$, so we have
\[
  - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi' \psi_m(u_n) dx =
  - \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi' \psi_m(u_n) dx.
\]
Since
\[
  - \int_{\{|u| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi' \psi_m(u_n) dx =
  - \int_{\{|u| > k\}} l_{m+1} \nabla T_k(u) \psi_m(u) dx + \epsilon_5(n, j),
\]
obsc{observing that $\nabla T_k(u) = 0$ on the subset $\{|u| > k\}$}, one has
\[
  - \int_{\{|u| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi' \psi_m(u_n) dx = \epsilon_6(n, j). \quad (4.17)
\]

For the last term of (4.13), we have
\[
  \left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^j_n) \psi'_m(u_n) dx \right|
  = \left| \int_{\{|m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^j_n) \psi'_m(u_n) dx \right|
  \leq \phi(2k) \int_{\{|m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx.
\]
To estimate the last term of the previous inequality, we use
\((T_1(u_n - T_m(u_n)) \in W_0^1 L_M(\Omega))\) as test function in (4.2), to get
\[
\int_{|u_n| \leq m+1} a(x, u_n, \nabla u_n) \nabla u_n \, dx + \int_{|u_n| \leq m+1} \Phi_n(u_n) \nabla u_n \, dx \\
+ \int_{|u_n| \geq m} g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \, dx = \langle f_n, T_1(u_n - T_m(u_n)) \rangle \\
+ \int_{|u_n| \leq m+1} F \nabla u_n \, dx.
\]

By Divergence theorem, we have
\[
\int_{|u_n| \leq m+1} \Phi_n(u_n) \nabla u_n \, dx = 0.
\]

Using the fact that \(g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \geq 0\) on the subset \(|u_n| \geq m\) and Young's inequality, we get
\[
\int_{|u_n| \leq m+1} a(x, u_n, \nabla u_n) \nabla u_n \, dx \\
\leq \langle f_n, T_1(u_n - T_m(u_n)) \rangle + \int_{|u_n| \leq m+1} M(|F|) \, dx.
\]
It follows that
\[
\left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi'(\theta_n^j)(u_n) \, dx \right| \\
\leq 2\phi(2k) \left( \int_{|u_n| \leq m} |f_n| \, dx + \int_{|u_n| \leq m+1} M(|F|) \, dx \right) \tag{4.18}
\]
From (4.14), (4.16), (4.17) and (4.18) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{m,n} \, dx \\
\geq \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \right) \phi'(\theta_n^j) \, dx \\
- \alpha \phi(2k) \left( \int_{|u_n| \leq m} |f_n| \, dx + \int_{|u_n| \leq m+1} M(|F|) \, dx \right) \\
- \int_{\Omega \setminus \Omega^*} l_k \cdot T_k(u) \, dx + \epsilon_7(n,j).
\tag{4.19}
\]
Now, we turn to second term in the left-hand side of (4.12). We have
\[
\left| \int_{|u_n| \leq k} g_n(x, u_n, \nabla u_n) \phi'(\theta_n^j)(u_n) \, dx \right| \\
= \left| \int_{|u_n| \leq k} g_n(x, T_k(u_n), \nabla T_k(u_n)) \phi'(\theta_n^j)(u_n) \, dx \right| \\
\leq b(k) \int_{\Omega} M(|\nabla T_k(u_n)|) |\phi'(\theta_n^j)| \, dx + b(k) \int_{\Omega} d(x) |\phi'(\theta_n^j)| \, dx \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\phi'(\theta_n^j)| \, dx + \epsilon_8(n,j).
\]
Then
\[
\left| \int_{\{u_n \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^i) \, dx \right| \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(v_j), \nabla T_k(v_j) \chi_n^j)) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_n^j \right) |\phi(\theta_n^i)| \, dx \\
+ \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_n^j) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_n^j \right) |\phi(\theta_n^i)| \, dx \\
+ \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_n^j |\phi(\theta_n^i)| \, dx + \epsilon_9(n, j). 
\] 
(4.22)

We proceed as above to get
\[
\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_n^j) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_n^j \right) |\phi(\theta_n^i)| \, dx = \epsilon_9(n, j)
\]

and
\[
\frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_n^j |\phi(\theta_n^i)| \, dx = \epsilon_{10}(n, j).
\]

Hence, we have
\[
\left| \int_{\{u_n \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^i) \, dx \right| \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(v_j), \nabla T_k(v_j) \chi_n^j) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_n^j \right) |\phi(\theta_n^i)| \, dx + \epsilon_{11}(n, j). 
\] 
(4.21)

Combining (4.12), (4.19) and (4.21), we get
\[
\left( \phi(\theta_n^i) - \frac{b(k)}{\alpha} |\phi(\theta_n^i)| \right) \, dx \\
\leq \int_{\Omega \setminus \Omega^*} l_k \nabla T_k(u) \, dx + \alpha \phi(2k) \left( \int_{\{m \leq |u_n| \}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1 \}} M(|F|) \, dx \right) \\
+ \epsilon_{12}(n, j).
\]

By (4.10), we have
\[
\left( \phi(\theta_n^i) - \frac{b(k)}{\alpha} |\phi(\theta_n^i)| \right) \, dx \\
\leq 2 \int_{\Omega \setminus \Omega^*} l_k \nabla T_k(u) \, dx + 4 \alpha \phi(2k) \left( \int_{\{m \leq |u_n| \}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1 \}} M(|F|) \, dx \right) \\
+ \epsilon_{12}(n, j). 
\] 
(4.22)
On the other hand we can write
\[
\int_\Omega \left( a(x,T_k(u_n) , \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(u)\chi^s) \right) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx
\]
\[
= \int_\Omega \left( a(x,T_k(u_n), \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(v_j)\chi^s) \right) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s) \, dx
\]
\[
+ \int_\Omega a(x,T_k(u_n), \nabla T_k(u)\chi^s)(\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx
\]
\[
- \int_\Omega a(x,T_k(u_n), \nabla T_k(u)\chi^s)(\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx
\]
\[
+ \int_\Omega a(x,T_k(u_n), \nabla T_k(v_j)\chi^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s) \, dx
\]

We shall pass to the limit in \( n \) and then in \( j \) in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain
\[
\int_\Omega \left( a(x,T_k(u_n), \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(u)\chi^s) \right) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx = \epsilon_{13}(n,j),
\]
\[
\int_\Omega \left( a(x,T_k(u_n), \nabla T_k(u)\chi^s)(\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \right) \, dx = \epsilon_{14}(n,j),
\]
\[
\int_\Omega \left( a(x,T_k(u_n), \nabla T_k(v_j)\chi^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s) \right) \, dx = \epsilon_{15}(n,j).
\]
So that
\[
\int_\Omega \left( a(x,T_k(u_n), \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(u)\chi^s) \right) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx
\]
\[
= \int_\Omega \left( a(x,T_k(u_n), \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(v_j)\chi^s) \right) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s) \, dx
\]
\[
+ \epsilon_{16}(n,j).
\]
Let \( r \leq s \). Using (3.2), (4.22) and (4.24) we can write
\[
0 \leq \int_{\Omega^r} \left( a(x,T_k(u_n), \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(u)) \right) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx
\]
\[
\leq \int_{\Omega^r} \left( a(x,T_k(u_n), \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(u)) \right) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx
\]
\[
= \int_{\Omega^r} \left( a(x,T_k(u_n), \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(u)\chi^s) \right) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx
\]
\[
\leq \int_{\Omega} \left( a(x,T_k(u_n), \nabla T_k(u_n)) - a(x,T_k(u_n), \nabla T_k(u)\chi^s) \right) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx
\]
\[
+ \epsilon_{15}(n,j)
\]
\[
\leq 2 \int_{\Omega^r} \nabla T_k(u) \, dx + 2\alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} |\overline{M}(|F|)\right) \, dx
\]
\[
+ \epsilon_{17}(n,j).
\]
By passing to the superior limit over \( n \) and then over \( j \)

\[
0 \leq \limsup_{n \to +\infty} \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx
\]

\[
\leq 2 \int_{\Omega^r} l_k \nabla T_k(u) \, dx + 4 \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|f|) \, dx \right).
\]

Letting \( s \to +\infty \) and then \( m \to +\infty \), taking into account that \( l_k \nabla T_k(u) \in L^1(\Omega) \), \( f \in L^1(\Omega) \), \(|F| \in (F_{\Pi\bar{\Pi}}(\Omega))^N \), \(|\Omega \setminus \Omega^s| \to 0 \), and \(|\{m \leq |u| \leq m+1\}| \to 0 \), one has

\[
\int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx,
\]

(4.25)

tends to 0 as \( n \to +\infty \). As in [20, we deduce that there exists a subsequence of \( \{u_n\} \) still indexed by \( n \) such that

\[
\nabla u_n \rightharpoonup \nabla u \text{ a. e. in } \Omega.
\]

(4.26)

Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get

\[
a(x, u, \nabla u) \in (L_{\Pi\bar{\Pi}}(\Omega))^N
\]

and

\[
a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\Pi\bar{\Pi}}(\Omega))^N \text{ for } \sigma(\Pi L_{\Pi\bar{\Pi}}, \Pi E_M).
\]

(4.27)

**Step 6: Modular convergence of the truncations.** Going back to equation (4.22), we can write

\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx
\]

\[
\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s \, dx
\]

\[
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \, dx
\]

\[
+ 2a \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|f|) \, dx \right)
\]

\[
+ 2 \int_{\Omega^r} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx + \epsilon_{12}(n, j).
\]

By (4.23) we get

\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx
\]

\[
\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s \, dx
\]

\[
+ 2a \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|f|) \, dx \right)
\]

\[
+ 2 \int_{\Omega^r} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx + \epsilon_{18}(n, j).
\]
We now pass to the superior limit over $n$ in both sides of this inequality using (4.27), to obtain
\begin{align*}
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \chi^s \, dx \\
+ 2\alpha \phi(2k) \left( \int_{\{m \leq |u|\}} |f| \, dx + \int_{\{m \leq |u| \leq m+1\}} \frac{M(|F|)}{M} \, dx \right) \\
+ 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\end{align*}

We then pass to the limit in $j$ to get
\begin{align*}
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \chi^s \, dx \\
+ 2\alpha \phi(2k) \left( \int_{\{m \leq |u|\}} |f| \, dx + \int_{\{m \leq |u| \leq m+1\}} \frac{M(|F|)}{M} \, dx \right) \\
+ 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\end{align*}

Letting $s$ and then $m \to +\infty$, one has
\begin{align*}
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\end{align*}

On the other hand, by (3.3), (4.5), (4.26) and Fatou’s lemma, we have
\begin{align*}
\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \\
\leq \liminf_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx.
\end{align*}

It follows that
\begin{align*}
\lim_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
= \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\end{align*}

By Lemma 2.5 we conclude that for every $k > 0$
\begin{align*}
a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u),
\end{align*}

strongly in $L^1(\Omega)$. The convexity of the $N$-function $M$ and (3.3) allow us to have
\begin{align*}
M \left( \frac{|\nabla T_k(u_n)| - |\nabla T_k(u)|}{2} \right) \\
\leq \frac{1}{2} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) + \frac{1}{2} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u).
\end{align*}

From Vitali’s theorem we deduce
\begin{align*}
\lim_{|E| \to 0} \sup_{n} \int_{E} \frac{M \left( |\nabla T_k(u_n)| - |\nabla T_k(u)| \right)}{2} \, dx = 0.
\end{align*}

Thus, for every $k > 0$
\begin{align*}
T_k(u_n) \to T_k(u) \text{ in } W_0^1 L_M(\Omega),
\end{align*}
for the modular convergence.

Step 7: Compactness of the nonlinearities. We need to prove that
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \] (4.29)
By virtue of (4.7) and (4.26) one has
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a. e. in } \Omega. \] (4.30)
Let \( E \) be measurable subset of \( \Omega \) and let \( m > 0 \). Using (3.3) and (3.4) we can write
\[
\int_E |g_n(x, u_n, \nabla u_n)| dx \\
= \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| dx \\
\leq b(m) \int_E d(x)dx + b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)dx \\
+ \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx.
\]
From (3.5) and (4.6), we deduce that
\[ 0 \leq \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \leq C_3. \]
So
\[ 0 \leq \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \leq \frac{C_3}{m}. \]
Then
\[ \lim_{m \to +\infty} \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx = 0. \]
Thanks to (4.28) the sequence \( \{a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)\}_n \) is equi-integrable. This fact allows us to get
\[ \lim_{|E| \to 0} \sup_n \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) dx = 0. \]
This shows that \( g_n(x, u_n, \nabla u_n) \) is equi-integrable. Thus, Vitali’s theorem implies that \( g(x, u, \nabla u) \in L^1(\Omega) \) and
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]

Step 8: Renormalization identity for the solutions. In this step we prove that
\[ \lim_{m \to +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u dx = 0. \] (4.31)
Indeed, for any $m \geq 0$ we can write
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \\
= \int_\Omega a(x, u_n, \nabla u_n)(\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) \, dx \\
= \int_\Omega a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) \, dx \\
- \int_\Omega a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx.
\]

In view of (4.28), we can pass to the limit as $n$ tends to $+\infty$ for fixed $m \geq 0$
\[
\lim_{n \to +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \\
= \int_\Omega a(x, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_{m+1}(u) \, dx \\
- \int_\Omega a(x, T_m(u), \nabla T_m(u)) \nabla T_m(u) \, dx \\
= \int_\Omega a(x, u, \nabla u)(\nabla T_{m+1}(u) - \nabla T_m(u)) \, dx \\
= \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx.
\]

Having in mind (4.9), we can pass to the limit as $m$ tends to $+\infty$ to obtain
(4.31).

**Step 9: Passing to the limit.** Thanks to (4.28) and Lemma (2.5), we obtain
\[
a(x, u_n, \nabla u_n) \nabla u_n \to a(x, u, \nabla u) \nabla u \text{ strongly in } L^1(\Omega). \tag{4.32}
\]

Let $h \in C^1_c(\mathbb{R})$ and $\varphi \in \mathcal{D}(\Omega)$. Inserting $h(u_n)\varphi$ as test function in (4.2), we get
\[
\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \varphi' \, dx + \int_\Omega a(x, u_n, \nabla u_n) \varphi \, dx + \int_\Omega \Phi_n(u_n) \nabla (h(u_n)\varphi) \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx \\
= \langle f_n, h(u_n) \varphi \rangle + \int_\Omega F \nabla (h(u_n)\varphi) \, dx. \tag{4.33}
\]

We shall pass to the limit as $n \to +\infty$ in each term of the equality (4.33).

Since $h$ and $h'$ have compact support on $\mathbb{R}$, there exists a real number $\nu > 0$, such that $\text{supp } h \subset [-\nu, \nu]$ and $\text{supp } h' \subset [-\nu, \nu]$. For $n > \nu$, we can write
\[
\Phi_n(t) h(t) = \Phi(T_\nu(t)) h(t) \text{ and } \Phi_n(t) h'(t) = \Phi(T_\nu(t)) h'(t).
\]

Moreover, the functions $\Phi h$ and $\Phi h'$ belong to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$. Observe first that the sequence $\{h(u_n)\varphi\}_n$ is bounded in $W_0^1 L^1(\Omega)$. Indeed, let $\rho > 0$
be a positive constant such that \(\|h(u_n)\nabla \varphi\|_{\infty} \leq \rho\) and \(\|h'(u_n)\varphi\|_{\infty} \leq \rho\). Using the convexity of the \(N\)-function \(M\) and taking into account (4.5) we have
\[
\int_{\Omega} M\left(\left\|\nabla (h(u_n)\varphi)\right\|\right) dx \\
\leq \int_{\Omega} M\left(\frac{\|h(u_n)\nabla \varphi\| + \|h'(u_n)\varphi\|\nabla u_n\|}{2\rho}\right) dx \\
\leq \frac{1}{2} M(1)\|\varphi\| + \frac{1}{2} \int_{\Omega} M(\nabla |u_n|) dx \\
\leq \frac{1}{2} M(1)\|\varphi\| + \frac{1}{2} \|\varphi\|_{\infty}.
\]
This, together with (4.7), imply that
\[
h(u_n)\varphi \rightharpoonup h(u)\varphi \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\Omega}). \tag{4.34}
\]
This enables us to get
\[
\langle f_n, h(u_n)\varphi \rangle \to \langle f, h(u)\varphi \rangle.
\]
Let \(E\) be a measurable subset of \(\Omega\). Define \(c_\nu = \max_{|t| \leq \nu} \Phi(t)\). Let us denote by \(\|v\|_{\sigma M}\) the Orlicz norm of a function \(v \in L_M(\Omega)\). Using strengthened Hölder inequality with both Orlicz and Luxemburg norms, we get
\[
\|\Phi(T_\nu(u_n))\chi_E\|_{\sigma M} = \sup_{\|v\|_{\sigma M} \leq 1} \left| \int_E \Phi(T_\nu(u_n)) v dx \right| \\
\leq c_\nu \sup_{\|v\|_{\sigma M} \leq 1} \|\chi_E\|_{\sigma M} \|v\|_M \\
\leq c_\nu |E|M^{-1}\left(\frac{1}{|E|}\right).
\]
Thus, we get
\[
\lim_{|E| \to 0} \sup_n \|\Phi(T_\nu(u_n))\chi_E\|_{\sigma M} = 0.
\]
Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain
\[
\Phi(T_\nu(u_n)) \to \Phi(T_\nu(u)) \text{ strongly in } (E_{\Omega})^N,
\]
which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have
\[
\int_{\Omega} \Phi(T_\nu(u_n)) \nabla (h(u_n)\varphi) dx \to \int_{\Omega} \Phi(T_\nu(u)) \nabla (h(u)\varphi) dx.
\]
We remark that
\[
|a(x, u_n, \nabla u_n)\nabla u_n h'(u_n)| \leq \rho a(x, u_n, \nabla u_n)\nabla u_n.
\]
Consequently, using (4.32) and Vitali’s theorem, we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n)\nabla u_n h'(u_n)\varphi dx \to \int_{\Omega} a(x, u, \nabla u)\nabla uh'(u)\varphi dx.
\]
and
\[
\int_{\Omega} F\nabla u_nh'(u_n)\varphi dx \to \int_{\Omega} F\nabla uh'(u)\varphi dx.
\]
For the second term of (4.33), as above we have
\[
h(u_n)\nabla \varphi \rightharpoonup h(u)\nabla \varphi \text{ strongly in } (E_{\Omega})^N,
\]
which together with (4.27) give
\[ \int_\Omega a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) \, dx \to \int_\Omega a(x, u, \nabla u) \nabla \varphi h(u) \, dx \]
and
\[ \int_\Omega F \nabla \varphi h(u_n) \, dx \to \int_\Omega F \nabla \varphi h(u) \, dx. \]
The fact that \( h(u_n) \varphi \rightharpoonup h(u) \varphi \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get
\[ \int_\Omega g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx \to \int_\Omega g(x, u, \nabla u) h(u) \varphi \, dx. \]
At this point we can pass to the limit in each term of (4.33) to get
\[ \int_\Omega a(x, u, \nabla u)(\nabla \varphi h(u) + h'(u) \varphi \nabla u) \, dx + \int_\Omega \Phi(u) h'(u) \varphi \, dx + \int_\Omega g(x, u, \nabla u) h(u) \varphi \, dx \]
\[ = (f, h(u) \varphi) + \int_\Omega F(\nabla \varphi h(u) + h'(u) \varphi \nabla u) \, dx, \]
for all \( h \in C^1_c(\mathbb{R}) \) and for all \( \varphi \in \mathcal{D}(\Omega) \). Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou’s lemma to get \( g(x, u, \nabla u) \in L^1(\Omega) \). By virtue of (4.7), (4.27), (4.29), (4.31), the function \( u \) is a renormalized solution of problem (1.1).

ACKNOWLEDGMENTS

We would like to thank the referees for careful reading of the first version of this manuscript and providing many helpful suggestions and comments · · ·

REFERENCES


