Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces

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Abstract. The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems
\[-\text{div} \left(a(x, u, \nabla u) + \Phi(u)\right) + g(x, u, \nabla u) = f - \text{div} \, F,\]
in a bounded open set $\Omega$ and $u = 0$ on $\partial\Omega$, in the framework of Orlicz-Sobolev spaces without any restriction on the $M$-function of the Orlicz spaces, where $-\text{div} \left(a(x, u, \nabla u)\right)$ is a Leray-Lions operator defined from $W^{1,0}_0 L^M(\Omega)$ into its dual, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$. The function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition and the datum $\mu$ is assumed to belong to $L^1(\Omega) + W^{-1} E^M(\Omega)$.

Keywords: Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.


1. Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and let $M$ be an $N$-function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem
\[
\begin{align*}
A(u) - \text{div} \, \Phi(u) + g(x, u, \nabla u) &= f - \text{div} \, F & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega.
\end{align*}
\]
Here, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$, while the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The non everywhere defined nonlinear operator $A(u) = -\text{div} (a(x, u, \nabla u))$ acts from its domain $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$. The function $a(x, u, \nabla u)$ is assumed to satisfy, among others, $a(x, u, \nabla u)$ nonstandard growth condition governed by the $N$-function $M$, and the source term $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$.

We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with $\Phi \equiv 0$, $g \equiv 0$ and $L^1(\Omega)$-data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when $f$ is a bounded Radon measure datum and $g$ grows at most like $|\nabla u|^{p-1}$ by Beta et al. in [9, 10, 11] with $\Phi \equiv 0$ and by Guibé and Mercaldo in [23, 24] when $\Phi(u)$ behaves at most like $|u|^{p-1}$. Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in $L^1(\Omega)$. Recall that to get both existence and uniqueness of a solution to problems with $L^1$-data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall’Aglio [14].

The authors in [5] have dealt with the equation (1.1) with $g = g(x, u)$ and $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, under the restriction that the $N$-function $M$ satisfies the $\Delta_2$-condition. This work was then extended in [2] for $N$-functions not satisfying necessarily the $\Delta_2$-condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the $N$-function $M$. Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, an existence result has been proved in [3]. Specific examples to which our results apply include the following:

\[- \text{div}(|\nabla u|^{p-2} \nabla u + |u|^s u) + u|\nabla u|^p = \mu \text{ in } \Omega,\]
\[- \text{div}(|\nabla u|^{p-2} \nabla u \log(1 + |\nabla u|) + |u|^s u) = \mu \text{ in } \Omega,\]
\[- \text{div}\left(\frac{M(|\nabla u| \nabla u)}{|\nabla u|^2} + |u|^s u\right) + M(|\nabla u|) = \mu \text{ in } \Omega,\]

where $p > 1$, $s > 0$, $\beta > 0$ and $\mu$ is a given Radon measure on $\Omega$.

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form $f = \text{div} F$ with $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, in the setting of Orlicz spaces without any restriction on the $N$-functions $M$. The approximate equations provide a $W_0^1 L_M(\Omega)$ bound for the corresponding solution $u_n$. This allows us to obtain
a function $u$ as a limit of the sequence $u_n$. Hence, appear two difficulties. The first one is how to give a sense to $\Phi(u)$, the second difficulty lies in the need of the convergence almost everywhere of the gradients of $u_n$ in $\Omega$. This is done by using suitable test functions built upon $u_n$ which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an $N$-function, i.e., $M$ is continuous, increasing, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$, and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$. Equivalently, $M$ admits the representation:

$$M(t) = \int_0^t a(s) \, ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to $+\infty$ as $t \to +\infty$.

The conjugate of $M$ is also an $N$-function and it is defined by $\overline{M} = \int_0^t \bar{a}(s) \, ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is the function $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1]).

An $N$-function $M$ is said to satisfy the $\Delta_2$-condition if, for some $k$,

$$M(2t) \leq kM(t) \quad \forall t \geq 0,$$

(2.1)

when (2.1) holds only for $t \geq t_0 > 0$ then $M$ is said to satisfy the $\Delta_2$-condition near infinity. Moreover, we have the following Young’s inequality

$$st \leq M(t) + \overline{M}(s), \quad \forall s, t \geq 0.$$

Given two $N$-functions, we write $P \ll Q$ to indicate $P$ grows essentially less rapidly than $Q$; i.e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if

$$\lim_{t \to \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $k_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions $u$ on $\Omega$ such that

$$\int_\Omega M(|u(x)|) \, dx < +\infty \quad (\text{resp.} \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) \, dx < +\infty \text{ for some } \lambda > 0).$$
The set $L_M(\Omega)$ is a Banach space under the norm
\[
\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\},
\]
and $k_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{M^*}(\Omega)$ by means of the pairing $\int_{\Omega} uv \, dx$, and the dual norm of $L_{M^*}(\Omega)$ is equivalent to $\|u\|_{M^*,\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm
\[
\|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}.
\]

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod L_{M^*})$ and $\sigma(\prod L_M, \prod L_{M^*})$.

The space $W^1_0E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W^1_0L_M(\Omega)$ as the $\sigma(\prod L_M, \prod L_{M^*})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that $u_n$ converges to $u$ for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, \[
\int_{\Omega} \left( \sum_{|\alpha| \leq 1} \frac{\|D^\alpha u_n - D^\alpha u\|_{M,\Omega}}{\lambda} \right) \, dx \rightarrow 0 \text{ for all } |\alpha| \leq 1.
\]
This implies convergence for $\sigma(\prod L_M, \prod L_{M^*})$. If $M$ satisfies the $\Delta_2$ condition on $\mathbb{R}^d$ (near infinity only when $\Omega$ has finite measure), then modular convergence coincides with norm convergence.

Let $W^1L_{M^*}(\Omega)$ [resp. $W^1E_{M^*}(\Omega)$] denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{M^*}(\Omega)$ [resp. $E_{M^*}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain $\Omega$ has the segment property if for every $x \in \partial \Omega$ there exists an open set $G_x$ and a nonzero vector $y_x$ such that $x \in G_x$ and $z \in \overline{G_x}$, then $z + ty_x \in \Omega$ for all $0 < t < 1$. The following lemmas can be found in [6].

**Lemma 2.1.** Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $M$ be an $N$-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then
\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u, & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0, & \text{a.e. in } \{x \in \Omega : u(x) \in D\} \end{cases}
\]
Lemma 2.2. Let \( F : \mathbb{R} \to \mathbb{R} \) be uniformly Lipschitzian, with \( F(0) = 0 \). We suppose that the set of discontinuity points of \( F' \) is finite. Let \( M \) be an \( N \)-function, then the mapping \( F : W^{1,1}(\Omega) \to W^{1,1}(\Omega) \) is sequentially continuous with respect to the weak* topology \( \sigma(\prod L_M, \prod E_{\mathbb{RT}}) \).

Lemma 2.3. ([21]) Let \( \Omega \) have the segment property. Then for each \( \nu \in W^{1,1}_0(\Omega) \), there exists a sequence \( \nu_n \in D(\Omega) \) such that \( \nu_n \) converges to \( \nu \) for the modular convergence in \( W^{1,1}_0(\Omega) \). Furthermore, if \( \nu \in W^{1,1}_0(\Omega) \), then

\[
\|\nu_n\|_{L^\infty(\Omega)} \leq (N + 1)\|\nu\|_{L^\infty(\Omega)}.
\]

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.4. Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) with finite measure. Let \( M, P, Q \) be \( N \)-functions such that \( Q \ll P \), and let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function such that, for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R} \):

\[
|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|),
\]

where \( k_1, k_2 \) are real constants and \( c(x) \in E_Q(\Omega) \).

Then the Nemytskii operator \( N_f \) defined by \( N_f(u)(x) = f(x, u(x)) \) is strongly continuous from \( P(E_M(\Omega), \frac{1}{k_1}) = \{ u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_1} \} \) into \( E_Q(\Omega) \).

We will also use the following technical lemma.

Lemma 2.5. ([26]) If \( \{ f_n \} \subset L^1(\Omega) \) with \( f_n \to f \in L^1(\Omega) \) a.e. in \( \Omega \), \( f_n, f \geq 0 \) a.e. in \( \Omega \) and \( \int_{\Omega} f_n(x) \, dx \to \int_{\Omega} f(x) \, dx \), then \( f_n \to f \) in \( L^1(\Omega) \).

3. Structural Assumptions and Main Result

Throughout the paper \( \Omega \) will be a bounded subset of \( \mathbb{R}^N \), \( N \geq 2 \), satisfying the segment property. Let \( M \) and \( P \) be two \( N \)-functions such that \( P \ll M \).

Let \( A \) be the non everywhere defined operator defined from its domain \( D(\Omega) \subset W^{1,1}_0(\Omega) \) into \( W^{-1,1}(\Omega) \) given by

\[
A(u) := - \text{div} \, a(\cdot, u, \nabla u),
\]

where \( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) is a Carathéodory function. We assume that there exist a nonnegative function \( c(x) \) in \( E_{\mathbb{RT}}(\Omega) \), \( \alpha > 0 \) and positive real constants \( k_1, k_2, k_3 \) and \( k_4 \), such that for every \( s \in \mathbb{R}, \xi \in \mathbb{R}^N, \xi' \in \mathbb{R}^N \) (\( \xi \neq \xi' \)) and for almost every \( x \in \Omega \)

\[
|a(x, s, \xi)| \leq c(x) + k_1 P^{-1} M(k_2 |s|) + k_3 M^{-1} M(k_4 |\xi|),
\]

(3.1)
\[ (a(x, s, \xi) - a(x, s, \xi'))(\xi - \xi') > 0, \quad (3.2) \]
\[ a(x, s, \xi)\xi \geq \alpha_M(|\xi|). \quad (3.3) \]

Here, \( g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is a Carathéodory function satisfying for almost every \( x \in \Omega \) and for all \( s \in \mathbb{R}, \xi \in \mathbb{R}^N, \)
\[ |g(x, s, \xi)| \leq b(|s|)(d(x) + M(|\xi|)), \quad (3.4) \]
\[ g(x, s, \xi)s \geq 0, \quad (3.5) \]
where \( b : \mathbb{R} \rightarrow \mathbb{R}^+ \) is a continuous and increasing function while \( d \) is a given nonnegative function in \( L^1(\Omega). \)

The right-hand side of (1.1) and \( \Phi : \mathbb{R} \rightarrow \mathbb{R}^N, \) are assumed to satisfy
\[ f \in L^1(\Omega) \text{ and } |F| \in E_{M}^{L^1} \Omega, \]
\[ \Phi \in C^0(\mathbb{R}, \mathbb{R}^N). \quad (3.7) \]

Our aim in this paper is to give a meaning to a possible solution of (1.1).

In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution \( u \) of problem (1.1) is the Orlicz-Sobolev space \( \text{W}^{1, L_M} \Omega \). But when \( u \) is only in \( \text{W}^{1, L_M} \Omega \) there is no reason for \( \Phi(u) \) to be in \( (L^1(\Omega))^N \) since no growth hypothesis is assumed on the function \( \Phi \). Thus, the term \( \text{div} (\Phi(u)) \) may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally through a pointwise multiplication of equation (1.1) by \( h(u) \) where \( h \) belongs to \( C^1_c(\mathbb{R}), \) the class of \( C^1(\mathbb{R}) \) functions with compact support.

**Definition 3.1.** A measurable function \( u : \Omega \rightarrow \mathbb{R} \) is called a renormalized solution of (1.1) if \( u \in \text{W}^{1, L_M} \Omega, \) \( a(x, u, \nabla u) \in (L^1(\Omega))^N, \)
\[ g(x, u, \nabla u) = L_M^1, \quad g(x, u, \nabla u)u \in L^1(\Omega), \]
\[ \lim_{m \to +\infty} \int_{\{ x \in \Omega : m \leq |u(x)| \leq m + 1 \}} a(x, u, \nabla u) \nabla u \, dx = 0, \]
and
\[ \begin{cases} -\text{div} a(x, u, \nabla u)h(u) - \text{div} (\Phi(u)h(u)) + h'(u)\Phi(u)\nabla u \\ + g(x, u, \nabla u)h(u) = fh(u) - \text{div} (Fh(u)) + h'(u)F\nabla u \end{cases} \quad (3.8) \]
for every \( h \in C^1_c(\mathbb{R}). \)

**Remark 3.2.** Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for \( h \) in \( C^1_c(\mathbb{R}) \) and \( u \) in \( \text{W}^{1, L_M} \Omega, \) \( h(u) \) belongs to \( \text{W}^{1, L_M} \Omega \) and for \( \varphi \) in \( \mathcal{D}(\Omega) \) the function \( \varphi h(u) \) belongs to \( \text{W}^{1, L_M} \Omega. \) Since \( (-\text{div} a(x, u, \nabla u)) \in W^{-1}L^1(\Omega), \) we also have
\[ \langle -\text{div} a(x, u, \nabla u)h(u), \varphi \rangle_{\mathcal{D}(\Omega)} = \langle -\text{div} a(x, u, \nabla u), \varphi h(u) \rangle_{W^{-1}L^1(\Omega), \text{W}^{1, L_M} \Omega} \]
\[ \forall \varphi \in \mathcal{D}(\Omega). \]
Finally, since $\Phi h$ and $\Phi h' \in (C^0_0(\mathbb{R}))^N$, for any measurable function $u$ we have $\Phi(u)h(u)$ and $\Phi(u)h'(u) \in (L^\infty(\Omega))^N$ and then $\text{div} (\Phi(u)h(u)) \in W^{-1,\infty}(\Omega)$ and $\Phi(u)h'(u) \in L_M(\Omega)$.

Our main result is the following

**Theorem 3.3.** Suppose that assumptions (3.1)–(3.7) are fulfilled. Then, problem (1.1) has at least one renormalized solution.

**Remark 3.4.** The condition (3.4) can be replaced by the weaker one

$$|g(x, s, \xi)| \leq d(x) + b(|s|)M(|\xi|),$$

with $b : \mathbb{R} \to \mathbb{R}^+$ a continuous function belonging to $L^1(\mathbb{R})$ and $d(x) \in L^1(\Omega)$.

Actually the original equation (1.1) will be recovered whenever $h(u) \equiv 1$, but unfortunately this cannot happen in general strong additional requirements on $u$. Therefore, (3.8) is to be viewed as a weaker form of (1.1).

4. **Proof of the Main Result**

From now on, we will use the standard truncation function $T_k$, $k > 0$, defined for all $s \in \mathbb{R}$ by $T_k(s) = \max\{-k, \min\{k, s\}\}.$

**Step 1: Approximate problems.** Let $f_n$ be a sequence of $L^\infty(\Omega)$ functions that converge strongly to $f$ in $L^1(\Omega)$. For $n \in \mathbb{N}$, $n \geq 1$, let us consider the following sequence of approximate equations

$$-\text{div} a(x, u_n, \nabla u_n) + \text{div} \Phi_n(u_n) + g_n(x, u_n, \nabla u_n) = f_n - F \text{ in } D'(\Omega),$$

where we have set $\Phi_n(s) = \Phi(T_n(s))$ and $g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{2} |g(x, s, \xi)|}.$ For fixed $n > 0$, it’s obvious to observe that

$$g_n(x, s, \xi) \geq 0, \quad |g_n(x, s, \xi)| \leq |g(x, s, \xi)| \text{ and } |g_n(x, s, \xi)| \leq n.$$  

Moreover, since $\Phi$ is continuous one has $|\Phi_n(t)| \leq \max_{|t| \leq n} |\Phi(t)|$. Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduces that there exists at least one solution $u_n$ of the approximate Dirichlet problem (4.1) in the sense

$$\begin{cases}
  u_n \in W^1_0L_M(\Omega), a(x, u_n, \nabla u_n) \in (L^\infty(\Omega))^N \\
  \int_\Omega a(x, u_n, \nabla u_n) \nabla v dx + \int_\Omega \Phi_n(u_n) \nabla v dx \\
  + \int_\Omega g_n(x, u_n, \nabla u_n) v dx = (f_n, v) + \int_\Omega F \nabla v dx, \text{ for every } v \in W^1_0L_M(\Omega).
\end{cases}$$

(4.2)
Step 2: Estimation in $W^{1}_{0} L_{M}(\Omega)$. Taking $u_n$ as function test in problem (4.2), we obtain
\[
\int_{\Omega}a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\Omega} \Phi_n(u_n) \nabla u_n dx \\
+ \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx = (f_n, u_n) + \int_{\Omega} F \nabla u_n dx.
\] (4.3)

Define $\tilde{\Phi}_n \in (C^{1}(\mathbb{R}))^N$ as $\tilde{\Phi}_n(t) = \int_{0}^{t} \Phi_n(\tau) d\tau$. Then formally
\[
\text{div}(\tilde{\Phi}_n(u_n)) = \Phi_n(u_n) \nabla u_n, \quad u_n = 0 \text{ on } \partial \Omega,
\]
\[
\tilde{\Phi}_n(0) = 0 \quad \text{and by the Divergence theorem}
\]
\[
\int_{\Omega} \Phi_n(u_n) \nabla u_n dx = \int_{\Omega} \text{div}(\tilde{\Phi}_n(u_n)) dx = \int_{\partial \Omega} \tilde{\Phi}_n(u_n) \nu ds = 0,
\]
where $\nu$ is the outward pointing unit normal field of the boundary $\partial \Omega$ ($ds$ may be used as a shorthand for $\nu ds$). Thus, by virtue of (3.5) and Young’s inequality, we get
\[
\int_{\Omega}a(x, u_n, \nabla u_n) \nabla u_n dx \leq C_1 + \frac{\alpha}{2} \int_{\Omega} M(|\nabla u_n|) dx,
\] (4.4)
which, together with (3.3) give
\[
\int_{\Omega} M(|\nabla u_n|) dx \leq C_2.
\] (4.5)

Moreover, we also have
\[
\int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx \leq C_3.
\] (4.6)

As a consequence of (4.5) there exist a subsequence of $\{u_n\}_n$, still indexed by $n$, and a function $u \in W^{1}_0 L_{M}(\Omega)$ such that
\[
\begin{aligned}
& u_n \rightharpoonup u \text{ weakly in } W^{1}_0 L_{M}(\Omega) \text{ for } \sigma(\Pi_{L_{M}}(\Omega), \Pi_{E_{M}}(\Omega)), \\
& u_n \rightarrow u \text{ strongly in } E_{M}(\Omega) \text{ and a.e. in } \Omega.
\end{aligned}
\] (4.7)

Step 3: Boundedness of $(a(x, u_n, \nabla u_n))_n$ in $(L_{M}(\Omega))^N$. Let $w \in (E_{M}(\Omega))^N$ with $\|w\|_M \leq 1$. Thanks to (3.2), we can write
\[
(a(x, u_n, \nabla u_n) - a(x, u_n, \frac{w}{k_4}))(\nabla u_n - \frac{w}{k_4}) \geq 0,
\]
which implies
\[
\frac{1}{k_4} \int_{\Omega} a(x, u_n, \nabla u_n) w dx \leq \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \\
+ \int_{\Omega} a(x, u_n, \frac{w}{k_4})(\frac{w}{k_4} - \nabla u_n) dx.
\]

Thanks to (4.4) and (4.5), one has
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq C_5.
\]
Define \( \lambda = 1 + k_1 + k_3 \). By the growth condition (3.1) and Young’s inequality, one can write
\[
\left| \int_\Omega a(x, u_n, \frac{w}{k_4}) \left( \frac{w}{k_4} - \nabla u_n \right) dx \right| 
\leq \left( 1 + \frac{1}{k_4} \right) \left( \int_\Omega M(c(x)) dx + k_1 \int_\Omega M(\lambda k_2 |u_n|) dx \right.
\]
\[+ k_3 \int_\Omega M(|w|) dx + \frac{\lambda}{k_4} \int_\Omega M(|w|) dx + \lambda \int_\Omega M(|\nabla u_n|) dx. \]

By virtue of [18] and Lemma 4.14 of [20], there exists an \( N \)-function \( Q \) such that \( M \ll Q \) and the space \( W_0^1 L_M(\Omega) \) is continuously embedded into \( L_Q(\Omega) \). Thus, by (4.5) there exists a constant \( c_0 > 0 \), not depending on \( n \), satisfying \( \|u_n\|_Q \leq c_0 \). Since \( M \ll Q \), we can write \( M(k_2 t) \leq Q \left( \frac{t}{c_0} \right) \), for \( t > 0 \) large enough. As \( P \ll M \), we can find a constant \( c_1 \), not depending on \( n \), such that
\[\int_\Omega M^{-1}(k_2 |u_n|) dx \leq \int_\Omega Q \left( \frac{|u_n|}{c_0} \right) + c_1.\]
Hence, we conclude that the quantity \( \int_\Omega a(x, u_n, \nabla u_n) dx \) is bounded from above for all \( w \in (E_M(\Omega))^N \) with \( \|w\|_M \leq 1 \). Using the Orlicz norm we deduce that
\[
\left( a(x, u_n, \nabla u_n) \right)_n \text{ is bounded in } (L_{\overline{MT}}(\Omega))^N. \quad (4.8)
\]

**Step 4:** Renormalization identity for the approximate solutions. For any \( m \geq 1 \), define \( \theta_m(r) = T_{m+1}(r) - T_m(r) \). Observe that by [19, Lemma 2] one has \( \theta_m(u_n) \in W_0^1 L_M(\Omega) \). The use of \( \theta_m(u_n) \) as test function in (4.2) yields
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n dx,
\]
By Hölder’s inequality and 4.5 we have
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle + C_0 \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx.
\]
It’s not hard to see that
\[
\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.
\]
So that by (4.5) and (4.7) one can deduce that
\[
\theta_m(u_n) \rightharpoonup \theta_m(u) \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{MT}}(\Omega)).
\]
Note that as \( m \) goes to \( \infty \), \( \theta_m(u) \rightharpoonup 0 \) weakly in \( W_0^1 L_M(\Omega) \) for \( \sigma(\Pi L_M(\Omega), \Pi E_{\overline{MT}}(\Omega)) \), and since \( f_n \) converges strongly in \( L^1(\Omega) \), by Lebesgue’s theorem we have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx = \lim_{m \to \infty} \lim_{n \to \infty} \langle f_n, \theta_m(u_n) \rangle = 0.
\]
By (3.3) we finally have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \tag{4.9}
\]

**Step 5:** Almost everywhere convergence of the gradients. Define
\[
\phi(s) = se^{\lambda s^2} \text{ with } \lambda = \left(\frac{b(k)}{2\alpha}\right)^2. \quad \text{One can easily verify that for all } s \in \mathbb{R}
\]
\[
\phi'(s) - \frac{b(k)}{\alpha} |\phi(s)| \geq \frac{1}{2}. \tag{4.10}
\]
For \(m \geq k\), we define the function \(\psi_m\) by
\[
\left\{ \begin{array}{ll}
\psi_m(s) = 1 & \text{if } |s| \leq m, \\
\psi_m(s) = m + 1 - |s| & \text{if } m \leq |s| \leq m + 1, \\
\psi_m(s) = 0 & \text{if } |s| \geq m + 1.
\end{array} \right.
\]
By virtue of [21, Theorem 4] there exists a sequence \(\{v_j\}_j \subset D(\Omega)\) such that
\[
v_j \to u \text{ in } W^1_0 L^1(\Omega) \text{ for the modular convergence and a.e. in } \Omega.
\]
Let us define the following functions \(\theta^j_n = T_k(u_n) - T_k(v_j), \theta^j = T_k(u) - T_k(v_j)\) and \(z^j_{n,m} = \phi(\theta^j_n)\psi_m(u_n)\). Using \(z^j_{n,m} \in W^1_0 L^1(\Omega)\) as test function in (4.2) we get
\[
\int_\Omega a(x, u_n, \nabla u_n) \nabla z^j_{n,m} dx + \int_\Omega f(x, u_n, \nabla u_n) \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx \\
+ \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \nabla \psi_m(u_n) \phi(T_k(u_n) - T_k(v_j)) dx \\
+ \int_\Omega g_n(x, u_n, \nabla u_n) z^j_{n,m} dx = \int_\Omega f_n z^j_{n,m} dx + \int_\Omega F \nabla z^j_{n,m} dx. \tag{4.11}
\]
From now on we denote by \(\epsilon_i(n, j), i = 0, 1, 2, ..., \) various sequences of real numbers which tend to zero, when \(n \to +\infty\), i.e.
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \epsilon_i(n, j) = 0.
\]
In view of (4.7), we have \(z^j_{n,m} \to \phi(\theta^j)\psi_m(u)\) weakly in \(L^\infty(\Omega)\) for \(\sigma^*(L^\infty, L^1)\) as \(n \to +\infty\), which yields
\[
\lim_{n \to +\infty} \int_\Omega f_n z^j_{n,m} dx = \int_\Omega f \phi(\theta^j)\psi_m(u) dx,
\]
and since \(\phi(\theta^j) \to 0\) weakly in \(L^\infty(\Omega)\) for \(\sigma(L^\infty, L^1)\) as \(j \to +\infty\), we have
\[
\lim_{j \to +\infty} \int_\Omega f \phi(\theta^j)\psi_m(u) dx = 0.
\]
Thus, we write
\[
\int_\Omega f_n z^j_{n,m} dx = \epsilon_0(n, j).
\]
Thanks to (4.5) and (4.7), we have as \(n \to +\infty,\)
\[
z^j_{n,m} \to \phi(\theta^j)\psi_m(u) \text{ in } W^1_0 L^1(\Omega) \text{ for } \sigma(IL_M(\Omega), IE_{M'}(\Omega)),
\]
which implies that
\[
\lim_{n \to +\infty} \int_{\Omega} F \nabla z_{n,m}^j dx = \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx + \int_{\Omega} F \nabla u \phi'(\theta^j) \psi_m(u) dx
\]
On the one hand, by Lebesgue’s theorem we get
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx = 0
\]
on the other hand, we write
\[
\int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx = \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx - \int_{\Omega} F \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx,
\]
so that, by Lebesgue’s theorem one has
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) dx.
\]
Let \( \lambda > 0 \) such that \( M\left(\nabla v_j - \nabla u\right) \to 0 \) strongly in \( L^1(\Omega) \) as \( j \to +\infty \) and
\[
M\left(\frac{\nabla v_j - \nabla u}{\lambda}\right) \in L^1(\Omega), \text{ the convexity of the N-function } M \text{ allows us to have}
\]
\[
M\left(\frac{\nabla v_j - \nabla u}{\lambda}\right) \leq \frac{1}{4} M\left(\frac{\nabla v_j - \nabla u}{\lambda}\right) + \frac{1}{4} \left(1 + \frac{1}{\sigma(2\xi)}\right) M\left(\frac{\nabla v_j - \nabla u}{\lambda}\right).
\]
Then, by using the modular convergence of \( \{\nabla v_j\} \) in \((L_M(\Omega))^N\) and Vitali’s theorem, we obtain
\[
\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) = 0 \text{ in } \Omega, \quad \text{as } j \text{ tends to } +\infty,
\]
for the modular convergence, and then
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) dx.
\]
We have proved that
\[
\int_{\Omega} F \nabla z_{n,m}^j dx = \epsilon_1(n,j).
\]
It’s easy to see that by the modular convergence of the sequence \( \{v_j\} \), one has
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \psi_m'(u_n) \psi(T_k(u_n) - T_k(v_j)) dx = 0,
\]
while for the third term in the left-hand side of (4.11) we can write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx
\]
\[
= \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) dx - \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx.
\]
Firstly, we have
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta^j_n) \psi_m(u_n) dx = 0.
\]
In view of (4.7), one has
\[
\Phi_n(u_n) \phi'(\theta^j_n) \psi_m(u_n) \to \Phi(u) \phi'(\theta^j) \psi_m(u),
\]
almost everywhere in \( \Omega \) as \( n \) tends to +\( \infty \). Furthermore, we can check that
\[
\| \Phi_n(u_n) \phi'(\theta^j_n) \psi_m(u_n) \|_{\mathcal{M}} \leq \overline{M} (c_m \phi'(2k)|\Omega| + 1),
\]
where \( c_m = \max_{t \leq m+1} \Phi(t) \). Applying [27, Theorem 14.6] we get
\[
\lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx.
\]
Using the modular convergence of the sequence \( \{v_j\}_j \), we obtain
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx.
\]
Then, using again the Divergence theorem we get
\[
\int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx = 0.
\]
Therefore, we write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx = \epsilon_2(n, j).
\]
Since \( g_n(x, u_n, \nabla u_n) z^j_{n,m} \geq 0 \) on the set \( \{ |u_n| > k \} \) and \( \psi_m(u_n) = 0 \) on the set \( \{ |u_n| \leq k \} \), from (4.11) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z^j_{n,m} dx + \int_{\{|u_n| \leq k \}} g_n(x, u_n, \nabla u_n) \phi(\theta^j_n) dx \leq \epsilon_3(n, j). \tag{4.12}
\]
We now evaluate the first term of the left-hand side of (4.12) by writing
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z^j_{n,m} dx
\]
\[
= \int_{\Omega} a(x, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta^j_n) \psi_m(u_n) dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^j_n) \psi_m(u_n) dx
\]
\[
= \int_{\{|u_n| > k \}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta^j_n) dx
\]
\[
- \int_{\{|u_n| \leq k \}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \psi_m(u_n) dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^j_n) \psi_m(u_n) dx.
\]
and then
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_{n,m} \, dx
= \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(v)) - a(x, T_k(u_n), \nabla T_k(v_j)) \right)
\n\left( \nabla T_k(u_n) - \nabla T_k(v_j) \right) \phi'(\theta_n^j) \, dx
\n+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \right) \phi'(\theta_n^j) \, dx
\n- \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \phi'(\theta_n^j) \, dx
\n- \int_{\{u_n > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx
\n+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) \, dx,
\]
(4.13)

where by \( \chi^s_j, s > 0 \), we denote the characteristic function of the subset
\[
\Omega^s_j = \{ x \in \Omega : |\nabla T_k(v_j)| \leq s \}.
\]

For fixed \( m \) and \( s \), we will pass to the limit in \( n \) and then in \( j \) in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \right) \phi'(\theta_n^j) \, dx
\n\to \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u) - \nabla T_k(v_j) \right) \phi'(\theta') \, dx,
\]
as \( n \to +\infty \). Since by lemma (2.4) one has
\[
a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) \to a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \phi'(\theta'),
\]
strongly in \( (E_{2,\Omega})^N \) as \( n \to \infty \), while by (4.5)
\[
\nabla T_k(u_n) \to \nabla T_k(u),
\]
weakly in \( (L_M(\Omega))^N \). Let \( \chi^s \) denote the characteristic function of the subset
\[
\Omega^s = \{ x \in \Omega : |\nabla T_k(u)| \leq s \}.
\]
As \( \nabla T_k(v_j) \chi_j^s \to \nabla T_k(u) \chi^s \) strongly in \( (E_M(\Omega))^N \) as \( j \to +\infty \), one has
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u) - \nabla T_k(v_j) \chi_j^s \right) \phi'(\theta') \, dx \to 0,
\]
as \( j \to \infty \). Then
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) \phi'(\theta_n^j) \, dx = \epsilon_4(n, j). \quad (4.14)
\]

We now estimate the third term of (4.13). It’s easy to see that by (3.3), \( a(x, s, 0) = 0 \) for almost everywhere \( x \in \Omega \) and for all \( s \in \mathbb{R} \). Thus, from (4.8) we have that \( (a(x, T_k(u_n), \nabla T_k(u_n)) \, n \) is bounded in \( (L_{\infty}(\Omega))^N \) for all \( k \geq 0 \).
Therefore, there exist a subsequence still indexed by \( n \) and a function \( l_k \) in \((L^\infty(\Omega))^N\) such that

\[
a(x, T_k(u_n), \nabla T_k(u_n)) \to l_k \text{ weakly in } (L^\infty(\Omega))^N \text{ for } \sigma(\Pi L^\infty, \Pi E_M). \quad (4.15)
\]

Then, since \( \nabla T_k(v_j) \chi_{\Omega \setminus \Omega_j^c} \in (E^\infty(\Omega))^N \), we obtain

\[
\int_{\Omega \setminus \Omega_j^c} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx \to \int_{\Omega \setminus \Omega_j^c} l_k \nabla T_k(v_j) \phi'(\theta^j) dx,
\]
as \( n \to +\infty \). The modular convergence of \( \{v_j\} \) allows us to get

\[
- \int_{\Omega \setminus \Omega_j^c} l_k \nabla T_k(v_j) \phi'(\theta^j) dx \to - \int_{\Omega \setminus \Omega} l_k \nabla T_k(u) dx,
\]
as \( j \to +\infty \). This proves

\[
- \int_{\Omega \setminus \Omega_j^c} a(x, T_k(u_n), \nabla T_k(v_j)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = - \int_{\Omega \setminus \Omega} l_k \nabla T_k(u) dx + \epsilon_5(n, j).
\]

(4.16)

As regards the fourth term, observe that \( \psi_m(u_n) = 0 \) on the subset \( \{|u_n| \geq m + 1\} \), so we have

\[
- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = - \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx.
\]

Since

\[
- \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = - \int_{\{|u| > k\}} l_{m+1} \nabla T_k(u) \psi_m(u) dx + \epsilon_5(n, j),
\]
observing that \( \nabla T_k(u) = 0 \) on the subset \( \{|u| > k\} \), one has

\[
- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \epsilon_6(n, j). \quad (4.17)
\]

For the last term of (4.13), we have

\[
\left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \right| = \left| \int_{\{|m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \right| \leq \phi(2k) \int_{\{|m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx.
\]
To estimate the last term of the previous inequality, we use $(T_1(u_n - T_m(u_n)) \in W^1_0 L^1(\Omega))$ as test function in (4.2), to get
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\{|u_n| \geq m\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) dx = \langle f_n, T_1(u_n - T_m(u_n)) \rangle
\]
\[
+ \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n dx.
\]
By Divergence theorem, we have
\[
\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n dx = 0.
\]
Using the fact that $g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \geq 0$ on the subset $\{|u_n| \geq m\}$ and Young’s inequality, we get
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx
\]
\[
\leq \langle f_n, T_1(u_n - T_m(u_n)) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx.
\]
It follows that
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi'(\theta^n) \psi'_m(u_n) dx
\]
\[
\leq 2\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right). 
\]
From (4.14), (4.16), (4.17) and (4.18) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z^{j}_{n,m} dx
\]
\[
\geq \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)) \psi'_m(u_n) \right)
\]
\[
(\nabla T_k(u_n) - \nabla T_k(v_j) \chi^j) \phi'(\theta^n) dx
\]
\[
- \alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right)
\]
\[
- \int_{\Omega \setminus \Omega'} l_k \cdot \nabla T_k(u) dx + \epsilon_r(n,j).
\]
(4.19)
Now, we turn to second term in the left-hand side of (4.12). We have
\[
\int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta^n) dx
\]
\[
= \int_{\{|u_n| \leq k\}} g_n(x, T_k(u_n), \nabla T_k(u_n)) \phi(\theta^n) dx
\]
\[
\leq \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \phi(\theta^n) dx + \epsilon_s(n,j).
\]
Then
\[
\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta^i_n) \, dx \right| \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} \left( a(x, T_k(u_n)), \nabla T_k(u_n) \right) - a(x, T_k(u_n)), \nabla T_k(v_j) \chi_j^k) \\
(\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^k) \phi(\theta^i_n) \, dx \\
+ \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n)), \nabla T_k(v_j) \chi_j^k) \, dx \\
+ \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n)), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^k \phi(\theta^i_n) \, dx \\
\leq b(k) \alpha \int_{\Omega} \left( a(x, T_k(u_n)), \nabla T_k(u_n) \right) - a(x, T_k(u_n)), \nabla T_k(v_j) \chi_j^k) \\
\phi(\theta^i_n) \, dx = \epsilon_9(n, j). 
\]

We proceed as above to get
\[
\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n)), \nabla T_k(v_j) \chi_j^k) \, dx = \epsilon_{10}(n, j). 
\]

Hence, we have
\[
\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta^i_n) \, dx \right| \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} \left( a(x, T_k(u_n)), \nabla T_k(u_n) \right) - a(x, T_k(u_n)), \nabla T_k(v_j) \chi_j^k) \\
(\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^k) \phi(\theta^i_n) \, dx + \epsilon_{11}(n, j). 
\]

Combining (4.12), (4.19) and (4.21), we get
\[
\left( \phi(\theta^i_n) - \frac{b(k)}{\alpha} \phi(\theta^i_n) \right) \, dx \\
\leq \int_{\Omega \setminus \Omega^*} l_k \nabla T_k(u) \, dx + \alpha \phi(2k) \left( \int_{\{|u_n| \leq m\}} f_n \, dx + \int_{\{|u_n| \leq m+1\}} \overline{M}(|F|) \, dx \right) \\
+ \epsilon_{12}(n, j). 
\]

By (4.10), we have
\[
\left| \int_{\Omega \setminus \Omega^*} (a(x, T_k(u_n)), \nabla T_k(v_j) \chi_j^k) \, dx \\
\right| \leq 2 \int_{\Omega \setminus \Omega^*} l_k \nabla T_k(u) \, dx + 4\alpha \phi(2k) \left( \int_{\{|u_n| \leq m\}} f_n \, dx + \int_{\{|u_n| \leq m+1\}} \overline{M}(|F|) \, dx \right) \\
+ \epsilon_{12}(n, j). 
\]

(4.22)
On the other hand we can write
\[
\int_{\Omega} (a(x, T_k(u_n)), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi^s) \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx = 0
\]
\[
\int_{\Omega} (a(x, T_k(u_n)), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_{j}^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_{j}^s \right) dx
\]
\[
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_{j}^s) \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx
\]
\[
- \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u) \chi^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_{j}^s \right) dx
\]
\[
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_{j}^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_{j}^s \right) dx
\]

We shall pass to the limit in \( n \) and then in \( j \) in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \left( \nabla T_k(v_j) \chi_{j}^s - \nabla T_k(u) \chi^s \right) dx = \epsilon_{13}(n, j),
\]
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u) \chi^s) \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx = \epsilon_{14}(n, j),
\]
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_{j}^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_{j}^s \right) dx = \epsilon_{15}(n, j).
\]

So that
\[
\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi^s)) \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx
\]
\[
= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_{j}^s)) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_{j}^s \right) dx
\]
\[
+ \epsilon_{16}(n, j).
\]

Let \( r \leq s \). Using (3.2), (4.22) and (4.24) we can write
\[
0 \leq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \left( \nabla T_k(u_n) - \nabla T_k(u) \right) dx
\]
\[
\leq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \left( \nabla T_k(u_n) - \nabla T_k(u) \right) dx
\]
\[
= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi^s)) \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx
\]
\[
\leq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi^s)) \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx
\]
\[
+ \epsilon_{15}(n, j)
\]
\[
\leq 2 \int_{\Omega} l_k \nabla T_k(u) dx + 2\alpha\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right)
\]
\[
+ \epsilon_{17}(n, j).
\]
By passing to the superior limit over \( n \) and then over \( j \)
\[
0 \leq \limsup_{n \to +\infty} \int_{\Omega'} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u), \nabla T_k(u)) \right) (\nabla T_k(u_n) - \nabla T_k(u)) dx
\]
\[
\leq 2 \int_{\Omega' \setminus \Omega} l_k \nabla T_k(u) dx + 4\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right).
\]
Letting \( s \to +\infty \) and then \( m \to +\infty \), taking into account that \( l_k \nabla T_k(u) \in L^1(\Omega), f \in L^1(\Omega), |F| \in (E_{\overline{M}}(\Omega))^N, |\Omega \setminus \Omega'| \to 0, \) \( \{m \leq |u| \leq m+1\} \to 0, \) one has
\[
\int_{\Omega'} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right) (\nabla T_k(u_n) - \nabla T_k(u)) dx,
\]
(4.25)
tends to 0 as \( n \to +\infty \). As in [20], we deduce that there exists a subsequence of \( \{u_n\} \) still indexed by \( n \) such that
\[
\nabla u_n \rightharpoonup \nabla u \text{ a. e. in } \Omega.
\]
(4.26)
Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get
\[
a(x, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N
\]
and
\[
a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M).
\]
(4.27)

**Step 6: Modular convergence of the truncations.** Going back to equation (4.22), we can write
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx
\]
\[
\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx
\]
\[
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx
\]
\[
+ 2\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right)
\]
\[
+ 2 \int_{\Omega' \setminus \Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{12}(n,j).
\]
By (4.23) we get
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx
\]
\[
\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx
\]
\[
+ 2\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right)
\]
\[
+ 2 \int_{\Omega' \setminus \Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{18}(n,j).
\]
We now pass to the superior limit over $n$ in both sides of this inequality using (4.27), to obtain
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \\
+ 2\alpha(2k) \left( \int_{\{m \leq |u|\}} |f| \, dx + \int_{\{m \leq |u| \leq m+1\}} M(|F|) \, dx \right) \\
+ 2\int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]
We then pass to the limit in $j$ to get
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \\
+ 2\alpha(2k) \left( \int_{\{m \leq |u|\}} |f| \, dx + \int_{\{m \leq |u| \leq m+1\}} M(|F|) \, dx \right) \\
+ 2\int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]
Letting $s$ and then $m \to +\infty$, one has
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]
On the other hand, by (3.3), (4.5), (4.26) and Fatou’s lemma, we have
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \leq \liminf_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx.
\]
It follows that
\[
\lim_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx = \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]
By Lemma 2.5 we conclude that for every $k > 0$
\[
a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u),
\]
strongly in $L^1(\Omega)$. The convexity of the $N$-function $M$ and (3.3) allow us to have
\[
M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) \\
\leq \frac{1}{2\alpha} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) + \frac{1}{2\alpha} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u).
\]
From Vitali’s theorem we deduce
\[
\limsup_{|E| \to 0} \int_{E} M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) \, dx = 0.
\]
Thus, for every $k > 0$
\[
T_k(u_n) \to T_k(u) \text{ in } W^1_0 L_M(\Omega),
\]
for the modular convergence.

**Step 7: Compactness of the nonlinearities.** We need to prove that

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]  

(4.29)

By virtue of (4.7) and (4.26) one has

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a.e. in } \Omega. \]  

(4.30)

Let \( E \) be measurable subset of \( \Omega \) and let \( m > 0 \). Using (3.3) and (3.4) we can write

\[
\int_E |g_n(x, u_n, \nabla u_n)| \, dx \\
= \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| \, dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| \, dx \\
\leq b(m) \int_E d(x) \, dx + b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx \\
+ \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx.
\]

From (3.5) and (4.6), we deduce that

\[ 0 \leq \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx \leq C_3. \]

So

\[ 0 \leq \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx \leq \frac{C_3}{m}. \]

Then

\[ \lim_{m \to +\infty} \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx = 0. \]

Thanks to (4.28) the sequence \( \{a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)\}_n \) is equi-integrable. This fact allows us to get

\[ \lim_{|E| \to 0} \sup_n \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) \, dx = 0. \]

This shows that \( g_n(x, u_n, \nabla u_n) \) is equi-integrable. Thus, Vitali’s theorem implies that \( g(x, u, \nabla u) \in L^1(\Omega) \) and

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]

**Step 8: Renormalization identity for the solutions.** In this step we prove that

\[ \lim_{m \to +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \, dx = 0. \]  

(4.31)
Indeed, for any $m \geq 0$ we can write
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) \, dx
= \int_{\Omega} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) \, dx
- \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx.
\]

In view of (4.28), we can pass to the limit as $n$ tends to $+\infty$ for fixed $m \geq 0$
\[
\lim_{n \to +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
= \int_{\Omega} a(x, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_{m+1}(u) \, dx
- \int_{\Omega} a(x, T_m(u), \nabla T_m(u)) \nabla T_m(u) \, dx
= \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx.
\]

Having in mind (4.9), we can pass to the limit as $m$ tends to $+\infty$ to obtain (4.31).

**Step 9: Passing to the limit.** Thanks to (4.28) and Lemma (2.5), we obtain
\[
a(x, u_n, \nabla u_n) \nabla u_n \to a(x, u, \nabla u) \nabla u \text{ strongly in } L^1(\Omega). \tag{4.32}
\]

Let $h \in C^1_0(\mathbb{R})$ and $\varphi \in \mathcal{D}(\Omega)$. Inserting $h(u_n)\varphi$ as test function in (4.2), we get
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi \, dx + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) \, dx
+ \int_{\Omega} \Phi_n(u_n) \nabla (h(u_n) \varphi) \, dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx
= (f_n, h(u_n) \varphi) + \int_{\Omega} F \nabla (h(u_n) \varphi) \, dx. \tag{4.33}
\]

We shall pass to the limit as $n \to +\infty$ in each term of the equality (4.33).

Since $h$ and $h'$ have compact support on $\mathbb{R}$, there exists a real number $\nu > 0$, such that $\text{supp } h \subset [-\nu, \nu]$ and $\text{supp } h' \subset [-\nu, \nu]$. For $n > \nu$, we can write
\[
\Phi_n(t) h(t) = \Phi(T_v(t)) h(t) \text{ and } \Phi_n(t) h'(t) = \Phi(T_v(t)) h'(t).
\]

Moreover, the functions $\Phi h$ and $\Phi h'$ belong to $(C^0(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))^N$. Observe first that the sequence $\{h(u_n)\varphi\}_n$ is bounded in $W^1_0 L^\rho(\Omega)$. Indeed, let $\rho > 0$
be a positive constant such that \( \|h(u_n)\nabla \phi\|_\infty \leq \rho \) and \( \|h'(u_n)\phi\|_\infty \leq \rho \). Using the convexity of the \( N \)-function \( M \) and taking into account (4.5) we have
\[
\int_\Omega M \left( \frac{\|h(u_n)\nabla \phi\|}{2\rho} \right) dx \leq \int_\Omega M \left( \frac{\|h(u_n)\nabla \phi\| + \|h'(u_n)\phi\|\nabla u_n\|}{2\rho} \right) dx
\]
\[
\leq \frac{1}{2} M(1)\|\phi\|_\Omega + \frac{1}{2} \int_\Omega (\|\nabla u_n\|) dx
\]
\[
\leq \frac{1}{2} M(1)\|\phi\|_\Omega + \frac{1}{2} C_2.
\]
This, together with (4.7), imply that
\[
\phi(u_n) \phi \to \phi(u) \phi \text{ weakly in } W^1_0 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{M}).
\]
(4.34)
This enables us to get
\[
\langle f_n, h(u_n)\phi \rangle \to \langle f, h(u)\phi \rangle.
\]
Let \( E \) be a measurable subset of \( \Omega \). Define \( c_\nu = \max_{|t| \leq 1} \Phi(t) \). Let us denote by \( \|v\|_M \) the Orlicz norm of a function \( v \in L_M(\Omega) \). Using strengthened Hölder inequality with both Orlicz and Luxemburg norms, we get
\[
\|\Phi(T_\nu(u_n))\chi_E\|_{(M)} = \sup_{\|v\|_M \leq 1} \left| \int_E \Phi(T_\nu(u_n))\chi_E \right|
\]
\[
\leq c_\nu \sup_{\|v\|_M \leq 1} \|\chi_E\|_{(M)} \|v\|_M
\]
\[
\leq c_\nu |E|M^{-1} \left( \frac{1}{|E|} \right).
\]
Thus, we get
\[
\lim_{|E| \to 0} \sup_n \|\Phi(T_\nu(u_n))\chi_E\|_{(M)} = 0.
\]
Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain
\[
\Phi(T_\nu(u_n)) \to \Phi(T_\nu(u)) \text{ strongly in } (E_{M^\ast})^N,
\]
which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have
\[
\int_\Omega \Phi(T_\nu(u_n))\nabla (h(u_n)\phi) dx \to \int_\Omega \Phi(T_\nu(u))\nabla (h(u)\phi) dx.
\]
We remark that
\[
|a(x, u_n, \nabla u_n) \nabla u_n h'(u_n)\phi| \leq \rho a(x, u_n, \nabla u_n) \nabla u_n.
\]
Consequently, using (4.32) and Vitali’s theorem, we obtain
\[
\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n h'(u_n)\phi dx \to \int_\Omega a(x, u, \nabla u) \nabla uh'(u)\phi dx.
\]
and
\[
\int_\Omega F\nabla u_n h'(u_n)\phi dx \to \int_\Omega F\nabla uh'(u)\phi dx.
\]
For the second term of (4.33), as above we have
\[
h(u_n)\nabla \phi \to h(u)\nabla \phi \text{ strongly in } (E_M(\Omega))^N.
\]
which together with (4.27) give
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) \, dx \to \int_{\Omega} a(x, u, \nabla u) \nabla \varphi h(u) \, dx
\]
and
\[
\int_{\Omega} F \nabla \varphi h(u_n) \, dx \to \int_{\Omega} F \nabla \varphi h(u) \, dx.
\]
The fact that \( h(u_n) \varphi \rightharpoonup h(u) \varphi \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get
\[
\int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \, dx \to \int_{\Omega} g(x, u, \nabla u) h(u) \, dx.
\]
At this point we can pass to the limit in each term of (4.33) to get
\[
\int_{\Omega} a(x, u, \nabla u)(\nabla \varphi h(u) + h'(u) \varphi \nabla u) \, dx + \int_{\Omega} \Phi(u) h'(u) \varphi \nabla u \, dx + \int_{\Omega} g(x, u, \nabla u) h(u) \, dx = (f, h(u) \varphi) + \int_{\Omega} F(\nabla \varphi h(u) + h'(u) \varphi \nabla u) \, dx,
\]
for all \( h \in C^1_c(\mathbb{R}) \) and for all \( \varphi \in \mathcal{D}(\Omega) \). Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou’s lemma to get \( g(x, u, \nabla u) u \in L^1(\Omega) \). By virtue of (4.7), (4.27), (4.29), (4.31), the function \( u \) is a renormalized solution of problem (1.1).

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REFERENCES


