

## Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces

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ABSTRACT. The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems

$$-\operatorname{div} (a(x, u, \nabla u) + \Phi(u)) + g(x, u, \nabla u) = f - \operatorname{div} F,$$

in a bounded open set  $\Omega$  and  $u = 0$  on  $\partial\Omega$ , in the framework of Orlicz-Sobolev spaces without any restriction on the  $M$   $N$ -function of the Orlicz spaces, where  $-\operatorname{div} (a(x, u, \nabla u))$  is a Leray-Lions operator defined from  $W_0^1 L_M(\Omega)$  into its dual,  $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$ . The function  $g(x, u, \nabla u)$  is a non linear lower order term with natural growth with respect to  $|\nabla u|$ , satisfying the sign condition and the datum  $\mu$  is assumed to belong to  $L^1(\Omega) + W^{-1} E_{\overline{M}}(\Omega)$ .

**Keywords:** Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.

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### 1. INTRODUCTION

Let  $\Omega$  be a bounded open set of  $\mathbb{R}^N$ ,  $N \geq 2$ , and let  $M$  be an  $N$ -function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem

$$\begin{cases} A(u) - \operatorname{div} \Phi(u) + g(x, u, \nabla u) = f - \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here,  $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$ , while the function  $g(x, u, \nabla u)$  is a non linear lower order term with natural growth with respect to  $|\nabla u|$  and satisfying the sign condition. The non everywhere defined nonlinear operator  $A(u) = -\operatorname{div}(a(x, u, \nabla u))$  acts from its domain  $D(A) \subset W_0^1 L_M(\Omega)$  into  $W^{-1} L_{\overline{M}}(\Omega)$ . The function  $a(x, u, \nabla u)$  is assumed to satisfy, among others,  $a(x, u, \nabla u)$  nonstandard growth condition governed by the  $N$ -function  $M$ , and the source term  $f \in L^1(\Omega)$  and  $|F| \in E_{\overline{M}}(\Omega)$ ,  $\overline{M}$  stands for the conjugate of  $M$ .

We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with  $\Phi \equiv 0$ ,  $g \equiv 0$  and  $L^1(\Omega)$ -data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when  $f$  is a bounded Radon measure datum and  $g$  grows at most like  $|\nabla u|^{p-1}$  by Beta et al. in [9, 10, 11] with  $\Phi \equiv 0$  and by Guibé and Mercaldo in [23, 24] when  $\Phi(u)$  behaves at most like  $|u|^{p-1}$ . Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in  $L^1(\Omega)$ . Recall that to get both existence and uniqueness of a solution to problems with  $L^1$ -data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall'Aglio [14].

The authors in [5] have dealt with the equation (1.1) with  $g = g(x, u)$  and  $\mu \in W^{-1} E_{\overline{M}}(\Omega)$ , under the restriction that the  $N$ -function  $M$  satisfies the  $\Delta_2$ -condition. This work was then extended in [2] for  $N$ -functions not satisfying necessarily the  $\Delta_2$ -condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the  $N$ -function  $M$ . Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is  $\mu \in W^{-1} E_{\overline{M}}(\Omega)$ , an existence result has been proved in [3], Specific examples to which our results apply include the following:

$$\begin{aligned} & - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u + |u|^s u\right) + u |\nabla u|^p = \mu \text{ in } \Omega, \\ & - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \log^\beta(1 + |\nabla u|) + |u|^s u\right) = \mu \text{ in } \Omega, \\ & - \operatorname{div}\left(\frac{M(|\nabla u|) \nabla u}{|\nabla u|^2} + |u|^s u\right) + M(|\nabla u|) = \mu \text{ in } \Omega, \end{aligned}$$

where  $p > 1$ ,  $s > 0$ ,  $\beta > 0$  and  $\mu$  is a given Radon measure on  $\Omega$ .

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form  $f - \operatorname{div} F$  with  $f \in L^1(\Omega)$  and  $|F| \in E_{\overline{M}}(\Omega)$ , in the setting of Orlicz spaces without any restriction on the  $N$ -functions  $M$ . The approximate equations provide a  $W_0^1 L_M(\Omega)$  bound for the corresponding solution  $u_n$ . This allows us to obtain

a function  $u$  as a limit of the sequence  $u_n$ . Hence, appear two difficulties. The first one is how to give a sense to  $\Phi(u)$ , the second difficulty lies in the need of the convergence almost everywhere of the gradients of  $u_n$  in  $\Omega$ . This is done by using suitable test functions built upon  $u_n$  which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

## 2. PRELIMINARIES

Let  $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be an  $N$ -function, i. e.,  $M$  is continuous, increasing, convex, with  $M(t) > 0$  for  $t > 0$ ,  $\frac{M(t)}{t} \rightarrow 0$  as  $t \rightarrow 0$ , and  $\frac{M(t)}{t} \rightarrow +\infty$  as  $t \rightarrow +\infty$ . Equivalently,  $M$  admits the representation:

$$M(t) = \int_0^t a(s) ds,$$

where  $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is increasing, right continuous, with  $a(0) = 0$ ,  $a(t) > 0$  for  $t > 0$  and  $a(t)$  tends to  $+\infty$  as  $t \rightarrow +\infty$ .

The conjugate of  $M$  is also an  $N$ -function and it is defined by  $\bar{M} = \int_0^t \bar{a}(s) ds$ , where  $\bar{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is the function  $\bar{a}(t) = \sup\{s : a(s) \leq t\}$  (see [1]).

An  $N$ -function  $M$  is said to satisfy the  $\Delta_2$ -condition if, for some  $k$ ,

$$M(2t) \leq kM(t) \quad \forall t \geq 0, \tag{2.1}$$

when (2.1) holds only for  $t \geq t_0 > 0$  then  $M$  is said to satisfy the  $\Delta_2$ -condition near infinity. Moreover, we have the following Young's inequality

$$st \leq M(t) + \bar{M}(s), \quad \forall s, t \geq 0.$$

Given two  $N$ -functions, we write  $P \ll Q$  to indicate  $P$  grows essentially less rapidly than  $Q$ ; i. e. for each  $\epsilon > 0$ ,  $\frac{P(t)}{Q(\epsilon t)} \rightarrow 0$  as  $t \rightarrow +\infty$ . This is the case if and only if

$$\lim_{t \rightarrow \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let  $\Omega$  be an open subset of  $\mathbb{R}^N$ . The Orlicz class  $k_M(\Omega)$  (resp. the Orlicz space  $L_M(\Omega)$ ) is defined as the set of (equivalence classes of) real valued measurable functions  $u$  on  $\Omega$  such that

$$\int_{\Omega} M(|u(x)|) dx < +\infty \quad (\text{resp. } \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0).$$

The set  $L_M(\Omega)$  is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M \left( \frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\},$$

and  $k_M(\Omega)$  is a convex subset of  $L_M(\Omega)$ .

The closure in  $L_M(\Omega)$  of the set of bounded measurable functions with compact support in  $\bar{\Omega}$  is denoted by  $E_M(\Omega)$ . The dual of  $E_M(\Omega)$  can be identified with  $L_{\bar{M}}(\Omega)$  by means of the pairing  $\int_{\Omega} uv dx$ , and the dual norm of  $L_{\bar{M}}(\Omega)$  is equivalent to  $\|\cdot\|_{\bar{M},\Omega}$ . We now turn to the Orlicz-Sobolev space,  $W^1 L_M(\Omega)$  [resp.  $W^1 E_M(\Omega)$ ] is the space of all functions  $u$  such that  $u$  and its distributional derivatives up to order 1 lie in  $L_M(\Omega)$  [resp.  $E_M(\Omega)$ ]. It is a Banach space under the norm

$$\|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^{\alpha} u\|_{M,\Omega}.$$

Thus,  $W^1 L_M(\Omega)$  and  $W^1 E_M(\Omega)$  can be identified with subspaces of product of  $N + 1$  copies of  $L_M(\Omega)$ . Denoting this product by  $\prod L_M$ , we will use the weak topologies  $\sigma(\prod L_M, \prod E_{\bar{M}})$  and  $\sigma(\prod L_M, \prod L_{\bar{M}})$ .

The space  $W_0^1 E_M(\Omega)$  is defined as the (norm) closure of the Schwartz space  $\mathcal{D}(\Omega)$  in  $W^1 E_M(\Omega)$  and the space  $W_0^1 L_M(\Omega)$  as the  $\sigma(\prod L_M, \prod E_{\bar{M}})$  closure of  $\mathcal{D}(\Omega)$  in  $W^1 L_M(\Omega)$ . We say that  $u_n$  converges to  $u$  for the modular convergence in  $W^1 L_M(\Omega)$  if for some  $\lambda > 0$ ,  $\int_{\Omega} M \left( \frac{D^{\alpha} u_n - D^{\alpha} u}{\lambda} \right) dx \rightarrow 0$  for all  $|\alpha| \leq 1$ . This implies convergence for  $\sigma(\prod L_M, \prod L_{\bar{M}})$ . If  $M$  satisfies the  $\Delta_2$  condition on  $\mathbb{R}^+$  (near infinity only when  $\Omega$  has finite measure), then modular convergence coincides with norm convergence.

Let  $W^{-1} L_{\bar{M}}(\Omega)$  [resp.  $W^{-1} E_{\bar{M}}(\Omega)$ ] denote the space of distributions on  $\Omega$  which can be written as sums of derivatives of order  $\leq 1$  of functions in  $L_{\bar{M}}(\Omega)$  [resp.  $E_{\bar{M}}(\Omega)$ ]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain  $\Omega$  has the segment property if for every  $x \in \partial\Omega$  there exists an open set  $G_x$  and a nonzero vector  $y_x$  such that  $x \in G_x$  and if  $z \in \bar{\Omega} \cap G_x$ , then  $z + ty_x \in \Omega$  for all  $0 < t < 1$ . The following lemmas can be found in [6].

**Lemma 2.1.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly Lipschitzian, with  $F(0) = 0$ . Let  $M$  be an  $N$ -function and let  $u \in W^1 L_M(\Omega)$  (resp.  $W^1 E_M(\Omega)$ ). Then  $F(u) \in W^1 L_M(\Omega)$  (resp.  $W^1 E_M(\Omega)$ ). Moreover, if the set  $D$  of discontinuity points of  $F'$  is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

**Lemma 2.2.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly Lipschitzian, with  $F(0) = 0$ . We suppose that the set of discontinuity points of  $F'$  is finite. Let  $M$  be an  $N$ -function, then the mapping  $F : W^1 L_M(\Omega) \rightarrow W^1 L_M(\Omega)$  is sequentially continuous with respect to the weak\* topology  $\sigma(\prod L_M, \prod E_{\overline{M}})$ .*

**Lemma 2.3.** ([21]) *Let  $\Omega$  have the segment property. Then for each  $\nu \in W_0^1 L_M(\Omega)$ , there exists a sequence  $\nu_n \in \mathcal{D}(\Omega)$  such that  $\nu_n$  converges to  $\nu$  for the modular convergence in  $W_0^1 L_M(\Omega)$ . Furthermore, if  $\nu \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ , then*

$$\|\nu_n\|_{L^\infty(\Omega)} \leq (N + 1)\|\nu\|_{L^\infty(\Omega)}.$$

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

**Lemma 2.4.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^N$  with finite measure. Let  $M, P, Q$  be  $N$ -functions such that  $Q \ll P$ , and let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that, for a.e.  $x \in \Omega$  and all  $s \in \mathbb{R}$ :*

$$|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|),$$

where  $k_1, k_2$  are real constants and  $c(x) \in E_Q(\Omega)$ .

Then the Nemytskii operator  $N_f$  defined by  $N_f(u)(x) = f(x, u(x))$  is strongly continuous from  $\mathcal{P}(E_M(\Omega), \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$  into  $E_Q(\Omega)$ .

We will also use the following technical lemma.

**Lemma 2.5.** ([26]) *If  $\{f_n\} \subset L^1(\Omega)$  with  $f_n \rightarrow f \in L^1(\Omega)$  a.e. in  $\Omega$ ,  $f_n, f \geq 0$  a.e. in  $\Omega$  and  $\int_\Omega f_n(x) dx \rightarrow \int_\Omega f(x) dx$ , then*

$$f_n \rightarrow f \text{ in } L^1(\Omega).$$

### 3. STRUCTURAL ASSUMPTIONS AND MAIN RESULT

Throughout the paper  $\Omega$  will be a bounded subset of  $\mathbb{R}^N$ ,  $N \geq 2$ , satisfying the segment property. Let  $M$  and  $P$  be two  $N$ -functions such that  $P \ll M$ . Let  $A$  be the non everywhere defined operator defined from its domain  $\mathcal{D}(\Omega) \subset W_0^1 L_M(\Omega)$  into  $W^{-1} L_{\overline{M}}(\Omega)$  given by

$$A(u) := - \operatorname{div} a(\cdot, u, \nabla u),$$

where  $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a Carathéodory function. We assume that there exist a nonnegative function  $c(x)$  in  $E_{\overline{M}}(\Omega)$ ,  $\alpha > 0$  and positive real constants  $k_1, k_2, k_3$  and  $k_4$ , such that for every  $s \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^N$ ,  $\xi' \in \mathbb{R}^N$  ( $\xi \neq \xi'$ ) and for almost every  $x \in \Omega$

$$|a(x, s, \xi)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2 |s|) + k_3 \overline{M}^{-1} M(k_4 |\xi|), \tag{3.1}$$

$$(a(x, s, \xi) - a(x, s, \xi'))(\xi - \xi') > 0, \quad (3.2)$$

$$a(x, s, \xi)\xi \geq \alpha M(|\xi|). \quad (3.3)$$

Here,  $g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a Carathéodory function satisfying for almost every  $x \in \Omega$  and for all  $s \in \mathbb{R}$ ,  $\xi \in \mathbb{R}^N$ ,

$$|g(x, s, \xi)| \leq b(|s|)(d(x) + M(|\xi|)), \quad (3.4)$$

$$g(x, s, \xi)s \geq 0, \quad (3.5)$$

where  $b : \mathbb{R} \rightarrow \mathbb{R}^+$  is a continuous and increasing function while  $d$  is a given nonnegative function in  $L^1(\Omega)$ .

The right-hand side of (1.1) and  $\Phi : \mathbb{R} \rightarrow \mathbb{R}^N$ , are assumed to satisfy

$$f \in L^1(\Omega) \text{ and } |F| \in E_{\overline{M}}(\Omega), \quad (3.6)$$

$$\Phi \in C^0(\mathbb{R}, \mathbb{R}^N). \quad (3.7)$$

Our aim in this paper is to give a meaning to a possible solution of (1.1). In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution  $u$  of problem (1.1) is the Orlicz-Sobolev space  $W_0^1 L_M(\Omega)$ . But when  $u$  is only in  $W_0^1 L_M(\Omega)$  there is no reason for  $\Phi(u)$  to be in  $(L^1(\Omega))^N$  since no growth hypothesis is assumed on the function  $\Phi$ . Thus, the term  $\operatorname{div}(\Phi(u))$  may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally through a pointwise multiplication of equation (1.1) by  $h(u)$  where  $h$  belongs to  $C_c^1(\mathbb{R})$ , the class of  $C^1(\mathbb{R})$  functions with compact support.

**Definition 3.1.** A measurable function  $u : \Omega \rightarrow \mathbb{R}$  is called a renormalized solution of (1.1) if  $u \in W_0^1 L_M(\Omega)$ ,  $a(x, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N$ ,  $g(x, u, \nabla u) \in L^1(\Omega)$ ,  $g(x, u, \nabla u)u \in L^1(\Omega)$ ,

$$\lim_{m \rightarrow +\infty} \int_{\{x \in \Omega : m \leq |u(x)| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx = 0,$$

and

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u)h(u) - \operatorname{div}(\Phi(u)h(u)) + h'(u)\Phi(u)\nabla u \\ +g(x, u, \nabla u)h(u) = fh(u) - \operatorname{div}(Fh(u)) + h'(u)F\nabla u \text{ in } \mathcal{D}'(\Omega), \end{cases} \quad (3.8)$$

for every  $h \in C_c^1(\mathbb{R})$ .

*Remark 3.2.* Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for  $h$  in  $C_c^1(\mathbb{R})$  and  $u$  in  $W_0^1 L_M(\Omega)$ ,  $h(u)$  belongs to  $W^1 L_M(\Omega)$  and for  $\varphi$  in  $\mathcal{D}(\Omega)$  the function  $\varphi h(u)$  belongs to  $W_0^1 L_M(\Omega)$ . Since  $(-\operatorname{div} a(x, u, \nabla u)) \in W^{-1} L_{\overline{M}}(\Omega)$ , we also have

$$\begin{aligned} & \langle -\operatorname{div} a(x, u, \nabla u)h(u), \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} \\ &= \langle -\operatorname{div} a(x, u, \nabla u), \varphi h(u) \rangle_{W^{-1} L_{\overline{M}}(\Omega), W_0^1 L_M(\Omega)} \\ & \forall \varphi \in \mathcal{D}(\Omega). \end{aligned}$$

Finally, since  $\Phi h$  and  $\Phi h' \in (C_c^0(\mathbb{R}))^N$ , for any measurable function  $u$  we have  $\Phi(u)h(u)$  and  $\Phi(u)h'(u) \in (L^\infty\Omega)^N$  and then  $\operatorname{div}(\Phi(u)h(u)) \in W^{-1,\infty}(\Omega)$  and  $\Phi(u)h'(u) \in L_M(\Omega)$ .

Our main result is the following

**Theorem 3.3.** *Suppose that assumptions (3.1)–(3.7) are fulfilled. Then, problem (1.1) has at least one renormalized solution.*

*Remark 3.4.* The condition (3.4) can be replaced by the weaker one

$$|g(x, s, \xi)| \leq d(x) + b(|s|)M(|\xi|),$$

with  $b: \mathbb{R} \rightarrow \mathbb{R}^+$  a continuous function belonging to  $L^1(\mathbb{R})$  and  $d(x) \in L^1(\Omega)$ .

Actually the original equation (1.1) will be recovered whenever  $h(u) \equiv 1$ , but unfortunately this cannot happen in general strong additional requirements on  $u$ . Therefore, (3.8) is to be viewed as a weaker form of (1.1).

#### 4. PROOF OF THE MAIN RESULT

From now on, we will use the standard truncation function  $T_k, k > 0$ , defined for all  $s \in \mathbb{R}$  by  $T_k(s) = \max\{-k, \min\{k, s\}\}$ .

**Step 1: Approximate problems.** Let  $f_n$  be a sequence of  $L^\infty(\Omega)$  functions that converge strongly to  $f$  in  $L^1(\Omega)$ . For  $n \in \mathbb{N}, n \geq 1$ , let us consider the following sequence of approximate equations

$$-\operatorname{div} a(x, u_n, \nabla u_n) + \operatorname{div} \Phi_n(u_n) + g_n(x, u_n, \nabla u_n) = f_n - \operatorname{div} F \text{ in } \mathcal{D}'(\Omega), \tag{4.1}$$

where we have set  $\Phi_n(s) = \Phi(T_n(s))$  and  $g_n(x, s, \xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}|g(x,s,\xi)|}$ . For fixed  $n > 0$ , it's obvious to observe that

$$g_n(x, s, \xi)s \geq 0, \quad |g_n(x, s, \xi)| \leq |g(x, s, \xi)| \text{ and } |g_n(x, s, \xi)| \leq n.$$

Moreover, since  $\Phi$  is continuous one has  $|\Phi_n(t)| \leq \max_{|t| \leq n} |\Phi(t)|$ . Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduces that there exists at least one solution  $u_n$  of the approximate Dirichlet problem (4.1) in the sense

$$\left\{ \begin{array}{l} u_n \in W_0^1 L_M(\Omega), a(x, u_n, \nabla u_n) \in (L_{\overline{M}}(\Omega))^N \text{ and} \\ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla v dx + \int_{\Omega} \Phi_n(u_n) \nabla v dx \\ + \int_{\Omega} g_n(x, u_n, \nabla u_n) v dx = \langle f_n, v \rangle + \int_{\Omega} F \nabla v dx, \text{ for every } v \in W_0^1 L_M(\Omega). \end{array} \right. \tag{4.2}$$

**Step 2: Estimation in  $W_0^1 L_M(\Omega)$ .** Taking  $u_n$  as function test in problem (4.2), we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\Omega} \Phi_n(u_n) \nabla u_n dx \\ & + \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx = \langle f_n, u_n \rangle + \int_{\Omega} F \nabla u_n dx. \end{aligned} \quad (4.3)$$

Define  $\tilde{\Phi}_n \in (C^1(\mathbb{R}))^N$  as  $\tilde{\Phi}_n(t) = \int_0^t \Phi_n(\tau) d\tau$ . Then formally

$\operatorname{div}(\tilde{\Phi}_n(u_n)) = \Phi_n(u_n) \nabla u_n$ ,  $u_n = 0$  on  $\partial\Omega$ ,  $\tilde{\Phi}_n(0) = 0$  and by the Divergence theorem

$$\int_{\Omega} \Phi_n(u_n) \nabla u_n dx = \int_{\Omega} \operatorname{div}(\tilde{\Phi}_n(u_n)) dx = \int_{\partial\Omega} \tilde{\Phi}_n(u_n) \vec{n} ds = 0,$$

where  $\vec{n}$  is the outward pointing unit normal field of the boundary  $\partial\Omega$  ( $ds$  may be used as a shorthand for  $\vec{n} ds$ ). Thus, by virtue of (3.5) and Young's inequality, we get

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq C_1 + \frac{\alpha}{2} \int_{\Omega} M(|\nabla u_n|) dx, \quad (4.4)$$

which, together with (3.3) give

$$\int_{\Omega} M(|\nabla u_n|) dx \leq C_2. \quad (4.5)$$

Moreover, we also have

$$\int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx \leq C_3. \quad (4.6)$$

As a consequence of (4.5) there exist a subsequence of  $\{u_n\}_n$ , still indexed by  $n$ , and a function  $u \in W_0^1 L_M(\Omega)$  such that

$$\begin{aligned} u_n & \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_M^-(\Omega)), \\ u_n & \rightarrow u \text{ strongly in } E_M(\Omega) \text{ and a. e. in } \Omega. \end{aligned} \quad (4.7)$$

**Step 3: Boundedness of  $(a(x, u_n, \nabla u_n))_n$  in  $(L_M^-(\Omega))^N$ .** Let  $w \in (E_M(\Omega))^N$  with  $\|w\|_M \leq 1$ . Thanks to (3.2), we can write

$$\left( a(x, u_n, \nabla u_n) - \left( a(x, u_n, \frac{w}{k_4}) \right) \right) \left( \nabla u_n - \frac{w}{k_4} \right) \geq 0,$$

which implies

$$\begin{aligned} \frac{1}{k_4} \int_{\Omega} a(x, u_n, \nabla u_n) w dx & \leq \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \\ & + \int_{\Omega} a\left(x, u_n, \frac{w}{k_4}\right) \left( \frac{w}{k_4} - \nabla u_n \right) dx. \end{aligned}$$

Thanks to (4.4) and (4.5), one has

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq C_5.$$



Define  $\lambda = 1 + k_1 + k_3$ . By the growth condition (3.1) and Young's inequality, one can write

$$\begin{aligned} & \left| \int_{\Omega} a\left(x, u_n, \frac{w}{k_4}\right) \left(\frac{w}{k_4} - \nabla u_n\right) dx \right| \\ & \leq \left(1 + \frac{1}{k_4}\right) \left( \int_{\Omega} \overline{M}(c(x)) dx + k_1 \int_{\Omega} \overline{M} \overline{P}^{-1} M(k_2|u_n|) dx \right. \\ & \quad \left. + k_3 \int_{\Omega} M(|w|) dx \right) + \frac{\lambda}{k_4} \int_{\Omega} M(|w|) dx + \lambda \int_{\Omega} M(|\nabla u_n|) dx. \end{aligned}$$

By virtue of [18] and Lemma 4.14 of [20], there exists an  $N$ -function  $Q$  such that  $M \ll Q$  and the space  $W_0^1 L_M(\Omega)$  is continuously embedded into  $L_Q(\Omega)$ . Thus, by (4.5) there exists a constant  $c_0 > 0$ , not depending on  $n$ , satisfying  $\|u_n\|_Q \leq c_0$ . Since  $M \ll Q$ , we can write  $M(k_2 t) \leq Q(\frac{t}{c_0})$ , for  $t > 0$  large enough. As  $P \ll M$ , we can find a constant  $c_1$ , not depending on  $n$ , such that  $\int_{\Omega} \overline{M} \overline{P}^{-1} M(k_2|u_n|) dx \leq \int_{\Omega} Q(\frac{|u_n|}{c_0}) + c_1$ . Hence, we conclude that the quantity  $\left| \int_{\Omega} a(x, u_n, \nabla u_n w) dx \right|$  is bounded from above for all  $w \in (E_M(\Omega))^N$  with  $\|w\|_M \leq 1$ . Using the Orlicz norm we deduce that

$$\left( a(x, u_n, \nabla u_n) \right)_n \text{ is bounded in } (L_{\overline{M}}(\Omega))^N. \tag{4.8}$$

**Step 4: Renormalization identity for the approximate solutions.** For any  $m \geq 1$ , define  $\theta_m(r) = T_{m+1}(r) - T_m(r)$ . Observe that by [19, Lemma2] one has  $\theta_m(u_n) \in W_0^1 L_M(\Omega)$ . The use of  $\theta_m(u_n)$  as test function in (4.2) yields

$$\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n dx,$$

By Hölder's inequality and 4.5 we have

$$\begin{aligned} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx & \leq \langle f_n, \theta_m(u_n) \rangle \\ & + C_6 \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx. \end{aligned}$$

It's not hard to see that

$$\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.$$

So that by (4.5) and (4.7) one can deduce that

$$\theta_m(u_n) \rightharpoonup \theta_m(u) \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)).$$

Note that as  $m$  goes to  $\infty$ ,  $\theta_m(u) \rightharpoonup 0$  weakly in  $W_0^1 L_M(\Omega)$  for  $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$ , and since  $f_n$  converges strongly in  $L^1(\Omega)$ , by Lebesgue's theorem we have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle f_n, \theta_m(u_n) \rangle = 0.$$

By (3.3) we finally have

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \quad (4.9)$$

**Step 5: Almost everywhere convergence of the gradients.** Define

$\phi(s) = se^{\lambda s^2}$  with  $\lambda = \left(\frac{b(k)}{2\alpha}\right)^2$ . One can easily verify that for all  $s \in \mathbb{R}$

$$\phi'(s) - \frac{b(k)}{\alpha} |\phi(s)| \geq \frac{1}{2}. \quad (4.10)$$

For  $m \geq k$ , we define the function  $\psi_m$  by

$$\begin{cases} \psi_m(s) = 1 & \text{if } |s| \leq m, \\ \psi_m(s) = m + 1 - |s| & \text{if } m \leq |s| \leq m + 1, \\ \psi_m(s) = 0 & \text{if } |s| \geq m + 1. \end{cases}$$

By virtue of [21, Theorem 4] there exists a sequence  $\{v_j\}_j \subset D(\Omega)$  such that  $v_j \rightarrow u$  in  $W_0^1 L_M(\Omega)$  for the modular convergence and a.e. in  $\Omega$ . Let us define the following functions  $\theta_n^j = T_k(u_n) - T_k(v_j)$ ,  $\theta^j = T_k(u) - T_k(v_j)$  and  $z_{n,m}^j = \phi(\theta_n^j) \psi_m(u_n)$ . Using  $z_{n,m}^j \in W_0^1 L_M(\Omega)$  as test function in (4.2) we get

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx + \int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx \\ & + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \psi_m'(u_n) \phi(T_k(u_n) - T_k(v_j)) dx \\ & + \int_{\Omega} g_n(x, u_n, \nabla u_n) z_{n,m}^j dx = \int_{\Omega} f_n z_{n,m}^j dx + \int_{\Omega} F \nabla z_{n,m}^j dx. \end{aligned} \quad (4.11)$$

From now on we denote by  $\epsilon_i(n, j)$ ,  $i = 0, 1, 2, \dots$ , various sequences of real numbers which tend to zero, when  $n$  and  $j \rightarrow +\infty$ , i. e.

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \epsilon_i(n, j) = 0.$$

In view of (4.7), we have  $z_{n,m}^j \rightharpoonup \phi(\theta^j) \psi_m(u)$  weakly in  $L^\infty(\Omega)$  for  $\sigma^*(L^\infty, L^1)$  as  $n \rightarrow +\infty$ , which yields

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n z_{n,m}^j dx = \int_{\Omega} f \phi(\theta^j) \psi_m(u) dx,$$

and since  $\phi(\theta^j) \rightharpoonup 0$  weakly in  $L^\infty(\Omega)$  for  $\sigma(L^\infty, L^1)$  as  $j \rightarrow +\infty$ , we have

$$\lim_{j \rightarrow +\infty} \int_{\Omega} f \phi(\theta^j) \psi_m(u) dx = 0.$$

Thus, we write

$$\int_{\Omega} f_n z_{n,m}^j dx = \epsilon_0(n, j).$$

Thanks to (4.5) and (4.7), we have as  $n \rightarrow +\infty$ ,

$$z_{n,m}^j \rightharpoonup \phi(\theta^j) \psi_m(u) \text{ in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)),$$

which implies that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} F \nabla z_{n,m}^j dx = \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx + \int_{\Omega} F \nabla u \phi(\theta^j) \psi'_m(u) dx$$

On the one hand, by Lebesgue's theorem we get

$$\lim_{j \rightarrow +\infty} \int_{\Omega} F \nabla u \phi(\theta^j) \psi'_m(u) dx = 0,$$

on the other hand, we write

$$\begin{aligned} \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx &= \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx \\ &\quad - \int_{\Omega} F \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx, \end{aligned}$$

so that, by Lebesgue's theorem one has

$$\lim_{j \rightarrow +\infty} \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) dx.$$

Let  $\lambda > 0$  such that  $M\left(\frac{|\nabla v_j - \nabla u|}{\lambda}\right) \rightarrow 0$  strongly in  $L^1(\Omega)$  as  $j \rightarrow +\infty$  and  $M\left(\frac{|\nabla u|}{\lambda}\right) \in L^1(\Omega)$ , the convexity of the  $N$ -function  $M$  allows us to have

$$\begin{aligned} M\left(\frac{|\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) - \nabla T_k(u) \psi_m(u)|}{4\lambda \phi'(2k)}\right) \\ = \frac{1}{4} M\left(\frac{|\nabla v_j - \nabla u|}{\lambda}\right) + \frac{1}{4} \left(1 + \frac{1}{\phi'(2k)}\right) M\left(\frac{|\nabla u|}{\lambda}\right). \end{aligned}$$

Then, by using the modular convergence of  $\{\nabla v_j\}$  in  $(L_M(\Omega))^N$  and Vitali's theorem, we obtain

$$\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) \rightarrow \nabla T_k(u) \psi_m(u) \text{ in } (L_M(\Omega))^N, \text{ as } j \text{ tends to } +\infty,$$

for the modular convergence, and then

$$\lim_{j \rightarrow +\infty} \int_{\Omega} F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) dx.$$

We have proved that

$$\int_{\Omega} F \nabla z_{n,m}^j dx = \epsilon_1(n, j).$$

It's easy to see that by the modular convergence of the sequence  $\{v_j\}_j$ , one has

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \psi'_m(u_n) \phi(T_k(u_n) - T_k(v_j)) dx = 0,$$

while for the third term in the left-hand side of (4.11) we can write

$$\begin{aligned} &\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx \\ &= \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) dx - \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx. \end{aligned}$$

Firstly, we have

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) dx = 0.$$

In view of (4.7), one has

$$\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n) \rightarrow \Phi(u) \phi'(\theta^j) \psi_m(u),$$

almost everywhere in  $\Omega$  as  $n$  tends to  $+\infty$ . Furthermore, we can check that

$$\|\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n)\|_{\overline{M}} \leq \overline{M} (c_m \phi'(2k)) |\Omega| + 1,$$

where  $c_m = \max_{|t| \leq m+1} \Phi(t)$ . Applying [27, Theorem 14.6] we get

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx.$$

Using the modular convergence of the sequence  $\{v_j\}_j$ , we obtain

$$\lim_{j \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx.$$

Then, using again the Divergence theorem we get

$$\int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx = 0.$$

Therefore, we write

$$\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx = \epsilon_2(n, j).$$

Since  $g_n(x, u_n, \nabla u_n) z_{n,m}^j \geq 0$  on the set  $\{|u_n| > k\}$  and  $\psi_m(u_n) = 1$  on the set  $\{|u_n| \leq k\}$ , from (4.11) we obtain

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \leq \epsilon_3(n, j). \quad (4.12)$$

We now evaluate the first term of the left-hand side of (4.12) by writing

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx \\ &= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) \psi_m(u_n) dx \\ & \quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m'(u_n) dx \\ &= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) dx \\ & \quad - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx \\ & \quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m'(u_n) dx, \end{aligned}$$

and then

$$\begin{aligned}
 & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx \\
 &= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) \\
 &\quad (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx \\
 &\quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx \\
 &\quad - \int_{\Omega \setminus \Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx \\
 &\quad - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx \\
 &\quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx,
 \end{aligned} \tag{4.13}$$

where by  $\chi_j^s$ ,  $s > 0$ , we denote the characteristic function of the subset

$$\Omega_j^s = \{x \in \Omega : |\nabla T_k(v_j)| \leq s\}.$$

For fixed  $m$  and  $s$ , we will pass to the limit in  $n$  and then in  $j$  in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have

$$\begin{aligned}
 & \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx \\
 & \rightarrow \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j) dx,
 \end{aligned}$$

as  $n \rightarrow +\infty$ . Since by lemma (2.4) one has

$$a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) \rightarrow a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j),$$

strongly in  $(E_{\overline{M}}(\Omega))^N$  as  $n \rightarrow \infty$ , while by (4.5)

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u),$$

weakly in  $(L_M(\Omega))^N$ . Let  $\chi^s$  denote the characteristic function of the subset

$$\Omega^s = \{x \in \Omega : |\nabla T_k(u)| \leq s\}.$$

As  $\nabla T_k(v_j) \chi_j^s \rightarrow \nabla T_k(u) \chi^s$  strongly in  $(E_M(\Omega))^N$  as  $j \rightarrow +\infty$ , one has

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \cdot (\nabla T_k(u) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j) dx \rightarrow 0,$$

as  $j \rightarrow \infty$ . Then

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx = \epsilon_4(n, j). \tag{4.14}$$

We now estimate the third term of (4.13). It's easy to see that by (3.3),  $a(x, s, 0) = 0$  for almost everywhere  $x \in \Omega$  and for all  $s \in \mathbb{R}$ . Thus, from (4.8) we have that  $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$  is bounded in  $(L_{\overline{M}}(\Omega))^N$  for all  $k \geq 0$ .

Therefore, there exist a subsequence still indexed by  $n$  and a function  $l_k$  in  $(L_{\overline{M}}(\Omega))^N$  such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup l_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \quad (4.15)$$

Then, since  $\nabla T_k(v_j)\chi_{\Omega \setminus \Omega_j^s} \in (E_{\overline{M}}(\Omega))^N$ , we obtain

$$\int_{\Omega \setminus \Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx \rightarrow \int_{\Omega \setminus \Omega_j^s} l_k \nabla T_k(v_j) \phi'(\theta^j) dx,$$

as  $n \rightarrow +\infty$ . The modular convergence of  $\{v_j\}$  allows us to get

$$- \int_{\Omega \setminus \Omega_j^s} l_k \nabla T_k(v_j) \phi'(\theta^j) dx \rightarrow - \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx,$$

as  $j \rightarrow +\infty$ . This, proves

$$- \int_{\Omega \setminus \Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) dx = - \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + \epsilon_5(n, j). \quad (4.16)$$

As regards the fourth term, observe that  $\psi_m(u_n) = 0$  on the subset  $\{|u_n| \geq m+1\}$ , so we have

$$\begin{aligned} & - \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \\ & \quad - \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx. \end{aligned}$$

Since

$$\begin{aligned} & - \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \\ & \quad - \int_{\{|u| > k\}} l_{m+1} \nabla T_k(u) \psi_m(u) dx + \epsilon_5(n, j), \end{aligned}$$

observing that  $\nabla T_k(u) = 0$  on the subset  $\{|u| > k\}$ , one has

$$- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \epsilon_6(n, j). \quad (4.17)$$

For the last term of (4.13), we have

$$\begin{aligned} & \left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \right| \\ & = \left| \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \right| \\ & \leq \phi(2k) \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx. \end{aligned}$$

To estimate the last term of the previous inequality, we use  $(T_1(u_n - T_m(u_n))) \in W_0^1 L_M(\Omega)$  as test function in (4.2), to get

$$\begin{aligned} & \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n dx \\ & + \int_{\{|u_n| \geq m\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) dx = \langle f_n, T_1(u_n - T_m(u_n)) \rangle \\ & \quad + \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n dx. \end{aligned}$$

By Divergence theorem, we have

$$\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n dx = 0.$$

Using the fact that  $g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \geq 0$  on the subset  $\{|u_n| \geq m\}$  and Young's inequality, we get

$$\begin{aligned} & \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \\ & \leq \langle f_n, T_1(u_n - T_m(u_n)) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx. \end{aligned}$$

It follows that

$$\begin{aligned} & \left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) dx \right| \\ & \leq 2\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right). \end{aligned} \tag{4.18}$$

From (4.14), (4.16), (4.17) and (4.18) we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx \\ & \geq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) \\ & \quad (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^j) dx \\ & \quad - \alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\ & \quad - \int_{\Omega \setminus \Omega^s} l_k \cdot \nabla T_k(u) dx + \epsilon_7(n, j). \end{aligned} \tag{4.19}$$

Now, we turn to second term in the left-hand side of (4.12). We have

$$\begin{aligned} & \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \right| \\ & = \left| \int_{\{|u_n| \leq k\}} g_n(x, T_k(u_n), \nabla T_k(u_n)) \phi(\theta_n^j) dx \right| \\ & \leq b(k) \int_{\Omega} M(|\nabla T_k(u_n)|) |\phi(\theta_n^j)| dx + b(k) \int_{\Omega} d(x) |\phi(\theta_n^j)| dx \\ & \leq \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) |\phi(\theta_n^j)| dx + \epsilon_8(n, j). \end{aligned}$$

Then

$$\begin{aligned}
& \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \right| \\
& \leq \frac{b(k)}{\alpha} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) \\
& \quad (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) |\phi(\theta_n^j)| dx \\
& \quad + \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) |\phi(\theta_n^j)| dx \\
& \quad + \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s |\phi(\theta_n^j)| dx + \epsilon_9(n, j).
\end{aligned} \tag{4.20}$$

We proceed as above to get

$$\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) |\phi(\theta_n^j)| dx = \epsilon_9(n, j)$$

and

$$\frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s |\phi(\theta_n^j)| dx = \epsilon_{10}(n, j).$$

Hence, we have

$$\begin{aligned}
& \left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \right| \\
& \leq \frac{b(k)}{\alpha} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) \\
& \quad (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) |\phi(\theta_n^j)| dx + \epsilon_{11}(n, j).
\end{aligned} \tag{4.21}$$

Combining (4.12), (4.19) and (4.21), we get

$$\begin{aligned}
& \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \\
& \quad \left( \phi'(\theta_n^j) - \frac{b(k)}{\alpha} |\phi(\theta_n^j)| \right) dx \\
& \leq \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + \alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\
& \quad + \epsilon_{12}(n, j).
\end{aligned}$$

By (4.10), we have

$$\begin{aligned}
& \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx \\
& \leq 2 \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + 4\alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\
& \quad + \epsilon_{12}(n, j).
\end{aligned} \tag{4.22}$$



On the other hand we can write

$$\begin{aligned} & \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\ &= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \\ & \quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s) dx \\ & \quad - \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\ & \quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \end{aligned}$$

We shall pass to the limit in  $n$  and then in  $j$  in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain

$$\begin{aligned} & \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s) dx = \epsilon_{13}(n, j), \\ & \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx = \epsilon_{14}(n, j), \\ & \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \\ &= \epsilon_{15}(n, j). \end{aligned} \tag{4.23}$$

So that

$$\begin{aligned} & \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\ &= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \\ & \quad + \epsilon_{16}(n, j). \end{aligned} \tag{4.24}$$

Let  $r \leq s$ . Using (3.2), (4.22) and (4.24) we can write

$$\begin{aligned} 0 &\leq \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\ &\leq \int_{\Omega^s} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\ &= \int_{\Omega^s} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\ &\leq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx \\ &= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx \\ & \quad + \epsilon_{15}(n, j) \\ &\leq 2 \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + 2\alpha\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\ & \quad + \epsilon_{17}(n, j). \end{aligned}$$

By passing to the superior limit over  $n$  and then over  $j$

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow +\infty} \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\ &\leq 2 \int_{\Omega \setminus \Omega^s} l_k \nabla T_k(u) dx + 4\alpha\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right). \end{aligned}$$

Letting  $s \rightarrow +\infty$  and then  $m \rightarrow +\infty$ , taking into account that  $l_k \nabla T_k(u) \in L^1(\Omega)$ ,  $f \in L^1(\Omega)$ ,  $|F| \in (E_{\overline{M}}(\Omega))^N$ ,  $|\Omega \setminus \Omega^s| \rightarrow 0$ , and  $|\{m \leq |u| \leq m+1\}| \rightarrow 0$ , one has

$$\int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx, \quad (4.25)$$

tends to 0 as  $n \rightarrow +\infty$ . As in [20], we deduce that there exists a subsequence of  $\{u_n\}$  still indexed by  $n$  such that

$$\nabla u_n \rightarrow \nabla u \text{ a. e. in } \Omega. \quad (4.26)$$

Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get

$$a(x, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N$$

and

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \quad (4.27)$$

**Step 6: Modular convergence of the truncations.** Going back to equation (4.22), we can write

$$\begin{aligned} &\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ &\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx \\ &\quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx \\ &\quad + 2\alpha\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\ &\quad + 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{12}(n, j). \end{aligned}$$

By (4.23) we get

$$\begin{aligned} &\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ &\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx \\ &\quad + 2\alpha\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) \\ &\quad + 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{18}(n, j). \end{aligned}$$

We now pass to the superior limit over  $n$  in both sides of this inequality using (4.27), to obtain

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ & \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(v_j) \chi_j^s dx \\ & \quad + 2\alpha\phi(2k) \left( \int_{\{m \leq |u|\}} |f| dx + \int_{\{m \leq |u| \leq m+1\}} \overline{M}(|F|) dx \right) \\ & \quad + 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx. \end{aligned}$$

We then pass to the limit in  $j$  to get

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ & \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \chi^s dx \\ & \quad + 2\alpha\phi(2k) \left( \int_{\{m \leq |u|\}} |f| dx + \int_{\{m \leq |u| \leq m+1\}} \overline{M}(|F|) dx \right) \\ & \quad + 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx. \end{aligned}$$

Letting  $s$  and then  $m \rightarrow +\infty$ , one has

$$\limsup_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.$$

On the other hand, by (3.3), (4.5), (4.26) and Fatou's lemma, we have

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx.$$

It follows that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx = \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.$$

By Lemma 2.5 we conclude that for every  $k > 0$

$$a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \rightarrow a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u), \tag{4.28}$$

strongly in  $L^1(\Omega)$ . The convexity of the  $N$ -function  $M$  and (3.3) allow us to have

$$\begin{aligned} & M\left(\frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) \\ & \leq \frac{1}{2\alpha} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) + \frac{1}{2\alpha} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u). \end{aligned}$$

From Vitali's theorem we deduce

$$\lim_{|E| \rightarrow 0} \sup_n \int_E M\left(\frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) dx = 0.$$

Thus, for every  $k > 0$

$$T_k(u_n) \rightarrow T_k(u) \text{ in } W_0^1 L_M(\Omega),$$

for the modular convergence.

**Step 7: Compactness of the nonlinearities.** We need to prove that

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \quad (4.29)$$

By virtue of (4.7) and (4.26) one has

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \quad \text{a. e. in } \Omega. \quad (4.30)$$

Let  $E$  be measurable subset of  $\Omega$  and let  $m > 0$ . Using (3.3) and (3.4) we can write

$$\begin{aligned} & \int_E |g_n(x, u_n, \nabla u_n)| dx \\ &= \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| dx \\ &\leq b(m) \int_E d(x) dx + b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) dx \\ &\quad + \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx. \end{aligned}$$

From (3.5) and (4.6), we deduce that

$$0 \leq \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \leq C_3.$$

So

$$0 \leq \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \leq \frac{C_3}{m}.$$

Then

$$\lim_{m \rightarrow +\infty} \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx = 0.$$

Thanks to (4.28) the sequence  $\{a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)\}_n$  is equi-integrable. This fact allows us to get

$$\lim_{|E| \rightarrow 0} \sup_n \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) dx = 0.$$

This shows that  $g_n(x, u_n, \nabla u_n)$  is equi-integrable. Thus, Vitali's theorem implies that  $g(x, u, \nabla u) \in L^1(\Omega)$  and

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \text{ strongly in } L^1(\Omega).$$

**Step 8: Renormalization identity for the solutions.** In this step we prove that

$$\lim_{m \rightarrow +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u dx = 0. \quad (4.31)$$

Indeed, for any  $m \geq 0$  we can write

$$\begin{aligned} & \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \\ &= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) dx \\ &= \int_{\Omega} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) dx \\ &\quad - \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) dx. \end{aligned}$$

In view of (4.28), we can pass to the limit as  $n$  tends to  $+\infty$  for fixed  $m \geq 0$

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \\ &= \int_{\Omega} a(x, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_{m+1}(u) dx \\ &\quad - \int_{\Omega} a(x, T_m(u), \nabla T_m(u)) \nabla T_m(u) dx \\ &= \int_{\Omega} a(x, u, \nabla u) (\nabla T_{m+1}(u) - \nabla T_m(u)) dx \\ &= \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u dx. \end{aligned}$$

Having in mind (4.9), we can pass to the limit as  $m$  tends to  $+\infty$  to obtain (4.31).

**Step 9: Passing to the limit.** Thanks to (4.28) and Lemma (2.5), we obtain

$$a(x, u_n, \nabla u_n) \nabla u_n \rightarrow a(x, u, \nabla u) \nabla u \text{ strongly in } L^1(\Omega). \tag{4.32}$$

Let  $h \in \mathcal{C}_c^1(\mathbb{R})$  and  $\varphi \in \mathcal{D}(\Omega)$ . Inserting  $h(u_n)\varphi$  as test function in (4.2), we get

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi dx + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) dx \\ &+ \int_{\Omega} \Phi_n(u_n) \nabla (h(u_n) \varphi) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi dx \\ &= \langle f_n, h(u_n) \varphi \rangle + \int_{\Omega} F \nabla (h(u_n) \varphi) dx. \end{aligned} \tag{4.33}$$

We shall pass to the limit as  $n \rightarrow +\infty$  in each term of the equality (4.33). Since  $h$  and  $h'$  have compact support on  $\mathbb{R}$ , there exists a real number  $\nu > 0$ , such that  $\text{supp } h \subset [-\nu, \nu]$  and  $\text{supp } h' \subset [-\nu, \nu]$ . For  $n > \nu$ , we can write

$$\Phi_n(t)h(t) = \Phi(T_\nu(t))h(t) \text{ and } \Phi_n(t)h'(t) = \Phi(T_\nu(t))h'(t).$$

Moreover, the functions  $\Phi h$  and  $\Phi h'$  belong to  $(\mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$ . Observe first that the sequence  $\{h(u_n)\varphi\}_n$  is bounded in  $W_0^1 L_M(\Omega)$ . Indeed, let  $\rho > 0$

be a positive constant such that  $\|h(u_n)\nabla\varphi\|_\infty \leq \rho$  and  $\|h'(u_n)\varphi\|_\infty \leq \rho$ . Using the convexity of the  $N$ -function  $M$  and taking into account (4.5) we have

$$\begin{aligned} \int_{\Omega} M\left(\frac{|\nabla(h(u_n)\varphi)|}{2\rho}\right) dx &\leq \int_{\Omega} M\left(\frac{|h(u_n)\nabla\varphi| + |h'(u_n)\varphi||\nabla u_n|}{2\rho}\right) dx \\ &\leq \frac{1}{2}M(1)|\Omega| + \frac{1}{2}\int_{\Omega} M(|\nabla u_n|)dx \\ &\leq \frac{1}{2}M(1)|\Omega| + \frac{1}{2}C_2. \end{aligned}$$

This, together with (4.7), imply that

$$h(u_n)\varphi \rightharpoonup h(u)\varphi \text{ weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}). \quad (4.34)$$

This enables us to get

$$\langle f_n, h(u_n)\varphi \rangle \rightarrow \langle f, h(u)\varphi \rangle.$$

Let  $E$  be a measurable subset of  $\Omega$ . Define  $c_\nu = \max_{|t|\leq\nu} \Phi(t)$ . Let us denote by  $\|v\|_{(M)}$  the Orlicz norm of a function  $v \in L_M(\Omega)$ . Using strengthened Hölder inequality with both Orlicz and Luxemburg norms, we get

$$\begin{aligned} \|\Phi(T_\nu(u_n))\chi_E\|_{(\overline{M})} &= \sup_{\|v\|_M \leq 1} \left| \int_E \Phi(T_\nu(u_n))v dx \right| \\ &\leq c_\nu \sup_{\|v\|_M \leq 1} \|\chi_E\|_{(\overline{M})} \|v\|_M \\ &\leq c_\nu |E| M^{-1}\left(\frac{1}{|E|}\right). \end{aligned}$$

Thus, we get

$$\lim_{|E|\rightarrow 0} \sup_n \|\Phi(T_\nu(u_n))\chi_E\|_{(\overline{M})} = 0.$$

Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain

$$\Phi(T_\nu(u_n)) \rightarrow \Phi(T_\nu(u)) \text{ strongly in } (E_{\overline{M}})^N,$$

which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have

$$\int_{\Omega} \Phi(T_\nu(u_n))\nabla(h(u_n)\varphi) dx \rightarrow \int_{\Omega} \Phi(T_\nu(u))\nabla(h(u)\varphi) dx.$$

We remark that

$$|a(x, u_n, \nabla u_n)\nabla u_n h'(u_n)\varphi| \leq \rho a(x, u_n, \nabla u_n)\nabla u_n.$$

Consequently, using (4.32) and Vitali's theorem, we obtain

$$\int_{\Omega} a(x, u_n, \nabla u_n)\nabla u_n h'(u_n)\varphi dx \rightarrow \int_{\Omega} a(x, u, \nabla u)\nabla u h'(u)\varphi dx.$$

and

$$\int_{\Omega} F\nabla u_n h'(u_n)\varphi dx \rightarrow \int_{\Omega} F\nabla u h'(u)\varphi dx.$$

For the second term of (4.33), as above we have

$$h(u_n)\nabla\varphi \rightarrow h(u)\nabla\varphi \text{ strongly in } (E_M(\Omega))^N,$$

which together with (4.27) give

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) dx \rightarrow \int_{\Omega} a(x, u, \nabla u) \nabla \varphi h(u) dx$$

and

$$\int_{\Omega} F \nabla \varphi h(u_n) dx \rightarrow \int_{\Omega} F \nabla \varphi h(u) dx.$$

The fact that  $h(u_n)\varphi \rightharpoonup h(u)\varphi$  weakly in  $L^\infty(\Omega)$  for  $\sigma^*(L^\infty, L^1)$  and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get

$$\int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi dx \rightarrow \int_{\Omega} g(x, u, \nabla u) h(u) \varphi dx.$$

At this point we can pass to the limit in each term of (4.33) to get

$$\begin{aligned} & \int_{\Omega} a(x, u, \nabla u) (\nabla \varphi h(u) + h'(u) \varphi \nabla u) dx + \int_{\Omega} \Phi(u) h'(u) \varphi \nabla u dx \\ & + \int_{\Omega} \Phi(u) h(u) \nabla \varphi dx + \int_{\Omega} g(x, u, \nabla u) h(u) \varphi dx \\ & = \langle f, h(u) \varphi \rangle + \int_{\Omega} F (\nabla \varphi h(u) + h'(u) \varphi \nabla u) dx, \end{aligned}$$

for all  $h \in C_c^1(\mathbb{R})$  and for all  $\varphi \in \mathcal{D}(\Omega)$ . Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou's lemma to get  $g(x, u, \nabla u)u \in L^1(\Omega)$ . By virtue of (4.7), (4.27), (4.29), (4.31), the function  $u$  is a renormalized solution of problem (1.1).

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