Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces

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Abstract. The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems

\[ -\text{div} \left( a(x, u, \nabla u) + \Phi(u) \right) + g(x, u, \nabla u) = f - \text{div} F, \]

in a bounded open set \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), in the framework of Orlicz-Sobolev spaces without any restriction on the \( M \)-function of the Orlicz spaces, where \( -\text{div} \left( a(x, u, \nabla u) \right) \) is a Leray-Lions operator defined from \( W_{10}^{1}L_{M}(\Omega) \) into its dual, \( \Phi \in C^{0}(\mathbb{R}, \mathbb{R}^{N}) \). The function \( g(x, u, \nabla u) \) is a non linear lower order term with natural growth with respect to \( |\nabla u| \), satisfying the sign condition and the datum \( \mu \) is assumed to belong to \( L^{1}(\Omega) + W^{-1,E}_{1}(\Omega) \).

Keywords: Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.


1. Introduction

Let \( \Omega \) be a bounded open set of \( \mathbb{R}^{N} \), \( N \geq 2 \), and let \( M \) be an \( N \)-function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem

\[
\begin{cases}
A(u) - \text{div} \Phi(u) + g(x, u, \nabla u) = f - \text{div} F & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

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Here, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$, while the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The non everywhere defined nonlinear operator $A(u) = -\text{div} \left( a(x, u, \nabla u) \right)$ acts from its domain $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$. The function $a(x, u, \nabla u)$ is assumed to satisfy, among others, $a(x, u, \nabla u)$ nonstandard growth condition governed by the $N$-function $M$, and the source term $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$.

We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with $\Phi \equiv 0$, $g \equiv 0$ and $L^1(\Omega)$-data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when $f$ is a bounded Radon measure datum and $g$ grows at most like $|\nabla u|^{p-1}$ by Beta et al. in [9, 10, 11] with $\Phi \equiv 0$ and by Guibé and Mercaldo in [23, 24] when $\Phi(u)$ behaves at most like $|u|^{p-1}$. Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in $L^1(\Omega)$. Recall that to get both existence and uniqueness of a solution to problems with $L^1$-data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall’Aglio [14].

The authors in [5] have dealt with the equation (1.1) with $g = g(x, u)$ and $\mu \in W^{-1} L_{\overline{M}}(\Omega)$, under the restriction that the $N$-function $M$ satisfies the $\Delta_2$-condition. This work was then extended in [2] for $N$-functions not satisfying necessarily the $\Delta_2$-condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the $N$-function $M$. Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, an existence result has been proved in [3]. Specific examples to which our results apply include the following:

$$- \text{div} \left( |\nabla u|^{p-2} \nabla u + |u|^s u \right) + u |\nabla u|^p = \mu \text{ in } \Omega,$$

$$- \text{div} \left( |\nabla u|^{p-2} \nabla u \log(1 + |\nabla u|) + |u|^s u \right) = \mu \text{ in } \Omega,$$

$$- \text{div} \left( \frac{M(|\nabla u|) \nabla u}{|\nabla u|^2} + |u|^s u \right) + M(|\nabla u|) = \mu \text{ in } \Omega,$$

where $p > 1$, $s > 0$, $\beta > 0$ and $\mu$ is a given Radon measure on $\Omega$.

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form $f = -\text{div} F$ with $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, in the setting of Orlicz spaces without any restriction on the $N$-functions $M$. The approximate equations provide a $W_0^1 L_M(\Omega)$ bound for the corresponding solution $u_n$. This allows us to obtain
a function \( u \) as a limit of the sequence \( u_n \). Hence, appear two difficulties. The first one is how to give a sense to \( \Phi(u) \), the second difficulty lies in the need of the convergence almost everywhere of the gradients of \( u_n \) in \( \Omega \). This is done by using suitable test functions built upon \( u_n \) which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

2. Preliminaries

Let \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) be an \( N \)-function, i.e., \( M \) is continuous, increasing, convex, with \( M(t) > 0 \) for \( t > 0 \), \( \frac{M(t)}{t} \to 0 \) as \( t \to 0 \), and \( \frac{M(t)}{t} \to +\infty \) as \( t \to +\infty \). Equivalently, \( M \) admits the representation:

\[
M(t) = \int_0^t a(s) \, ds,
\]

where \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) is increasing, right continuous, with \( a(0) = 0 \), \( a(t) > 0 \) for \( t > 0 \) and \( a(t) \) tends to \( +\infty \) as \( t \to +\infty \).

The conjugate of \( M \) is also an \( N \)-function and it is defined by \( \overline{M} = \int_0^t \bar{a}(s) \, ds \), where \( \bar{a} : \mathbb{R}^+ \to \mathbb{R}^+ \) is the function \( \bar{a}(t) = \sup\{ s : a(s) \leq t \} \) (see [1]).

An \( N \)-function \( M \) is said to satisfy the \( \Delta_2 \)-condition if, for some \( k \),

\[
M(2t) \leq kM(t) \quad \forall t \geq 0,
\]

(2.1)

when (2.1) holds only for \( t \geq t_0 > 0 \) then \( M \) is said to satisfy the \( \Delta_2 \)-condition near infinity. Moreover, we have the following Young’s inequality

\[
st \leq M(t) + \overline{M}(s), \quad \forall s, t \geq 0.
\]

Given two \( N \)-functions, we write \( P \ll Q \) to indicate \( P \) grows essentially less rapidly than \( Q \); i.e. for each \( \epsilon > 0 \), \( \frac{P(t)}{Q(\epsilon t)} \to 0 \) as \( t \to +\infty \). This is the case if and only if

\[
\lim_{t \to \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.
\]

Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). The Orlicz class \( k_M(\Omega) \) (resp. the Orlicz space \( L_M(\Omega) \)) is defined as the set of (equivalence classes of) real valued measurable functions \( u \) on \( \Omega \) such that

\[
\int_{\Omega} M(|u(x)|) \, dx < +\infty \quad \text{(resp.} \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right) \, dx < +\infty \text{ for some } \lambda > 0).\]
The set $L_M(\Omega)$ is a Banach space under the norm
\[
\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\},
\]
and $k_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_{\Omega} uv \, dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1 L_M(\Omega)$ [resp. $W^1 E_M(\Omega)$] is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm
\[
\|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}.
\]

Thus, $W^1 L_M(\Omega)$ and $W^1 E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W^1_0 E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1 E_M(\Omega)$ and the space $W^1_0 L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1 L_M(\Omega)$. We say that $u_n$ converges to $u$ for the modular convergence in $W^1 L_M(\Omega)$ if for some $\lambda > 0$, $\int_{\Omega} M \left( \frac{D^\alpha u_n - D^\alpha u}{\lambda} \right) \, dx \to 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\prod L_M, \prod L_{\overline{M}})$. If $M$ satisfies the $\Delta_2$ condition on $\mathbb{R}^+$ (near infinity only when $\Omega$ has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1} L_{\overline{M}}(\Omega)$ [resp. $W^{-1} E_{\overline{M}}(\Omega)$] denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain $\Omega$ has the segment property if for every $x \in \partial \Omega$ there exists an open set $G_x$ and a nonzero vector $y_x$ such that $x \in G_x$ and if $z \in \overline{G} \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$. The following lemmas can be found in [6].

**Lemma 2.1.** Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $M$ be an $N$-function and let $u \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$). Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then

\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{ x \in \Omega : u(x) \notin D \}, \\ 0 & \text{a.e. in } \{ x \in \Omega : u(x) \in D \}. \end{cases}
\]
Lemma 2.2. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of $F'$ is finite. Let $M$ be an $N$-function, then the mapping $F : W^{1}L_{M}(\Omega) \to W^{1}L_{M}(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_{M}, \prod E_{\mathcal{M}})$.

Lemma 2.3. ([21]) Let $\Omega$ have the segment property. Then for each $\nu \in W_{0}^{1}L_{M}(\Omega)$, there exists a sequence $\nu_{n} \in \mathcal{D}(\Omega)$ such that $\nu_{n}$ converges to $\nu$ for the modular convergence in $W_{0}^{1}L_{M}(\Omega)$. Furthermore, if $\nu \in W_{0}^{1}L_{M}(\Omega) \cap L^{\infty}(\Omega)$, then

$$
\|\nu_{n}\|_{L^{\infty}(\Omega)} \leq (N + 1)\|\nu\|_{L^{\infty}(\Omega)}.
$$

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.4. Let $\Omega$ be an open subset of $\mathbb{R}^{N}$ with finite measure. Let $M, P, Q$ be $N$-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:

$$
|f(x, s)| \leq c(x) + k_{1}P^{-1}M(k_{2}|s|),
$$

where $k_{1}, k_{2}$ are real constants and $c(x) \in E_{Q}(\Omega)$.

Then the Nemytskii operator $N_{f}$ defined by $N_{f}(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_{M}(\Omega), \frac{1}{k_{2}}) = \{u \in L_{M}(\Omega) : d(u, E_{M}(\Omega)) < \frac{1}{k_{2}}\}$ into $E_{Q}(\Omega)$.

We will also use the following technical lemma.

Lemma 2.5. ([26]) If $\{f_{n}\} \subset L^{1}(\Omega)$ with $f_{n} \to f \in L^{1}(\Omega)$ a.e. in $\Omega$, $f_{n}, f \geq 0$ a.e. in $\Omega$ and $\int_{\Omega}f_{n}(x)\,dx \to \int_{\Omega}f(x)\,dx$, then

$$
f_{n} \to f \text{ in } L^{1}(\Omega).
$$

3. STRUCTURAL ASSUMPTIONS AND MAIN RESULT

Throughout the paper $\Omega$ will be a bounded subset of $\mathbb{R}^{N}$, $N \geq 2$, satisfying the segment property. Let $M$ and $P$ be two $N$-functions such that $P \ll M$. Let $A$ be the non everywhere defined operator defined from its domain $\mathcal{D}(\Omega) \subset W_{0}^{1}L_{M}(\Omega)$ into $W^{-1}L_{\mathcal{M}}(\Omega)$ given by

$$
A(u) := -\text{div } a(\cdot, u, \nabla u),
$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^{N} \to \mathbb{R}^{N}$ is a Carathéodory function. We assume that there exist a nonnegative function $c(x)$ in $E_{\mathcal{M}}(\Omega)$, $\alpha > 0$ and positive real constants $k_{1}, k_{2}, k_{3}$ and $k_{4}$, such that for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^{N}$, $\xi' \in \mathbb{R}^{N}$ ($\xi \neq \xi'$) and for almost every $x \in \Omega$

$$
|a(x, s, \xi)| \leq c(x) + k_{1}\mathcal{P}^{-1}M(k_{2}|s|) + k_{3}\mathcal{M}^{-1}M(k_{4}|\xi|),
$$

(3.1)
(a(x, s, ξ) − a(x, s, ξ′))(ξ − ξ′) > 0, \quad (3.2)
\quad a(x, s, ξ)ξ ≥ αM(|ξ|). \quad (3.3)

Here, \( g(x, s, ξ) : Ω × R × RN → R \) is a Carathéodory function satisfying for almost every \( x ∈ Ω \) and for all \( s ∈ R, ξ ∈ RN, \)
\[
|g(x, s, ξ)| ≤ b(|s|)(d(x) + M(|ξ|)), \quad (3.4)
\]
\[
g(x, s, ξ)s ≥ 0, \quad (3.5)
\]
where \( b : R → R^+ \) is a continuous and increasing function while \( d \) is a given nonnegative function in \( L^1(Ω) \).

The right-hand side of (1.1) and \( Φ : R → RN, \) are assumed to satisfy
\[
f ∈ L^1(Ω) \text{ and } |F| ∈ E^{M}(Ω), \quad (3.6)
\]
\[
Φ ∈ C^0(R, RN). \quad (3.7)
\]

Our aim in this paper is to give a meaning to a possible solution of (1.1).
In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution \( u \) of problem (1.1) is the Orlicz-Sobolev space \( W^1_0 L_M(Ω) \). But when \( u \) is only in \( W^1_0 L_M(Ω) \) there is no reason for \( Φ(u) \) to be in \( (L^1(Ω))^N \) since no growth hypothesis is assumed on the function \( Φ \). Thus, the term \( -\text{div} (Φ(u)) \) may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally through a pointwise multiplication of equation (1.1) by \( h(u) \) where \( h \) belongs to \( C^1_c(R) \), the class of \( C^1(R) \) functions with compact support.

Definition 3.1. A measurable function \( u : Ω → R \) is called a renormalized solution of (1.1) if \( u ∈ W^1_0 L_M(Ω) \), \( a(x, u, ∇u) ∈ (L^1(Ω))^N, \)
\( g(x, u, ∇u) ∈ L^1(Ω), g(x, u, ∇u)u ∈ L^1(Ω), \)
\[
\lim_{m→∞} \int_{\{x ∈ Ω : m ≤ |u(x)| ≤ m + 1\}} a(x, u, ∇u)∇u dx = 0,
\]
and
\[
-\text{div} a(x, u, ∇u)h(u) − \text{div} (Φ(u)h(u)) + h'(u)Φ(u)∇u
\]
\[
+ g(x, u, ∇u)h(u) = fh(u) − \text{div} (Fh(u)) + h'(u)F∇u \quad \text{in } D'(Ω),
\]
for every \( h ∈ C^1_c(R) \).

Remark 3.2. Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for \( h \) in \( C^1_c(R) \) and \( u \) in \( W^1_0 L_M(Ω) \), \( h(u) \) belongs to \( W^1 L_M(Ω) \) and for \( φ \) in \( D(Ω) \) the function \( φh(u) \) belongs to \( W^1_0 L_M(Ω) \). Since \( -\text{div} a(x, u, ∇u) \) ∈ \( W^{-1} L^1(Ω) \), we also have
\[
\langle -\text{div} a(x, u, ∇u)h(u), φ \rangle_{D'(Ω), D(Ω)} = \langle -\text{div} a(x, u, ∇u), φh(u) \rangle_{W^{-1} L^1(Ω), W^1_0 L_M(Ω)} \quad \forall φ \in D(Ω).
\]
Finally, since $\Phi h$ and $\Phi h' \in (C^0_c(\mathbb{R}))^N$, for any measurable function $u$ we have $\Phi(u)h(u)$ and $\Phi(u)h'(u) \in (L^\infty(\Omega))^N$ and then $\text{div} (\Phi(u)h(u)) \in W^{-1,\infty}(\Omega)$ and $\Phi(u)h'(u) \in L_M(\Omega)$.

Our main result is the following

**Theorem 3.3.** Suppose that assumptions (3.1)--(3.7) are fulfilled. Then, problem (1.1) has at least one renormalized solution.

**Remark 3.4.** The condition (3.4) can be replaced by the weaker one

$$|g(x, s, \xi)| \leq d(x) + b(|s|)M(|\xi|),$$

with $b: \mathbb{R} \to \mathbb{R}^+$ a continuous function belonging to $L^1(\mathbb{R})$ and $d(x) \in L^1(\Omega)$.

Actually the original equation (1.1) will be recovered whenever $h(u) \equiv 1$, but unfortunately this cannot happen in general strong additional requirements on $u$. Therefore, (3.8) is to be viewed as a weaker form of (1.1).

4. **Proof of the Main Result**

From now on, we will use the standard truncation function $T_k, k > 0$, defined for all $s \in \mathbb{R}$ by $T_k(s) = \max\{-k, \min\{k, s\}\}$.

**Step 1: Approximate problems.** Let $f_n$ be a sequence of $L^\infty(\Omega)$ functions that converge strongly to $f$ in $L^1(\Omega)$. For $n \in \mathbb{N}, n \geq 1$, let us consider the following sequence of approximate equations

$$-\text{div} a(x, u_n, \nabla u_n) + \text{div} \Phi_n(u_n) + g_n(x, u_n, \nabla u_n) = f_n - \text{div} F \text{ in } D'(\Omega),$$

where we have set $\Phi_n(s) = \Phi(T_n(s))$ and $g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{2}|g(x, s, \xi)|}$. For fixed $n > 0$, it’s obvious to observe that

$$g_n(x, s, \xi) s \geq 0, \quad |g_n(x, s, \xi)| \leq |g(x, s, \xi)| \quad \text{and} \quad |g_n(x, s, \xi)| \leq n.$$

Moreover, since $\Phi$ is continuous one has $|\Phi_n(t)| \leq \max_{|t| \leq n} |\Phi(t)|$. Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduce that there exists at least one solution $u_n$ of the approximate Dirichlet problem (4.1) in the sense

\[
\begin{align*}
\left\{ \begin{array}{l}
 u_n \in W^1_0 L_M(\Omega), a(x, u_n, \nabla u_n) \in (L^\infty(\Omega))^N \text{ and}
 \\
 \int_\Omega a(x, u_n, \nabla u_n) \nabla v dx + \int_\Omega \Phi_n(u_n) \nabla v dx
 \\
 + \int_\Omega g_n(x, u_n, \nabla u_n) v dx = \langle f_n, v \rangle + \int_\Omega F \nabla v dx, \text{ for every } v \in W^1_0 L_M(\Omega).
\end{array} \right.
\]
**Step 2: Estimation in** $W^1_0 L_M(\Omega)$. Taking $u_n$ as function test in problem (4.2), we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \, dx + \int_{\Omega} \Phi_n(u_n) \nabla u_n \, dx \\
+ \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx = (f_n, u_n) + \int_{\Omega} F \nabla u_n \, dx.
\] (4.3)

Define \( \tilde{\Phi}_n \in (C^1({\mathbb{R}}))^N \) as \( \tilde{\Phi}_n(t) = \int_0^t \Phi_n(\tau) \, d\tau \). Then formally
\[
\text{div}(\tilde{\Phi}_n(u_n)) = \Phi_n(u_n) \nabla u_n, \quad u_n = 0 \text{ on } \partial \Omega, \quad \tilde{\Phi}_n(0) = 0
\]
and by the Divergence theorem
\[
\int_{\Omega} \Phi_n(u_n) \nabla u_n \, dx = \int_{\Omega} \text{div}(\tilde{\Phi}_n(u_n)) \, dx = \int_{\partial \Omega} \tilde{\Phi}_n(u_n) \nu \, ds = 0,
\]
where \( \nu \) is the outward pointing unit normal field of the boundary \( \partial \Omega \) (\( ds \) may be used as a shorthand for \( \nu \, ds \)). Thus, by virtue of (3.5) and Young's inequality, we get
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq C_1 + \frac{\alpha}{2} \int_{\Omega} M(|\nabla u_n|) \, dx,
\] (4.4)

which, together with (3.3) give
\[
\int_{\Omega} M(|\nabla u_n|) \, dx \leq C_2.
\] (4.5)

Moreover, we also have
\[
\int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx \leq C_3.
\] (4.6)

As a consequence of (4.5) there exist a subsequence of \( \{u_n\}_n \), still indexed by \( n \), and a function \( u \in W^1_0 L_M(\Omega) \) such that
\[
u_n \rightharpoonup u \text{ weakly in } W^1_0 L_M(\Omega) \text{ for } \sigma(H^1_M(\Omega), H^1_M(\Omega)), \\
u_n \to u \text{ strongly in } E_M(\Omega) \text{ and a.e. in } \Omega.
\] (4.7)

**Step 3: Boundedness of** \( (a(x, u_n, \nabla u_n))_n \) **in** \( (L^M(\Omega))^N \). Let \( w \in (E_M(\Omega))^N \) with \( \|w\|_M \leq 1 \). Thanks to (3.2), we can write
\[
(a(x, u_n, \nabla u_n) - a(x, u_n, \frac{w}{k^4}))(\nabla u_n - \frac{w}{k^4}) \geq 0,
\]
which implies
\[
\frac{1}{k^4} \int_{\Omega} a(x, u_n, \nabla u_n) w \, dx \leq \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \, dx \\
+ \int_{\Omega} a(x, u_n, \frac{w}{k^4})(\frac{w}{k^4} - \nabla u_n) \, dx.
\]

Thanks to (4.4) and (4.5), one has
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq C_5.
\]
Define \( \lambda = 1 + k_1 + k_3 \). By the growth condition (3.1) and Young’s inequality, one can write
\[
\left| \int_{\Omega} a(x,u_n,\frac{u}{k_4})(\frac{u}{k_4} - \nabla u_n) \, dx \right| \\
\leq \left( 1 + \frac{1}{k_4} \right) \left( \int_{\Omega} M(c(x)) \, dx + k_1 \int_{\Omega} M^{-1}(k_2 |u_n|) \, dx \right) \\
+ k_3 \int_{\Omega} M(|u|) \, dx + \frac{\lambda}{k_4} \int_{\Omega} M(|u|) \, dx + \lambda \int_{\Omega} M(|\nabla u_n|) \, dx.
\]

By virtue of [18] and Lemma 4.14 of [20], there exists an \( N \)-function \( Q \) such that \( M \ll Q \) and the space \( W^{1}_0 L_M(\Omega) \) is continuously embedded into \( L_Q(\Omega) \). Thus, by (4.5) there exists a constant \( c_0 > 0 \), not depending on \( n \), satisfying \( \|u_n\|_Q \leq c_0 \). Since \( M \ll Q \), we can write \( M(k_2 t) \leq Q(\frac{c_0}{t}) \), for \( t > 0 \) large enough. As \( P \ll M \), we can find a constant \( c_1 \), not depending on \( n \), such that
\[
\int_{\Omega} M^{-1}(k_2 |u_n|) \, dx \leq \int_{\Omega} Q\left( \frac{|u_n|}{c_0} \right) + c_1.
\]
Hence, we conclude that the quantity \( \int_{\Omega} a(x,u_n,\nabla u_n) \, dx \) is bounded from above for all \( w \in (E_M(\Omega))^N \) with \( \|w\|_M \leq 1 \). Using the Orlicz norm we deduce that
\[
\left( a(x,u_n,\nabla u_n) \right)_n \quad \text{is bounded in} \quad (L_M(\Omega))^N. \tag{4.8}
\]

**Step 4: Renormalization identity for the approximate solutions.** For any \( m \geq 1 \), define \( \theta_m(r) = T_{m+1}(r) - T_m(r) \). Observe that by [19, Lemma 2] one has \( \theta_m(u_n) \in W^{1}_0 L_M(\Omega) \). The use of \( \theta_m(u_n) \) as test function in (4.2) yields
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x,u_n,\nabla u_n) \nabla u_n \, dx \leq \langle f_n, \theta_m(u_n) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n \, dx,
\]

By Hölder’s inequality and 4.5 we have
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x,u_n,\nabla u_n) \nabla u_n \, dx \leq \langle f_n, \theta_m(u_n) \rangle \\
+ C_0 \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx.
\]

It’s not hard to see that
\[
\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.
\]
So that by (4.5) and (4.7) one can deduce that
\[
\theta_m(u_n) \rightharpoonup \theta_m(u) \quad \text{weakly in} \quad W^{1}_0 L_M(\Omega) \quad \text{for} \sigma(\Pi L_M(\Omega),\Pi E_M(\Omega)).
\]

Note that as \( m \) goes to \( \infty \), \( \theta_m(u) \rightharpoonup 0 \) weakly in \( W^{1}_0 L_M(\Omega) \) for \( \sigma(\Pi L_M(\Omega),\Pi E_M(\Omega)) \), and since \( f_n \) converges strongly in \( L^1(\Omega) \), by Lebesgue’s theorem we have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx = \lim_{m \to \infty} \lim_{n \to \infty} \langle f_n, \theta_m(u_n) \rangle = 0.
\]
By (3.3) we finally have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx = 0. \tag{4.9}
\]

Step 5: Almost everywhere convergence of the gradients. Define \(\phi(s) = se^{\lambda s^2}\) with \(\lambda = \left(\frac{b(k)}{2a}\right)^2\). One can easily verify that for all \(s \in \mathbb{R}\)
\[
\phi'(s) - \frac{b(k)}{a} |\phi(s)| \geq \frac{1}{2}. \tag{4.10}
\]

For \(m \geq k\), we define the function \(\psi_m\) by
\[
\begin{cases}
\psi_m(s) = 1 & \text{if } |s| \leq m, \\
\psi_m(s) = m + 1 - |s| & \text{if } m \leq |s| \leq m + 1, \\
\psi_m(s) = 0 & \text{if } |s| \geq m + 1.
\end{cases}
\]

By virtue of [21, Theorem 4] there exists a sequence \(\{v_j\}_j \subset D(\Omega)\) such that \(v_j \rightharpoonup u\) in \(W^1_0 L^1(\Omega)\) for the modular convergence and a.e. in \(\Omega\). Let us define the following functions \(\theta^j_n = T_k(u_n) - T_k(v_j), \theta^j = T_k(u) - T_k(v_j)\) and \(z^j_{n,m} = \phi(\theta^j_n) \psi_m(u_n)\). Using \(z^j_{n,m} \in W^1_0 L^1(\Omega)\) as test function in (4.2) we get
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z^j_{n,m} \, dx + \int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) \, dx \\
+ \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \psi'(m) \phi(T_k(u_n) - T_k(v_j)) \, dx \\
+ \int_{\Omega} g_n(x, u_n, \nabla u_n) z^j_{n,m} \, dx = \int_{\Omega} f_n z^j_{n,m} \, dx + \int_{\Omega} F \nabla z^j_{n,m} \, dx. \tag{4.11}
\]

From now on we denote by \(\epsilon_i(n, j), i = 0, 1, 2, ...,\) various sequences of real numbers which tend to zero, when \(n \) and \(j \to +\infty\), i.e.
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \epsilon_i(n, j) = 0.
\]

In view of (4.7), we have \(z^j_{n,m} \rightharpoonup \phi(\theta^j) \psi_m(u)\) weakly in \(L^\infty(\Omega)\) for \(\sigma^*(L^\infty, L^1)\) as \(n \to +\infty\), which yields
\[
\lim_{n \to +\infty} \int_{\Omega} f_n z^j_{n,m} \, dx = \int_{\Omega} f \phi(\theta^j) \psi_m(u) \, dx,
\]
and since \(\phi(\theta^j) \to 0\) weakly in \(L^\infty(\Omega)\) for \(\sigma(L^\infty, L^1)\) as \(j \to +\infty\), we have
\[
\lim_{j \to +\infty} \int_{\Omega} f \phi(\theta^j) \psi_m(u) \, dx = 0.
\]

Thus, we write
\[
\int_{\Omega} f_n z^j_{n,m} \, dx = \epsilon_0(n, j).
\]

Thanks to (4.5) and (4.7), we have as \(n \to +\infty\),
\[
z^j_{n,m} \rightharpoonup \phi(\theta^j) \psi_m(u) \text{ in } W^1_0 L^1(\Omega) \text{ for } \sigma(\Pi L^1(\Omega), \Pi E^\infty(\Omega)).
\]
which implies that
\[
\lim_{n \to +\infty} \int_\Omega F \nabla z_{n,m}^j \, dx = \int_\Omega F \nabla \theta^j \phi'(\theta^j) \psi_m(u) \, dx + \int_\Omega F \nabla u \phi(\theta^j) \psi'_m(u) \, dx
\]
On the one hand, by Lebesgue’s theorem we get
\[
\lim_{j \to +\infty} \int_\Omega F \nabla u \phi(\theta^j) \psi'_m(u) \, dx = 0,
\]
on the other hand, we write
\[
\int_\Omega F \nabla \theta^j \phi'(\theta^j) \psi_m(u) \, dx = \int_\Omega F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) \, dx - \int_\Omega F \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) \, dx,
\]
so that, by Lebesgue’s theorem one has
\[
\lim_{j \to +\infty} \int_\Omega F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) \, dx = \int_\Omega F \nabla T_k(u) \psi_m(u) \, dx.
\]
Let \( \lambda > 0 \) such that \( M \left( \frac{\nabla v_j - \nabla u}{\lambda} \right) \to 0 \) strongly in \( L^1(\Omega) \) as \( j \to +\infty \) and \( M \left( \frac{|\nabla u|}{\lambda} \right) \in L^1(\Omega) \), the convexity of the \( N \)-function \( M \) allows us to have
\[
M \left( \frac{\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) - \nabla T_k(u) \psi_m(u)}{M(\frac{\nabla v_j - \nabla u}{\lambda})} \right) \approx \frac{1}{4} M \left( \frac{\nabla v_j - \nabla u}{\lambda} \right) + \frac{1}{4} \left( 1 + \frac{1}{\sigma(2^k)} \right) M \left( \frac{|\nabla u|}{\lambda} \right).
\]
Then, by using the modular convergence of \( \{\nabla v_j\} \) in \( (L_M(\Omega))^N \) and Vitali’s theorem, we obtain
\[
\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) \to \nabla T_k(u) \psi_m(u) \text{ in } (L_M(\Omega))^N,
\]
for the modular convergence, and then
\[
\lim_{j \to +\infty} \int_\Omega F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) \, dx = \int_\Omega F \nabla T_k(u) \psi_m(u) \, dx.
\]
We have proved that
\[
\int_\Omega F \nabla z_{n,m}^j \, dx = \epsilon_1(n,j).
\]
It’s easy to see that by the modular convergence of the sequence \( \{v_j\} \), one has
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \Phi_n(u_n) \nabla u_n \psi'_m(u_n) \phi(T_k(u_n) - T_k(v_j)) \, dx = 0,
\]
while for the third term in the left-hand side of (4.11) we can write
\[
\Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) \, dx
\]
\[
= \int_\Omega \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta^j_n) \psi_m(u_n) \, dx - \int_\Omega \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \psi_m(u_n) \, dx.
\]
Firstly, we have
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) dx = 0.
\]
In view of (4.7), one has
\[
\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n) \to \Phi(u) \phi'(\theta^j) \psi_m(u),
\]
almost everywhere in \( \Omega \) as \( n \) tends to \( +\infty \). Furthermore, we can check that
\[
\| \Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n) \|_{L^p} \leq M(c_m \phi'(2k)) \Omega + 1,
\]
where \( c_m = \max_{t \leq m+1} \Phi(t) \). Applying [27, Theorem 14.6] we get
\[
\lim_{n \to +\infty} \lim_{j \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx.
\]
Using the modular convergence of the sequence \( \{v_j\} \), we obtain
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx.
\]
Then, using again the Divergence theorem we get
\[
\int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx = 0.
\]
Therefore, we write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx = \epsilon_2(n, j).
\]
Since \( g_n(x, u_n, \nabla u_n) z_{n,m}^j \geq 0 \) on the set \( \{| u_n | > k\} \) and \( \psi_m(u_n) = 1 \) on the set \( \{| u_n | \leq k\} \), from (4.11) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx + \int_{\{| u_n | \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \leq \epsilon_3(n, j). \tag{4.12}
\]
We now evaluate the first term of the left-hand side of (4.12) by writing
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx
\]
\[
= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) \psi_m(u_n) dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) dx
\]
\[
= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi(\theta_n^j) dx
\]
\[
- \int_{\{| u_n | > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) dx.
\]
and then
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j \, dx \\
= \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \right) \\
\quad \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) \phi'(\theta_n^j) \, dx \\
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) \phi'(\theta_n^j) \, dx \\
- \int_{\Omega \backslash \Omega_j^k} a(x, T_k(u_n), \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx \\
- \int_{\{u_n > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx \\
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi'(\theta_n^j) \psi'_m(u_n) \, dx,
\]
(4.13)

where by \( \chi_j^s, s > 0 \), we denote the characteristic function of the subset
\[\Omega_j^s = \{ x \in \Omega : |\nabla T_k(v_j)| \leq s \} .\]

For fixed \( m \) and \( s \), we will pass to the limit in \( n \) and then in \( j \) in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) \phi'(\theta_n^j) \, dx \\
\rightarrow \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u) - \nabla T_k(v_j) \chi_j^s \right) \phi'(\theta^j) \, dx,
\]
as \( n \to +\infty \). Since by lemma (2.4) one has
\[a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) \to a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j),\]
strongly in \((E_{M}^N(\Omega))^N\) as \( n \to \infty \), while by (4.5)
\[\nabla T_k(u_n) \rightarrow \nabla T_k(u),\]
weakly in \((L_M(\Omega))^N\). Let \( \chi^s \) denote the characteristic function of the subset
\[\Omega^s = \{ x \in \Omega : |\nabla T_k(u)| \leq s \} .\]

As \( \nabla T_k(v_j) \chi_j^s \rightarrow \nabla T_k(u) \chi^s \) strongly in \((E_M(\Omega))^N\) as \( j \rightarrow +\infty \), one has
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \cdot \left( \nabla T_k(u) - \nabla T_k(v_j) \chi_j^s \right) \phi'(\theta^j) \, dx \rightarrow 0,
\]
as \( j \to \infty \). Then
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) \phi'(\theta_n^j) \, dx = \epsilon_4(n,j). \quad \text{(4.14)}
\]

We now estimate the third term of (4.13). It’s easy to see that by (3.3), \( a(x, s, 0) = 0 \) for almost everywhere \( x \in \Omega \) and for all \( s \in \mathbb{R} \). Thus, from (4.8) we have that \( (a(x, T_k(u_n), \nabla T_k(u_n)))_n \) is bounded in \((L_M(\Omega))^N\) for all \( k \geq 0 \).
Therefore, there exist a subsequence still indexed by $n$ and a function $l_k$ in $(L^N_\mathcal{M}(\Omega))^N$ such that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightarrow l_k \quad \text{weakly in} \quad (L^N_\mathcal{M}(\Omega))^N \quad \text{for} \quad \sigma(\Pi L^N_\mathcal{M}, E_M). \quad (4.15)$$

Then, since $\nabla T_k(v_j) \chi_{\Omega \setminus \Omega_j^c} \in (E^N_\mathcal{M}(\Omega))^N$, we obtain

$$\int_{\Omega \setminus \Omega_j^c} a(x,T_k(u_n),\nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx \rightarrow \int_{\Omega \setminus \Omega_j^c} l_k \nabla T_k(v_j) \phi'(\theta^j) dx,$$

as $n \rightarrow +\infty$. The modular convergence of $\{v_j\}$ allows us to get

$$- \int_{\Omega \setminus \Omega_j^c} l_k \nabla T_k(v_j) \phi'(\theta^j) dx \rightarrow - \int_{\Omega \setminus \Omega} l_k \nabla T_k(u) dx,$$

as $j \rightarrow +\infty$. This, proves

$$- \int_{\Omega \setminus \Omega_j^c} a(x,T_k(u_n),\nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = - \int_{\Omega \setminus \Omega} l_k \nabla T_k(u) dx + \epsilon_5(n,j). \quad (4.16)$$

As regards the fourth term, observe that $\psi_m(u_n) = 0$ on the subset $\{|u_n| \geq m + 1\}$, so we have

$$- \int_{\{u_n > k\}} a(x,u_n,\nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = - \int_{\{u_n > k\}} a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx.$$

Since

$$- \int_{\{u_n > k\}} a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = - \int_{\{u > k\}} l_{m+1} \nabla T_k(u) \psi_m(u) dx + \epsilon_5(n,j),$$

observing that $\nabla T_k(u) = 0$ on the subset $\{|u| > k\}$, one has

$$- \int_{\{u_n > k\}} a(x,u_n,\nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \epsilon_6(n,j). \quad (4.17)$$

For the last term of (4.13), we have

$$\left| \int_{\Omega} a(x,u_n,\nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m'(u_n) dx \right| = \left| \int_{\{m \leq |u_n| \leq m+1\}} a(x,u_n,\nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m'(u_n) dx \right| \leq \phi(2k) \int_{\{m \leq |u_n| \leq m+1\}} a(x,u_n,\nabla u_n) \nabla u_n dx.$$
To estimate the last term of the previous inequality, we use

\((T_1(u_n - T_m(u_n)) \in W^1_0 L^M(\Omega))\) as test function in (4.2), to get

\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \, dx
\]

\[
+ \int_{\{m \leq |u_n| \leq m+1\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \, dx = \langle f_n, T_1(u_n - T_m(u_n)) \rangle
\]

\[
+ \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n \, dx.
\]

By Divergence theorem, we have

\[
\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \, dx = 0.
\]

Using the fact that \(g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \geq 0\) on the subset \(\{ |u_n| \geq m \}\) and Young’s inequality, we get

\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
\]

\[
\leq \langle f_n, T_1(u_n - T_m(u_n)) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx.
\]

It follows that

\[
\left| \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^k_n) \, dx \right|
\]

\[
\leq 2 \phi(2k) \left( \int_{\{m \leq |u_n| \leq m+1\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right). \tag{4.18}
\]

From (4.14), (4.16), (4.17) and (4.18) we obtain

\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z^{s,m}_{n} \, dx
\]

\[
\geq \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)) \right)
\]

\[
\chi_n^j \phi(\theta^k_n) \, dx
\]

\[
- \alpha \phi(2k) \left( \int_{\{m \leq |u_n| \leq m+1\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right)
\]

\[
- \int_{\Omega \setminus \Omega^*} l_k \cdot \nabla T_k(u) \, dx + \epsilon \gamma(n, j). \tag{4.19}
\]

Now, we turn to second term in the left-hand side of (4.12). We have

\[
\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta^k_n) \, dx \right|
\]

\[
= \left| \int_{\{ |u_n| \leq k \}} g_n(x, T_k(u_n), \nabla T_k(u_n)) \phi(\theta^k_n) \, dx \right|
\]

\[
\leq b(k) \int_{\Omega} \mathcal{M}(|\nabla T_k(u_n)|) |\phi(\theta^k_n)| \, dx + b(k) \int_{\Omega} d(x) |\phi(\theta^k_n)| \, dx
\]

\[
\leq \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\phi(\theta^k_n)| \, dx + \epsilon \gamma(n, j).
\]
Then
\[
\left| \int_{\{\|u_n\| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) \, dx \right| \\
\leq \frac{b(k)}{\alpha} \int_\Omega \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^k) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^k \right) \phi(\theta_n^j) \, dx \\
+ \frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^k) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^k \right) \phi(\theta_n^j) \, dx \\
+ \frac{b(k)}{\alpha} \int_\Omega a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^k \phi(\theta_n^j) \, dx + \epsilon_9(n, j).
\]

(4.20)

We proceed as above to get
\[
\frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^k) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^k \right) \phi(\theta_n^j) \, dx = \epsilon_9(n, j)
\]
and
\[
\frac{b(k)}{\alpha} \int_\Omega a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^k \phi(\theta_n^j) \, dx = \epsilon_{10}(n, j).
\]

Hence, we have
\[
\left| \int_{\{\|u_n\| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) \, dx \right| \\
\leq \frac{b(k)}{\alpha} \int_\Omega \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^k) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^k \right) \phi(\theta_n^j) \, dx + \epsilon_{11}(n, j).
\]

(4.21)

Combining (4.12), (4.19) and (4.21), we get
\[
\int_\Omega \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^k) \right) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^k \right) \\
\left( \phi(\theta_n^j) - \frac{b(k)}{\alpha} \phi(\theta_n^j) \right) \, dx \\
\leq \int_{\Omega \setminus \Omega_0} l_k \nabla T_k(u) \, dx + a(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right) \\
+ \epsilon_{12}(n, j).
\]

By (4.10), we have
\[
\int_\Omega \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^k) \right) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^k \right) \, dx \\
\leq 2 \int_{\Omega \setminus \Omega_0} l_k \nabla T_k(u) \, dx + 4a(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right) \\
+ \epsilon_{12}(n, j).
\]

(4.22)
On the other hand we can write
\[
\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx \\
= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s) \, dx \\
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx \\
- \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi^s) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s) \, dx \\
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi^s) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s) \, dx
\]

We shall pass to the limit in \( n \) and then in \( j \) in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(v_j)\chi^s_n - \nabla T_k(u)\chi^s) \, dx = \epsilon_{13}(n,j), \\
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx = \epsilon_{14}(n,j), \\
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi^s_n) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_n) \, dx = \epsilon_{15}(n,j)
\]

So that
\[
\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx \\
= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi^s_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_n) \, dx \\
+ \epsilon_{16}(n,j).
\]

Let \( r \leq s \). Using (3.2), (4.22) and (4.24) we can write
\[
0 \leq \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \\
\leq \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \\
= \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx \\
\leq \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx \\
\leq \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi^s_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_n) \, dx
\]

\[
+ \epsilon_{15}(n,j)
\]

\[
\leq 2 \int_{\Omega \setminus \Omega^r} l_k \nabla T_k(u) \, dx + 2\alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} \mathcal{M}(|F|) \, dx \right) \\
+ \epsilon_{17}(n,j).
\]
By passing to the superior limit over \( n \) and then over \( j \)
\[
0 \leq \limsup_{n \to +\infty} \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx
\]
\[
\leq 2 \int_{\Omega^r} l_k \nabla T_k(u) dx + 4 \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right).
\]
Letting \( s \to +\infty \) and then \( m \to +\infty \), taking into account that \( l_k \nabla T_k(u) \in L^1(\Omega) \), \( f \in L^1(\Omega) \), \( |F| \in (E_{TT^*}(\Omega))^N \), \( |\Omega \setminus \Omega^r| \to 0 \), and \( [|m \leq |u| \leq m+1|] \to 0 \), one has
\[
\int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx,
\]
(4.25)
tends to 0 as \( n \to +\infty \). As in [20], we deduce that there exists a subsequence of \( \{u_n\} \) still indexed by \( n \) such that
\[
\nabla u_n \rightharpoonup \nabla u \text{ a. e. in } \Omega.
\]
(4.26)
Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get
\[
a(x, u, \nabla u) \in (L_{TT^*}(\Omega))^N
\]
and
\[
a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{TT^*}(\Omega))^N \text{ for } \sigma(\Pi L_{TT^*}, \mathbb{C} E_M).
\]
(4.27)

**Step 6: Modular convergence of the truncations.** Going back to equation (4.22), we can write
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx
\]
\[
\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx
\]
\[
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \chi_j^s (\nabla T_k(u_n) - \nabla T_k(v_j)) dx
\]
\[
+ 2a \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right)
\]
\[
+ 2 \int_{\Omega^r} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{12}(n, j).
\]
By (4.23) we get
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx
\]
\[
\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx
\]
\[
+ 2a \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right)
\]
\[
+ 2 \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{18}(n, j).
\]
We now pass to the superior limit over \( n \) in both sides of this inequality using (4.27), to obtain
\[
\limsup_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \\
+ 2\alpha \phi(2k) \left( \int_{\{ m \leq |u| \}} |f| \, dx + \int_{\{ m \leq |u| \leq m+1 \}} M(|F|) \, dx \right) \\
+ 2 \int_{\Omega \setminus \Omega^*} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]

We then pass to the limit in \( j \) to get
\[
\limsup_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \\
+ 2\alpha \phi(2k) \left( \int_{\{ m \leq |u| \}} |f| \, dx + \int_{\{ m \leq |u| \leq m+1 \}} M(|F|) \, dx \right) \\
+ 2 \int_{\Omega \setminus \Omega^*} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]

Letting \( s \) and then \( m \to +\infty \), one has
\[
\limsup_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]

On the other hand, by (3.3), (4.5), (4.26) and Fatou’s lemma, we have
\[
\int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \leq \liminf_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx.
\]

It follows that
\[
\lim_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx = \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]

By Lemma 2.5 we conclude that for every \( k > 0 \)
\[
a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u),
\]

strongly in \( L^1(\Omega) \). The convexity of the \( N \)-function \( M \) and (3.3) allow us to have
\[
M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) \\
\leq \frac{1}{2\alpha} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) + \frac{1}{2\alpha} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u).
\]

From Vitali’s theorem we deduce
\[
\limsup_{n \to +\infty} \int_\Omega M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) \, dx = 0.
\]

Thus, for every \( k > 0 \)
\[
T_k(u_n) \to T_k(u) \text{ in } W^{1,1}_0 L_M(\Omega),
\]
for the modular convergence.

**Step 7: Compactness of the nonlinearities.** We need to prove that

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]  

(4.29)

By virtue of (4.7) and (4.26) one has

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a. e. in } \Omega. \]  

(4.30)

Let \( E \) be measurable subset of \( \Omega \) and let \( m > 0 \). Using (3.3) and (3.4) we can write

\[
\int_E |g_n(x, u_n, \nabla u_n)| \, dx \\
= \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| \, dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| \, dx \\
\leq b(m) \int_E d(x) \, dx + b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx \\
+ \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx.
\]

From (3.5) and (4.6), we deduce that

\[ 0 \leq \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx \leq C_3. \]

So

\[ 0 \leq \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx \leq \frac{C_3}{m}. \]

Then

\[ \lim_{m \to +\infty} \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx = 0. \]

Thanks to (4.28) the sequence \( \{a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)\}_n \) is equi-integrable. This fact allows us to get

\[ \lim_{|E| \to 0} \sup_n \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx = 0. \]

This shows that \( g_n(x, u_n, \nabla u_n) \) is equi-integrable. Thus, Vitali’s theorem implies that \( g(x, u, \nabla u) \in L^1(\Omega) \) and

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]

**Step 8: Renormalization identity for the solutions.** In this step we prove that

\[ \lim_{m \to +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \, dx = 0. \]  

(4.31)
Indeed, for any \( m \geq 0 \) we can write
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
\]
\[
= \int_{\Omega} a(x, u_n, \nabla u_n)(\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) \, dx
\]
\[
= \int_{\Omega} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) \, dx
\]
\[
- \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx.
\]

In view of (4.28), we can pass to the limit as \( n \) tends to \( +\infty \) for fixed \( m \geq 0 \)
\[
\lim_{n \to +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
\]
\[
= \int_{\Omega} a(x, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_{m+1}(u) \, dx
\]
\[
- \int_{\Omega} a(x, T_m(u), \nabla T_m(u)) \nabla T_m(u) \, dx
\]
\[
= \int_{\Omega} a(x, u, \nabla u)(\nabla T_{m+1}(u) - \nabla T_m(u)) \, dx
\]
\[
- \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx.
\]

Having in mind (4.9), we can pass to the limit as \( m \) tends to \( +\infty \) to obtain (4.31).

**Step 9: Passing to the limit.** Thanks to (4.28) and Lemma (2.5), we obtain
\[
a(x, u_n, \nabla u_n) \nabla u_n \to a(x, u, \nabla u) \nabla u \text{ strongly in } L^1(\Omega).
\]

Let \( h \in C_c^1(\mathbb{R}) \) and \( \varphi \in \mathcal{D}(\Omega) \). Inserting \( h(u_n)\varphi \) as test function in (4.2), we get
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi \, dx + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) \, dx
\]
\[
+ \int_{\Omega} \Phi_n(u_n) \nabla (h(u_n)\varphi) \, dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx
\]
\[
= (f_n, h(u_n)\varphi) + \int_{\Omega} F \nabla (h(u_n)\varphi) \, dx.
\]

We shall pass to the limit as \( n \to +\infty \) in each term of the equality (4.33).

Since \( h \) and \( h' \) have compact support on \( \mathbb{R} \), there exists a real number \( \nu > 0 \), such that \( \text{supp } h \subset [-\nu, \nu] \) and \( \text{supp } h' \subset [-\nu, \nu] \). For \( n > \nu \), we can write
\[
\Phi_n(t)h(t) = \Phi(T_\nu(t))h(t) \text{ and } \Phi_n(t)h'(t) = \Phi(T_\nu(t))h'(t).
\]

Moreover, the functions \( \Phi h \) and \( \Phi h' \) belong to \( (C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N \). Observe first that the sequence \( \{h(u_n)\varphi\}_n \) is bounded in \( W^1_0 L_\mathcal{M}(\Omega) \). Indeed, let \( \rho > 0 \).
be a positive constant such that \(\|h(u_n)\nabla \varphi\|_\infty \leq \rho\) and \(\|h'(u_n)\varphi\|_\infty \leq \rho\). Using the convexity of the \(N\)-function \(M\) and taking into account (4.5) we have
\[
\int_\Omega M\left(\frac{|\nabla (h(u_n)\varphi)|}{2\rho}\right) dx \leq \int_\Omega M\left(\frac{|h(u_n)\nabla \varphi| + |h'(u_n)\varphi||\nabla u_n|}{2\rho}\right) dx \\
\leq \frac{1}{2} M(1|\Omega|) + \frac{1}{2} \int_\Omega M(|\nabla u_n|) dx \\
\leq \frac{1}{2} M(1|\Omega|) + \frac{1}{2} C_2.
\]
This, together with (4.7), imply that
\[
h(u_n)\varphi \rightharpoonup h(u)\varphi \text{ weakly in } W^1_0 \Omega (\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_M).
\] (4.34)
This enables us to get
\[
(f_n, h(u_n)\varphi) \rightarrow (f, h(u)\varphi).
\]
Let \(E\) be a measurable subset of \(\Omega\). Define \(c_{\nu} = \max_{|t| \leq \nu} \Phi(t)\). Let us denote by \(\|v\|_{\ell_M}\) the Orlicz norm of a function \(v \in L_M(\Omega)\). Using strengthened Hölder inequality with both Orlicz and Luxemburg norms, we get
\[
\|\Phi(T_\nu(u_n))\chi_E\|_{(\ell_M)} = \sup_{\|v\|_{\ell_M} \leq 1} \left| \int_E \Phi(T_\nu(u_n)) v dx \right| \\
\leq c_{\nu} \sup_{\|v\|_{\ell_M} \leq 1} \|\chi_E\|_{(\ell_M)} \|v\|_{\ell_M} \\
\leq c_{\nu} |E|^{-1} \left( \frac{1}{|E|} \right).
\]
Thus, we get
\[
\lim_{|E| \rightarrow 0} \sup_n \|\Phi(T_\nu(u_n))\chi_E\|_{(\ell_M)} = 0.
\]
Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain
\[
\Phi(T_\nu(u_n)) \rightarrow \Phi(T_\nu(u)) \text{ strongly in } (E_M)^N,
\]
which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have
\[
\int_\Omega \Phi(T_\nu(u_n)) \nabla (h(u_n)\varphi) dx \rightarrow \int_\Omega \Phi(T_\nu(u)) \nabla (h(u)\varphi) dx.
\]
We remark that
\[
|a(x, u_n, \nabla u_n) \nabla u_n h'(u_n)\varphi| \leq \rho a(x, u_n, \nabla u_n) \nabla u_n.
\]
Consequently, using (4.32) and Vitali’s theorem, we obtain
\[
\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n h'(u_n)\varphi dx \rightarrow \int_\Omega a(x, u, \nabla u) \nabla uh'(u)\varphi dx.
\]
and
\[
\int_\Omega F \nabla u_n h'(u_n)\varphi dx \rightarrow \int_\Omega F \nabla uh'(u)\varphi dx.
\]
For the second term of (4.33), as above we have
\[
h(u_n)\nabla \varphi \rightharpoonup h(u)\nabla \varphi \text{ strongly in } (E_M(\Omega))^N,
\]
which together with (4.27) give
\[ \int_\Omega a(x, u_n, \nabla u_n) \nabla \varphi(u_n) dx \to \int_\Omega a(x, u, \nabla u) \nabla \varphi(u) dx \]
and
\[ \int_\Omega F \nabla \varphi(u_n) dx \to \int_\Omega F \nabla \varphi(u) dx. \]

The fact that \( h(u_n) \varphi \rightharpoonup h(u) \varphi \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get
\[ \int_\Omega g_n(x, u_n, \nabla u_n) h(u_n) \varphi dx \to \int_\Omega g(x, u, \nabla u) h(u) \varphi dx. \]

At this point we can pass to the limit in each term of (4.33) to get
\[
\int_\Omega a(x, u, \nabla u) (\nabla \varphi h(u) + h'(u) \varphi \nabla u) dx + \int_\Omega \Phi(u) h'(u) \varphi \nabla u dx \\
+ \int_\Omega \Phi(u) h(u) \nabla \varphi dx + \int_\Omega g(x, u, \nabla u) h(u) \varphi dx \\
= (f, h(u) \varphi) + \int_\Omega F(\nabla \varphi h(u) + h'(u) \varphi \nabla u) dx,
\]
for all \( h \in C^1_c(\mathbb{R}) \) and for all \( \varphi \in D(\Omega) \). Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou’s lemma to get \( g(x, u, \nabla u) u \in L^1(\Omega) \). By virtue of (4.7), (4.27), (4.29), (4.31), the function \( u \) is a renormalized solution of problem (1.1).

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**REFERENCES**


