Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces

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Abstract. The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems

\[-\text{div} \left( a(x, u, \nabla u) + \Phi(u) \right) + g(x, u, \nabla u) = f - \text{div} F,\]

in a bounded open set \(\Omega\) and \(u = 0\) on \(\partial\Omega\), in the framework of Orlicz-Sobolev spaces without any restriction on the \(M\)-function of the Orlicz spaces, where \(-\text{div} \left( a(x, u, \nabla u) \right)\) is a Leray-Lions operator defined from \(W_0^{1, \mathcal{M}}(\Omega)\) into its dual, \(\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)\). The function \(g(x, u, \nabla u)\) is a non linear lower order term with natural growth with respect to \(|\nabla u|\), satisfying the sign condition and the datum \(\mu\) is assumed to belong to \(L^1(\Omega) + W^{-1,E}_{\mathcal{M}}(\Omega)\).

Keywords: Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.


1. Introduction

Let \(\Omega\) be a bounded open set of \(\mathbb{R}^N, N \geq 2\), and let \(M\) be an \(N\)-function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem

\[
\begin{aligned}
&A(u) - \text{div} \Phi(u) + g(x, u, \nabla u) = f - \text{div} F & \text{in } \Omega, \\
&u = 0 & \text{on } \partial\Omega.
\end{aligned}
\]  (1.1)
Here, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$, while the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The non everywhere defined nonlinear operator $A(u) = -\text{div} (a(x, u, \nabla u))$ acts from its domain $D(A) \subset W_0^1L_M(\Omega)$ into $W^{-1}L_\overline{M}(\Omega)$. The function $a(x, u, \nabla u)$ is assumed to satisfy, among others, $a(x, u, \nabla u)$ nonstandard growth condition governed by the $N$-function $M$, and the source term $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$. $\overline{M}$ stands for the conjugate of $M$.

We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with $\Phi \equiv 0$, $g \equiv 0$ and $L^1(\Omega)$-data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when $f$ is a bounded Radon measure datum and $g$ grows at most like $|\nabla u|^{p-1}$ by Beta et al. in [9, 10, 11] with $\Phi \equiv 0$ and by Guibé and Mercaldo in [23, 24] when $\Phi(u)$ behaves at most like $|u|^{p-1}$. Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in $L^1(\Omega)$. Recall that to get both existence and uniqueness of a solution to problems with $L^1$-data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall’Aglio [14].

The authors in [5] have dealt with the equation (1.1) with $g = g(x, u)$ and $\mu \in W^{-1}E_{\overline{M}}(\Omega)$, under the restriction that the $N$-function $M$ satisfies the $\Delta_2$-condition. This work was then extended in [2] for $N$-functions not satisfying necessarily the $\Delta_2$-condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the $N$-function $M$. Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $\mu \in W^{-1}E_{\overline{M}}(\Omega)$, an existence result has been proved in [3]. Specific examples to which our results apply include the following:

$$
- \text{div} \left( |\nabla u|^{p-2} \nabla u + |u|^s u \right) + u |\nabla u|^p = \mu \text{ in } \Omega,
$$

$$
- \text{div} \left( |\nabla u|^{p-2} \nabla \log (1 + |\nabla u|) + |u|^s u \right) = \mu \text{ in } \Omega,
$$

$$
- \text{div} \left( \frac{M(|\nabla u|) \nabla u}{|\nabla u|^2} + |u|^s u \right) + M(|\nabla u|) = \mu \text{ in } \Omega,
$$

where $p > 1$, $s > 0$, $\beta > 0$ and $\mu$ is a given Radon measure on $\Omega$.

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form $f - \text{div} F$ with $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, in the setting of Orlicz spaces without any restriction on the $N$-functions $M$. The approximate equations provide a $W_0^1L_M(\Omega)$ bound for the corresponding solution $u_n$. This allows us to obtain...
a function $u$ as a limit of the sequence $u_n$. Hence, appear two difficulties. The first one is how to give a sense to $\Phi(u)$, the second difficulty lies in the need of the convergence almost everywhere of the gradients of $u_n$ in $\Omega$. This is done by using suitable test functions built upon $u_n$ which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

2. Preliminaries

 Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an $N$-function, i.e., $M$ is continuous, increasing, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$, and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$. Equivalently, $M$ admits the representation:

$$M(t) = \int_0^t a(s) \, ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to $+\infty$ as $t \to +\infty$.

The conjugate of $M$ is also an $N$-function and it is defined by

$$\overline{M} = \int_0^t \bar{a}(s) \, ds,$$

where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is the function $\bar{a}(t) = \sup \{ s : a(s) \leq t \}$ (see [1]).

An $N$-function $M$ is said to satisfy the $\Delta_2$-condition if, for some $k$,

$$M(2t) \leq k M(t) \quad \forall t \geq 0,$$

(2.1)

when (2.1) holds only for $t \geq t_0 > 0$ then $M$ is said to satisfy the $\Delta_2$-condition near infinity. Moreover, we have the following Young’s inequality

$$st \leq M(t) + \overline{M}(s), \quad \forall s, t \geq 0.$$

Given two $N$-functions, we write $P \ll Q$ to indicate $P$ grows essentially less rapidly than $Q$; i.e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if

$$\lim_{t \to \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $k_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions $u$ on $\Omega$ such that

$$\int_\Omega M(|u(x)|) \, dx < +\infty \quad \text{(resp. } \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) \, dx < +\infty \text{ for some } \lambda > 0).$$
The set $L_M(\Omega)$ is a Banach space under the norm
\[
\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\},
\]
and $k_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{\mathcal{M}}}(\Omega)$ by means of the pairing $\int_{\Omega} uv \, dx$, and the dual norm of $L_{\overline{\mathcal{M}}}(\Omega)$ is equivalent to $\| \cdot \|_{\overline{\mathcal{M}},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1 L_M(\Omega)$ [resp. $W^1 E_M(\Omega)$] is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm
\[
\|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}.
\]

Thus, $W^1 L_M(\Omega)$ and $W^1 E_M(\Omega)$ can be identified with subspaces of product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod \overline{\mathcal{M}})$ and $\sigma(\prod L_M, \prod \overline{\mathcal{M}})$.

The space $W^1_0 E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1 E_M(\Omega)$ and the space $W^1_0 L_M(\Omega)$ as the $\sigma(\prod L_M, \prod \overline{\mathcal{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1 L_M(\Omega)$. We say that $u_n$ converges to $u$ for the modular convergence in $W^1 L_M(\Omega)$ if for some $\lambda > 0$, $\int_{\Omega} M \left( \frac{D^\alpha u_n - D^\alpha u}{\lambda} \right) \, dx \to 0$ for all $|\alpha| \leq 1$.

This implies convergence for $\sigma(\prod L_M, \prod \overline{\mathcal{M}})$. If $M$ satisfies the $\Delta_2$ condition on $\mathbb{R}^+$ (near infinity only when $\Omega$ has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1} L_{\overline{\mathcal{M}}}(\Omega)$ [resp. $W^{-1} E_{\overline{\mathcal{M}}}(\Omega)$] denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\overline{\mathcal{M}}}(\Omega)$ [resp. $E_{\overline{\mathcal{M}}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain $\Omega$ has the segment property if for every $x \in \partial \Omega$ there exists an open set $G_x$ and a nonzero vector $y_x$ such that $x \in G_x$ and if $z \in \overline{\Omega} \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$. The following lemmas can be found in [6].

**Lemma 2.1.** Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $M$ be an $N$-function and let $u \in W^1 L_M(\Omega)$ [resp. $W^1 E_M(\Omega)$]. Then $F(u) \in W^1 L_M(\Omega)$ [resp. $W^1 E_M(\Omega)$]. Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then
\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\
0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}
\]
Lemma 2.2. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of $F'$ is finite. Let $M$ be an $N$-function, then the mapping $F : W^{1,1}_M(\Omega) \to W^{1,1}_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{\mathcal{M}}})$.

Lemma 2.3. ([21]) Let $\Omega$ have the segment property. Then for each $\nu \in W^{1,1}_M(\Omega)$, there exists a sequence $\nu_n \in \mathcal{D}(\Omega)$ such that $\nu_n$ converges to $\nu$ for the modular convergence in $W^{1,1}_M(\Omega)$. Furthermore, if $\nu \in W^{1,1}_M(\Omega) \cap L^\infty(\Omega)$, then

$$||\nu_n||_{L^\infty(\Omega)} \leq (N + 1)||\nu||_{L^\infty(\Omega)}.$$  

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.4. Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure. Let $M, P, Q$ be $N$-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:

$$|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

where $k_1, k_2$ are real constants and $c(x) \in E_Q(\Omega)$.

Then the Nemytskii operator $N_f$ defined by $N_f(u)(x) = f(x,u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), 1_{\overline{T}}) = \{u \in L_M(\Omega) : d(u,E_M(\Omega)) < 1_{\overline{T}}\}$ into $E_Q(\Omega)$.

We will also use the following technical lemma.

Lemma 2.5. ([26]) If $\{f_n\} \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in $\Omega$, $f_n, f \geq 0$ a.e. in $\Omega$ and $\int_{\Omega} f_n(x) \, dx \to \int_{\Omega} f(x) \, dx$, then $f_n \to f$ in $L^1(\Omega)$.

3. STRUCTURAL ASSUMPTIONS AND MAIN RESULT

Throughout the paper $\Omega$ will be a bounded subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property. Let $M$ and $P$ be two $N$-functions such that $P \ll M$. Let $A$ be the non everywhere defined operator defined from its domain $\mathcal{D}(\Omega) \subset W^{1,1}_M(\Omega)$ into $W^{-1}_M(\Omega)$ given by

$$A(u) := - \text{div} a(\cdot, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. We assume that there exist a nonnegative function $c(x)$ in $E_M(\Omega)$, $\alpha > 0$ and positive real constants $k_1, k_2, k_3$ and $k_4$, such that for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\xi' \in \mathbb{R}^N$ ($\xi \neq \xi'$) and for almost every $x \in \Omega$

$$|a(x,s,\xi)| \leq c(x) + k_1 P^{-1} M(k_2|s|) + k_3 M^{-1} M(k_4|\xi|),$$

(3.1)
\( (a(x, s, \xi) - a(x, s, \xi'))(\xi - \xi') > 0, \quad (3.2) \)
\[
a(x, s, \xi) \xi \geq \alpha M(|\xi|). \quad (3.3) \]

Here, \( g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is a Carathéodory function satisfying for almost every \( x \in \Omega \) and for all \( s \in \mathbb{R}, \xi \in \mathbb{R}^N, \)
\[
|g(x, s, \xi)| \leq b(|s|) (d(x) + M(|\xi|)), \quad (3.4) \]
\[
g(x, s, \xi)s \geq 0, \quad (3.5) \]
where \( b : \mathbb{R} \rightarrow \mathbb{R}^+ \) is a continuous and increasing function while \( d \) is a given nonnegative function in \( L^1(\Omega). \)

The right-hand side of (1.1) and \( \Phi : \mathbb{R} \rightarrow \mathbb{R}^N, \) are assumed to satisfy
\[
f \in L^1(\Omega) \text{ and } |F| \in E_{\mathcal{M}}(\Omega), \quad (3.6) \]
\[
\Phi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N). \quad (3.7) \]

Our aim in this paper is to give a meaning to a possible solution of (1.1).

In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution \( u \) of problem (1.1) is the Orlicz-Sobolev space \( W^1_0 L_M(\Omega). \) But when \( u \) is only in \( W^1_0 L_M(\Omega) \) there is no reason for \( \Phi(u) \) to be in \( (L^1(\Omega))^N \) since no growth hypothesis is assumed on the function \( \Phi. \)
Thus, the term \( \langle -\text{div} (\Phi(u)), \phi \rangle \) may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally through a pointwise one can seek for a solution \( u \) of problem (1.1). Indeed, for \( h \) in \( C^1_c(\mathbb{R}) \) and \( u \) in \( W^1_0 L_M(\Omega), \) \( h(u) \) belongs to \( W^1 L_M(\Omega) \) and for \( \varphi \) in \( D(\Omega) \) the function \( \varphi h(u) \) belongs to \( W^1_0 L_M(\Omega). \) Since
\[
(\langle -\text{div} a(x, u, \nabla u), \varphi \rangle_{D'((\Omega); D(\Omega))} = \langle -\text{div} a(x, u, \nabla u), \varphi h(u) \rangle_{W^{-1} L_M(\Omega), W^1_0 L_M(\Omega)} \forall \varphi \in D(\Omega), \]

\textbf{Definition 3.1.} A measurable function \( u : \Omega \rightarrow \mathbb{R} \) is called a renormalized solution of (1.1) if \( u \in W^1_0 L_M(\Omega), \) \( a(x, u, \nabla u) \in (L^1(\Omega))^N, \)
\[
g(x, u, \nabla u) \in L^1(\Omega), \quad g(x, u, \nabla u)u \in L^1(\Omega), \]
\[
\lim_{m \rightarrow +\infty} \int_{\{x \in \Omega : m \leq |u(x)| \leq m + 1\}} a(x, u, \nabla u) \nabla u \, dx = 0, \]
and
\[
\left\{ \begin{array}{l}
-\text{div} a(x, u, \nabla u)h(u) - \text{div} (\Phi(u)h(u)) + h'(u)\Phi(u)\n\vspace{1em}
+ g(x, u, \nabla u)h(u) = fh(u) - \text{div} (Fh(u)) + h'(u)F\n\end{array} \right. \quad (3.8)
\text{for every } h \in C^1_c(\mathbb{R}).
\]

\textbf{Remark 3.2.} Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for \( h \) in \( C^1_c(\mathbb{R}) \) and \( u \) in \( W^1_0 L_M(\Omega), \) \( h(u) \) belongs to \( W^1 L_M(\Omega) \) and for \( \varphi \) in \( D(\Omega) \) the function \( \varphi h(u) \) belongs to \( W^1_0 L_M(\Omega). \) Since
\[
(\langle -\text{div} a(x, u, \nabla u), \varphi \rangle_{D'((\Omega); D(\Omega))} = \langle -\text{div} a(x, u, \nabla u), \varphi h(u) \rangle_{W^{-1} L_M(\Omega), W^1_0 L_M(\Omega)} \forall \varphi \in D(\Omega), \]
Finally, since \( \Phi_h \) and \( \Phi_h' \in (C_0^\infty(\mathbb{R}))^N \), for any measurable function \( u \) we have \( \Phi(u)h(u) \) and \( \Phi(u)h'(u) \in (L^\infty(\Omega))^N \) and then \( \operatorname{div} (\Phi(u)h(u)) \in W^{-1,\infty}(\Omega) \) and \( \Phi(u)h'(u) \in L_M(\Omega) \).

Our main result is the following

**Theorem 3.3.** Suppose that assumptions (3.1)–(3.7) are fulfilled. Then, problem (1.1) has at least one renormalized solution.

**Remark 3.4.** The condition (3.4) can be replaced by the weaker one

\[
g(x,s,\xi) \leq d(x) + b(|s|)M(|\xi|),
\]

with \( b : \mathbb{R} \to \mathbb{R}^+ \) a continuous function belonging to \( L^1(\mathbb{R}) \) and \( d(x) \in L^1(\Omega) \).

Actually the original equation (1.1) will be recovered whenever \( h(u) \equiv 1 \), but unfortunately this cannot happen in general strong additional requirements on \( u \). Therefore, (3.8) is to be viewed as a weaker form of (1.1).

### 4. Proof of the Main Result

From now on, we will use the standard truncation function \( T_k, k > 0 \), defined for all \( s \in \mathbb{R} \) by \( T_k(s) = \max\{-k, \min\{k,s\}\} \).

**Step 1: Approximate problems.** Let \( f_n \) be a sequence of \( L^\infty(\Omega) \) functions that converge strongly to \( f \) in \( L^1(\Omega) \). For \( n \in \mathbb{N}, n \geq 1 \), let us consider the following sequence of approximate equations

\[
-\operatorname{div} a(x,u_n,\nabla u_n) + \operatorname{div} \Phi_n(u_n) + g_n(x,u_n,\nabla u_n) = f_n - \operatorname{div} F \quad \text{in } D'(\Omega),
\]

where we have set \( \Phi_n(s) = \Phi(T_n(s)) \) and \( g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{2}|g(x,s,\xi)|} \). For fixed \( n > 0 \), it’s obvious to observe that

\[
g_n(x,s,\xi) \geq 0, \quad |g_n(x,s,\xi)| \leq |g(x,s,\xi)| \quad \text{and} \quad |g_n(x,s,\xi)| \leq n.
\]

Moreover, since \( \Phi \) is continuous one has \( |\Phi_n(t)| \leq \max_{|t| \leq n} |\Phi(t)| \). Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduces that there exists at least one solution \( u_n \) of the approximate Dirichlet problem (4.1) in the sense

\[
\begin{cases}
  u_n \in W_0^1L_M(\Omega), a(x,u_n,\nabla u_n) \in (L^\infty(\Omega))^N \quad \text{and} \\
  \int_\Omega a(x,u_n,\nabla u_n)\nabla vdx + \int_\Omega \Phi_n(u_n)\nabla vdx \\
  + \int_\Omega g_n(x,u_n,\nabla u_n)vdx = (f_n,v) + \int_\Omega F\nabla vdx, \quad \text{for every } v \in W_0^1L_M(\Omega).
\end{cases}
\]
Step 2: Estimation in $W^1_0 L_M(\Omega)$. Taking $u_n$ as function test in problem (4.2), we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\Omega} \Phi_n(u_n) \nabla u_n dx \\
+ \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx = (f_n, u_n) + \int_{\Omega} F \nabla u_n dx.
\] (4.3)

Define $\tilde{\Phi}_n \in (C^1(\mathbb{R}))^N$ as $\tilde{\Phi}_n(t) = \int_0^t \Phi_n(\tau) d\tau$. Then formally
\[
\text{div}(\tilde{\Phi}_n(u_n)) = \Phi_n(u_n) \nabla u_n, \quad u_n = 0 \text{ on } \partial \Omega, \quad \tilde{\Phi}_n(0) = 0
\]
and by the Divergence theorem
\[
\int_{\Omega} \Phi_n(u_n) \nabla u_n dx = \int_{\Omega} \text{div} (\tilde{\Phi}_n(u_n)) dx = \int_{\partial \Omega} \tilde{\Phi}_n(u_n) \n \nu ds = 0,
\]
where $\n \nu$ is the outward pointing unit normal field of the boundary $\partial \Omega$ ($ds$ may be used as a shorthand for $\n \nu ds$). Thus, by virtue of (3.5) and Young’s inequality, we get
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq C_1 + \alpha \int_{\Omega} M(|\nabla u_n|) dx,
\] (4.4)
which, together with (3.3) give
\[
\int_{\Omega} M(|\nabla u_n|) dx \leq C_2.
\] (4.5)

Moreover, we also have
\[
\int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx \leq C_3.
\] (4.6)

As a consequence of (4.5) there exist a subsequence of $\{u_n\}_n$ still indexed by $n$, and a function $u \in W^1_0 L_M(\Omega)$ such that
\[
\begin{align*}
    u_n &\to u \text{ weakly in } W^1_0 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_M(\Omega)), \\
    u_n &\to u \text{ strongly in } E_M(\Omega) \text{ and a.e. in } \Omega.
\end{align*}
\] (4.7)

Step 3: Boundedness of $(a(x, u_n, \nabla u_n))_n$ in $(L_M(\Omega))^N$. Let $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Thanks to (3.2), we can write
\[
\left( a(x, u_n, \nabla u_n) - \left( a(x, u_n, \frac{w}{k_4}) \right) \left( \nabla u_n - \frac{w}{k_4} \right) \right) \geq 0,
\]
which implies
\[
\frac{1}{k_4} \int_{\Omega} a(x, u_n, \nabla u_n) w dx \leq \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \\
+ \int_{\Omega} a \left( x, u_n, \frac{w}{k_4} \right) \left( \frac{w}{k_4} - \nabla u_n \right) dx.
\]

Thanks to (4.4) and (4.5), one has
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n dx \leq C_5.
\]
Define $\lambda = 1 + k_1 + k_3$. By the growth condition (3.1) and Young’s inequality, one can write
\[
\left| \int_\Omega a\left( x, u_n, \frac{w}{k_4} \right) \left( \frac{w}{k_4} - \nabla u_n \right) dx \right| 
\leq \left( 1 + \frac{1}{k_4} \right) \left( \int_\Omega M(c(x)) dx + k_1 \int_\Omega \mathcal{M}^{-1}(M(k_2|u_n|)) dx 
+ k_3 \int_\Omega M(|w|) dx \right) + \frac{\lambda}{k_4} \int_\Omega M(|w|) dx + \lambda \int_\Omega M(|\nabla u_n|) dx.
\]

By virtue of [18] and Lemma 4.14 of [20], there exists an $N$-function $Q$ such that $M \ll Q$ and the space $W^{1}_0L_M(\Omega)$ is continuously embedded into $L_Q(\Omega)$. Thus, by (4.5) there exists a constant $c_0 > 0$, not depending on $n$, satisfying $\|u_n\|_Q \leq c_0$. Since $M \ll Q$, we can write $M(k_2t) \leq Q\left( \frac{t}{c_0} \right)$, for $t > 0$ large enough. As $P \ll M$, we can find a constant $c_1$, not depending on $n$, such that $\int_\Omega \mathcal{M}^{-1}(M(k_2|u_n|)) dx \leq \int_\Omega Q\left( \frac{|u_n|}{c_0} \right) + c_1$. Hence, we conclude that the quantity $\left| \int_\Omega a(x, u_n, \nabla u_n, w) dx \right|$ is bounded from above for all $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Using the Orlicz norm we deduce that
\[
\left( a(x, u_n, \nabla u_n) \right)_{n}\text{ is bounded in } (L_{\mathcal{M}}(\Omega))^N.
\]

**Step 4: Renormalization identity for the approximate solutions.**

For any $m \geq 1$, define $\theta_m(r) = T_{m+1}(r) - T_m(r)$. Observe that by [19, Lemma 2] one has $\theta_m(u_n) \in W^{1}_0L_M(\Omega)$. The use of $\theta_m(u_n)$ as test function in (4.2) yields
\[
\int_{\{|m| \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle + \int_{\{|m| \leq |u_n| \leq m+1\}} F \nabla u_n dx,
\]

By Hölder’s inequality and 4.5 we have
\[
\int_{\{|m| \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle
+ C_0 \int_{\{|m| \leq |u_n| \leq m+1\}} \mathcal{M}(|F|) dx.
\]

It’s not hard to see that
\[
\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.
\]

So that by (4.5) and (4.7) one can deduce that
\[
\theta_m(u_n) \rightharpoonup \theta_m(u) \text{ weakly in } W^{1}_0L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\mathcal{M}}(\Omega)).
\]

Note that as $m$ goes to $\infty$, $\theta_m(u) \to 0$ weakly in $W^{1}_0L_M(\Omega)$ for $\sigma(\Pi L_M(\Omega), \Pi E_{\mathcal{M}}(\Omega))$, and since $f_n$ converges strongly in $L^1(\Omega)$, by Lebesgue’s theorem we have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{|m| \leq |u_n| \leq m+1\}} \mathcal{M}(|F|) dx = \lim_{m \to \infty} \lim_{n \to \infty} \langle f_n, \theta_m(u_n) \rangle = 0.
\]
By (3.3) we finally have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \tag{4.9}
\]

**Step 5:** Almost everywhere convergence of the gradients. Define \( \phi(s) = se^{\lambda s^2} \) with \( \lambda = \left( \frac{b(k)}{2\sigma} \right)^2 \). One can easily verify that for all \( s \in \mathbb{R} \)
\[
\phi'(s) - \frac{b(k)}{\sigma} |\phi(s)| \geq \frac{1}{2}.
\tag{4.10}
\]

For \( m \geq k \), we define the function \( \psi_m \) by
\[
\begin{align*}
\psi_m(s) &= 1 & \text{if } & |s| \leq m, \\
\psi_m(s) &= m + 1 - |s| & \text{if } & m \leq |s| \leq m + 1, \\
\psi_m(s) &= 0 & \text{if } & |s| \geq m + 1.
\end{align*}
\]

By virtue of [21, Theorem 4] there exists a sequence \( \{v_j\}_j \subset D(\Omega) \) such that \( v_j \to u \) in \( W_0^1 L_M(\Omega) \) for the modular convergence and a.e. in \( \Omega \). Let us define the following functions \( \theta_n^j = T_k(u_n) - T_k(v_j), \theta^j = T_k(u) - T_k(v_j) \) and \( z_n^j = \phi(\theta_n^j) \psi_m(u_n) \). Using \( z_n^j \in W_0^1 L_M(\Omega) \) as test function in (4.2) we get
\[
\begin{align*}
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_n^j dx + \int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx \\
+ \int_{\Omega} \Phi_n(u_n) \nabla u_n \psi_m'(u_n) \phi(T_k(u_n) - T_k(v_j)) dx \\
+ \int_{\Omega} g_n(x, u_n, \nabla u_n) z_n^j dx = \int_{\Omega} f_n z_n^j dx + \int_{\Omega} F \nabla z_n^j dx.
\end{align*}
\tag{4.11}
\]

From now on we denote by \( \epsilon_i(n, j), i = 0, 1, 2, ..., \) various sequences of real numbers which tend to zero, when \( n \) and \( j \to +\infty \), i.e.
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \epsilon_i(n, j) = 0.
\]

In view of (4.7), we have \( z_n^j \to \phi(\theta^j) \psi_m(u) \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) as \( n \to +\infty \), which yields
\[
\lim_{n \to +\infty} \int_{\Omega} f_n z_n^j dx = \int_{\Omega} f \phi(\theta^j) \psi_m(u) dx,
\]
and since \( \phi(\theta^j) \to 0 \) weakly in \( L^\infty(\Omega) \) for \( \sigma(L^\infty, L^1) \) as \( j \to +\infty \), we have
\[
\lim_{j \to +\infty} \int_{\Omega} f \phi(\theta^j) \psi_m(u) dx = 0.
\]
Thus, we write
\[
\int_{\Omega} f_n z_n^j dx = \epsilon_0(n, j).
\]

Thanks to (4.5) and (4.7), we have as \( n \to +\infty \),
\[
z_n^j \to \phi(\theta^j) \psi_m(u) \text{ in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_M(\Omega)).
which implies that
\[
\lim_{n \to +\infty} \int_{\Omega} F \nabla z_{n,m}^j \, dx = \int_{\Omega} F \nabla \theta^j \phi'(\theta^j) \psi_m(u) \, dx + \int_{\Omega} F \nabla u \phi(\theta^j) \psi'_m(u) \, dx
\]
On the one hand, by Lebesgue’s theorem we get
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla \theta^j \phi' \psi'_m(u) \, dx = 0,
\]
on the other hand, we write
\[
\int_{\Omega} F \nabla \theta^j \phi' \psi'_m(u) \, dx = \int_{\Omega} F \nabla T_k(u) \phi' \psi_m(u) \, dx - \int_{\Omega} F \nabla T_k(v_j) \phi' \psi_m(u) \, dx,
\]
so that, by Lebesgue’s theorem one has
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla T_k(u) \phi' \psi_m(u) \, dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) \, dx.
\]
Let \( \lambda > 0 \) such that \( M \left( \frac{\nabla v_j - \nabla u}{\lambda} \right) \to 0 \) strongly in \( L^1(\Omega) \) as \( j \to +\infty \) and \( M \left( \frac{\nabla u}{\lambda} \right) \in L^1(\Omega) \), the convexity of the \( N \)-function \( M \) allows us to have
\[
M \left( \frac{\nabla T_k(v_j) \phi' \psi_m(u) - \nabla T_k(u) \psi_m(u)}{\lambda} \right)
= \frac{1}{4} M \left( \frac{\nabla v_j - \nabla u}{\lambda} \right) + \frac{1}{4} \left( 1 + \frac{1}{\sigma(2k)} \right) M \left( \frac{\nabla u}{\lambda} \right).
\]
Then, by using the modular convergence of \( \{ \nabla v_j \} \) in \( (L_M(\Omega))^N \) and Vitali’s theorem, we obtain
\[
\nabla T_k(v_j) \phi' \psi_m(u) \to \nabla T_k(u) \psi_m(u) \text{ in } (L_M(\Omega))^N,
\]
for the modular convergence, and then
\[
\lim_{j \to +\infty} \int_{\Omega} F \nabla T_k(u) \phi' \psi_m(u) \, dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) \, dx.
\]
We have proved that
\[
\int_{\Omega} F \nabla z_{n,m}^j \, dx = \epsilon_1(n, j).
\]
It’s easy to see that by the modular convergence of the sequence \( \{ v_j \} \), one has
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \Phi_n(u_n) \nabla u_n \psi_m'(u_n) \phi(T_k(u_n) - T_k(v_j)) \, dx = 0,
\]
while for the third term in the left-hand side of (4.11) we can write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) \, dx
= \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) \, dx - \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx.
\]
Firstly, we have
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n) \psi_m(u_n) dx = 0.
\]
In view of (4.7), one has
\[
\Phi_n(u_n) \phi'(\theta_n) \psi_m(u_n) \to \Phi(u) \phi'(\theta) \psi_m(u),
\]
almost everywhere in \( \Omega \) as \( n \) tends to +\( \infty \). Furthermore, we can check that
\[
\|\Phi_n(u_n) \phi'(\theta_n) \psi_m(u_n)\|_{T^*} \leq M\epsilon_m(2k)\Omega + 1,
\]
where \( \epsilon_m = \max_{|t| \leq m+1} \Phi(t) \). Applying [27, Theorem 14.6] we get
\[
\lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(v_j) \phi'(\theta) \psi_m(u) dx.
\]
Using the modular convergence of the sequence \( \{v_j\}_j \), we obtain
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx.
\]
Then, using again the Divergence theorem we get
\[
\int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx = 0.
\]
Therefore, we write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx = \epsilon_2(n, j).
\]
Since \( g_n(x, u_n, \nabla u_n) z_{n,m}^j \geq 0 \) on the set \( \{|u_n| > k\} \) and \( \psi_m(u_n) = 1 \) on the set \( \{|u_n| \leq k\} \), from (4.11) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx + \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n) dx \leq \epsilon_3(n, j). \quad (4.12)
\]
We now evaluate the first term of the left-hand side of (4.12) by writing
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx
\]
\[
= \int_{\Omega} a(x, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n) \psi_m(u_n) dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n) \psi_m(u_n) dx
\]
\[
= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n) dx
\]
\[
- \int_{\{|u_n| \leq k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n) \psi_m(u_n) dx
\]
\[
+ \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n) \psi_m(u_n) dx.
\]
and then
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j \, dx
= \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)) \right)
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi^s_j \right) \phi'(\theta_n^j) \, dx
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi^s_j \right) \phi'(\theta_n^j) \, dx
- \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta_n^j) \, dx
- \int_{[\chi^s_j > k]} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) \, dx,
\]
where by \( \chi^s_j, s > 0 \), we denote the characteristic function of the subset
\[\Omega^j_s = \{ x \in \Omega : |\nabla T_k(v_j)| \leq s \}.\]
For fixed \( m \) and \( s \), we will pass to the limit in \( n \) and then in \( j \) in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi^s_j) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi^s_j \right) \phi'(\theta_n^j) \, dx
\to \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi^s_j) \left( \nabla T_k(u) - \nabla T_k(v_j) \chi^s_j \right) \phi'(\theta^j) \, dx,
\]
as \( n \to +\infty \). Since by lemma (2.4) one has
\[
a(x, T_k(u_n), \nabla T_k(v_j) \chi^s_j) \phi'(\theta_n^j) \to a(x, T_k(u), \nabla T_k(v_j) \chi^s_j) \phi'(\theta^j),
\]
strongly in \((E_{2M}(\Omega))^N\) as \( n \to \infty \), while by (4.5)
\[
\nabla T_k(u_n) \rightharpoonup \nabla T_k(u),
\]
weakly in \((L_M(\Omega))^N\). Let \( \chi^s \) denote the characteristic function of the subset
\[\Omega^s = \{ x \in \Omega : |\nabla T_k(u)| \leq s \}.\]
As \( \nabla T_k(v_j) \chi^s_j \to \nabla T_k(u) \chi^s \) strongly in \((E_M(\Omega))^N\) as \( j \to +\infty \), one has
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi^s_j) \cdot \left( \nabla T_k(u) - \nabla T_k(v_j) \chi^s_j \right) \phi'(\theta^j) \, dx \to 0,
\]
as \( j \to \infty \). Then
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi^s_j) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi^s_j \right) \phi'(\theta_n^j) \, dx = \epsilon_4(n, j). \quad (4.14)
\]
We now estimate the third term of (4.13). It’s easy to see that by (3.3), \( a(x, s, 0) = 0 \) for almost everywhere \( x \in \Omega \) and for all \( s \in \mathbb{R} \). Thus, from (4.8) we have that \( \left( a(x, T_k(u_n), \nabla T_k(u_n)) \right)_n \) is bounded in \((L_{2M}(\Omega))^N\) for all \( k \geq 0 \).
Therefore, there exist a subsequence still indexed by \( n \) and a function \( l_k \) in 
\( (L_M(\Omega))^N \) such that
\[
a(x, T_k(u_n), \nabla T_k(u_n)) \rightarrow l_k \text{ weakly in } (L_M(\Omega))^N \text{ for } \sigma(\Pi L_M, H E_M). \tag{4.15}
\]

Then, since \( \nabla T_k(v_j) \chi_{\Omega \setminus \Omega_j^i} \in (E_M(\Omega))^N \), we obtain
\[
\int_{\Omega \setminus \Omega_j^i} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta^i_n) dx \rightarrow \int_{\Omega \setminus \Omega_j^i} l_k \nabla T_k(v_j) \phi'(\theta^i) dx,
\]
as \( n \rightarrow +\infty \). The modular convergence of \( \{v_j\} \) allows us to get
\[
- \int_{\Omega \setminus \Omega_j^i} l_k \nabla T_k(v_j) \phi'(\theta^i) dx \rightarrow - \int_{\Omega \setminus \Omega_j^i} l_k \nabla T_k(u) dx,
\]
as \( j \rightarrow +\infty \). This proves
\[
- \int_{\Omega \setminus \Omega_j^i} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta^i_n) \psi_m(u_n) dx = - \int_{\Omega \setminus \Omega_j^i} l_k \nabla T_k(u) dx + \epsilon_5(n, j). \tag{4.16}
\]

As regards the fourth term, observe that \( \psi_m(u_n) = 0 \) on the subset \( \{ |u_n| \geq m + 1 \} \), so we have
\[
- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta^i_n) \psi_m(u_n) dx = - \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta^i_n) \psi_m(u_n) dx.
\]

Since
\[
- \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta^i_n) \psi_m(u_n) dx = - \int_{\{|u| > k\}} l_{m+1} \nabla T_k(u) \psi_m(u) dx + \epsilon_5(n, j),
\]
observing that \( \nabla T_k(u) = 0 \) on the subset \( \{|u| > k\} \), one has
\[
- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta^i_n) \psi_m(u_n) dx = \epsilon_6(n, j). \tag{4.17}
\]

For the last term of \( (4.13) \), we have
\[
\left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi'_{\theta^i_n} \psi_m'(u_n) dx \right| \\
= \left| \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi'_{\theta^i_n} \psi_m'(u_n) dx \right| \\
\leq \phi(2k) \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx.
\]
To estimate the last term of the previous inequality, we use $(T_1(u_n - T_m(u_n)) \in W^1_0 L^\infty(\Omega))$ as test function in (4.2), to get
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n dx
+ \int_{\{|u_n| \geq m\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) dx = \langle f_n, T_1(u_n - T_m(u_n)) \rangle
+ \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n dx.
\]

By Divergence theorem, we have
\[
\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n dx = 0.
\]

Using the fact that $g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \geq 0$ on the subset $\{|u_n| \geq m\}$ and Young’s inequality, we get
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx
\leq \langle f_n, T_1(u_n - T_m(u_n)) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx.
\]

It follows that
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \phi'(\theta_n^j) u_n dx
\leq 2\phi(2k) \int_{\{m \leq |u_n| \leq m+1\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx.
\]

From (4.14), (4.16), (4.17) and (4.18) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx
\geq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^m))
\times (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^m) \phi(\theta_n^j) d\xi
\leq -\alpha \phi(2k) \left( \int_{\{m \leq |u_n| \leq m+1\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right)
- \int_{\Omega \setminus \Omega^*} l_k \cdot \nabla T_k(u) dx + \epsilon(n,j).
\]

Now, we turn to second term in the left-hand side of (4.12). We have
\[
\int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx
= \int_{\{|u_n| \leq k\}} g_n(x, T_k(u_n), \nabla T_k(u_n)) \phi(\theta_n^j) dx
\leq b(k) \int_{\Omega} M(|\nabla T_k(u_n)|) \phi(\theta_n^j) dx + b(k) \int_{\Omega} d(x) \phi(\theta_n^j) dx
\leq \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \phi(\theta_n^j) dx + \epsilon(n,j).
\]
Then
\[ \left| \int_{\{u_n \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) \, dx \right| \leq \frac{b(k)}{\alpha} \int_\Omega (a(x, T_k(u_n), \nabla T_k(v_j)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^*) \right) \right| \phi(\theta_n^j) \, dx \
+ \frac{b(k)}{\alpha} \int_\Omega (a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^*) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^*) \phi(\theta_n^j) \, dx \
+ \frac{b(k)}{\alpha} \int_\Omega a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^* \phi(\theta_n^j) \, dx + \epsilon_9(n, j). \]
\[ (4.22) \]

Hence, we have
\[ \frac{b(k)}{\alpha} \int_\Omega a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^*) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^*) \phi(\theta_n^j) \, dx = \epsilon_9(n, j) \]
and
\[ \frac{b(k)}{\alpha} \int_\Omega a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^* \phi(\theta_n^j) \, dx = \epsilon_{10}(n, j). \]

Combining (4.12), (4.19) and (4.21), we get
\[ \int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^*) \right) \right| \phi(\theta_n^j) \, dx \
+ \frac{b(k)}{\alpha} \int_\Omega a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^* \phi(\theta_n^j) \, dx + \epsilon_{11}(n, j). \]

By (4.10), we have
\[ \int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^*) \right) \right| \phi(\theta_n^j) \, dx \
+ 2 \frac{b(k)}{\alpha} \int_\Omega a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^* \phi(\theta_n^j) \, dx + \epsilon_{12}(n, j). \]
\[ (4.22) \]
On the other hand we can write
\[
\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx
\]
\[
= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s) \, dx \\
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi_s) (\nabla T_k(v_j)\chi_s - \nabla T_k(u)\chi_s) \, dx \\
- \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi_s) (\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx \\
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_s) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s) \, dx.
\]  
We shall pass to the limit in \( n \) and then in \( j \) in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(v_j)\chi_s - \nabla T_k(u)\chi_s) \, dx = \epsilon_{13}(n, j),
\]
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi_s) (\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx = \epsilon_{14}(n, j),
\]
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi_s) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s) \, dx = \epsilon_{15}(n, j).
\]  
So that
\[
\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)) (\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx \\
= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_s)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s) \, dx \\
+ \epsilon_{16}(n, j).
\]  
Let \( r \leq s \). Using (3.2), (4.22) and (4.24) we can write
\[
0 \leq \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \\
\leq \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \\
= \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s) \right) (\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx \\
\leq \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s) \right) (\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx \\
+ \epsilon_{15}(n, j) \\
\leq 2 \int_{\Omega} l_k \nabla T_k(u) \, dx + 2\alpha\phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} |M|(|F|) \, dx \right) \\
+ \epsilon_{17}(n, j).
\]
By passing to the superior limit over $n$ and then over $j$
\[
0 \leq \limsup_{n \to +\infty} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx
\]
\[
\leq 2 \int_{\Omega} l_k \nabla T_k(u) dx + 4\alpha \phi(2k) \left( \int_{|m \leq |u_n||} |f| dx + \int_{|m \leq |u_n| \leq |m+1|} \mathcal{M}(|F|) dx \right).
\]
Letting $s \to +\infty$ and then $m \to +\infty$, taking into account that $l_k \nabla T_k(u) \in L^1(\Omega)$, $f \in L^1(\Omega)$, $|F| \in (E_{\mathcal{M}}(\Omega))^N$, $|\Omega \setminus \Omega^s| \to 0$, and $|[m \leq |u| \leq m+1]| \to 0$, one has
\[
\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx,
\]
tends to 0 as $n \to +\infty$. As in [20], we deduce that there exists a subsequence of $\{u_n\}$ still indexed by $n$ such that
\[
\nabla u_n \rightharpoonup \nabla u \text{ a. e. in } \Omega.
\]
Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get
\[
a(x, u, \nabla u) \in (L_{\mathcal{M}}(\Omega))^N
\]
and
\[
a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \text{ weakly in } (L_{\mathcal{M}}(\Omega))^N \text{ for } \sigma(II_{\mathcal{M}}, II_E).
\]

**Step 6: Modular convergence of the truncations.** Going back to equation (4.22), we can write
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx
\]
\[
\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx
\]
\[
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx
\]
\[
+ 2\alpha \phi(2k) \left( \int_{|m \leq |u_n||} |f_n| dx + \int_{|m \leq |u_n| \leq |m+1|} \mathcal{M}(|F|) dx \right)
\]
\[
+ 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{12}(n, j).
\]
By (4.23) we get
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx
\]
\[
\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx
\]
\[
+ 2\alpha \phi(2k) \left( \int_{|m \leq |u_n||} |f_n| dx + \int_{|m \leq |u_n| \leq |m+1|} \mathcal{M}(|F|) dx \right)
\]
\[
+ 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{18}(n, j).
\]
We now pass to the superior limit over \( n \) in both sides of this inequality using (4.27), to obtain
\[
\limsup_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \\
+ 2\alpha \phi(2k) \left( \int_{\{|u| \leq 1\}} |f| \, dx + \int_{\{|m| \leq |u| \leq m+1\}} M(|F|) \, dx \right) \\
+ 2 \int_{\Omega \setminus \Omega^v} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]

We then pass to the limit in \( j \) to get
\[
\limsup_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
\leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \\
+ 2\alpha \phi(2k) \left( \int_{\{|u| \leq 1\}} |f| \, dx + \int_{\{|m| \leq |u| \leq m+1\}} M(|F|) \, dx \right) \\
+ 2 \int_{\Omega \setminus \Omega^v} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]

Letting \( s \) and then \( m \to +\infty \), one has
\[
\limsup_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]

On the other hand, by (3.3), (4.5), (4.26) and Fatou's lemma, we have
\[
\int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx \leq \liminf_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx.
\]

It follows that
\[
\lim_{n \to +\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx = \int_\Omega a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx.
\]

By Lemma 2.5 we conclude that for every \( k > 0 \)
\[
a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u),
\]

strongly in \( L^1(\Omega) \). The convexity of the \( N \)-function \( M \) and (3.3) allow us to have
\[
M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) \\
\leq \frac{1}{2} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) + \frac{1}{2} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u).
\]

From Vitali's theorem we deduce
\[
\limsup_{|E| \to 0} \sup_n \int_E M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) \, dx = 0.
\]

Thus, for every \( k > 0 \)
\[
T_k(u_n) \to T_k(u) \text{ in } W^1_0 L_M(\Omega),
\]
Step 7: Compactness of the nonlinearities. We need to prove that

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]  

(4.29)

By virtue of (4.7) and (4.26) one has

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \text{a. e. in } \Omega. \]  

(4.30)

Let \( E \) be measurable subset of \( \Omega \) and let \( m > 0 \). Using (3.3) and (3.4) we can write

\[
\int_E |g_n(x, u_n, \nabla u_n)| \, dx \\
= \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| \, dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| \, dx \\
\leq b(m) \int_E d(x) \, dx + b(m) \int_{E \cap \{|u_n| > m\}} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx \\
+ \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx.
\]

From (3.5) and (4.6), we deduce that

\[ 0 \leq \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx \leq C_3. \]

So

\[ 0 \leq \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx \leq \frac{C_3}{m}. \]

Then

\[
\lim_{m \to +\infty} \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n \, dx = 0.
\]

Thanks to (4.28) the sequence \( \{a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)\}_n \) is equi-integrable. This fact allows us to get

\[
\lim_{|E| \to 0} \sup_n \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) \, dx = 0.
\]

This shows that \( g_n(x, u_n, \nabla u_n) \) is equi-integrable. Thus, Vitali’s theorem implies that \( g(x, u, \nabla u) \in L^1(\Omega) \) and

\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]

Step 8: Renormalization identity for the solutions. In this step we prove that

\[
\lim_{m \to +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \, dx = 0. \]  

(4.31)
Indeed, for any $m \geq 0$ we can write
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx
\] 
\[
= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) dx 
\] 
\[
= \int_{\Omega} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) dx 
\] 
\[
- \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) dx.
\]
In view of (4.28), we can pass to the limit as $n$ tends to $+\infty$ for fixed $m \geq 0$
\[
\lim_{n \to +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx
\]
\[
= \int_{\Omega} a(x, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_{m+1}(u) dx 
\] 
\[
- \int_{\Omega} a(x, T_m(u), \nabla T_m(u)) \nabla T_m(u) dx
\]
\[
= \int_{\Omega} a(x, u, \nabla u)(\nabla T_{m+1}(u) - \nabla T_m(u)) dx 
\] 
\[
= \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u dx.
\]
Having in mind (4.9), we can pass to the limit as $m$ tends to $+\infty$ to obtain (4.31).

**Step 9: Passing to the limit.** Thanks to (4.28) and Lemma (2.5), we obtain
\[
a(x, u_n, \nabla u_n) \nabla u_n \to a(x, u, \nabla u) \nabla u \text{ strongly in } L^1(\Omega).
\] (4.32)
Let $h \in C^1_c(\mathbb{R})$ and $\varphi \in \mathcal{D}(\Omega)$. Inserting $h(u_n)\varphi$ as test function in (4.2), we get
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi dx + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) dx 
\] 
\[
+ \int_{\Omega} \Phi_n(u_n) \nabla (h(u_n)\varphi) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi dx 
\] 
\[
= \langle f_n, h(u_n)\varphi \rangle + \int_{\Omega} F \nabla (h(u_n)\varphi) dx.
\] (4.33)
We shall pass to the limit as $n \to +\infty$ in each term of the equality (4.33). Since $h$ and $h'$ have compact support on $\mathbb{R}$, there exists a real number $\nu > 0$, such that $\text{supp } h \subseteq [-\nu, \nu]$ and $\text{supp } h' \subseteq [-\nu, \nu]$. For $n > \nu$, we can write
\[
\Phi_n(t)h(t) = \Phi(T_{\nu}(t))h(t) \text{ and } \Phi_n(t)h'(t) = \Phi(T_{\nu}(t))h'(t).
\]
Moreover, the functions $\Phi h$ and $\Phi h'$ belong to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$. Observe first that the sequence $\{h(u_n)\varphi\}_n$ is bounded in $W^1_0 L_B(\Omega)$. Indeed, let $\rho > 0$
be a positive constant such that \( \|h(u_n)\nabla \varphi\|_\infty \leq \rho \) and \( \|h'(u_n)\varphi\|_\infty \leq \rho \). Using the convexity of the \( N \)-function \( M \) and taking into account (4.5) we have
\[
\int_\Omega M\left(\frac{|\nabla (h(u_n)\varphi)|}{2\rho}\right)dx \leq \int_\Omega M\left(\frac{|h(u_n)\nabla \varphi| + |h'(u_n)\varphi|\nabla u_n|}{2\rho}\right)dx \\
\leq \frac{1}{2} M(1)\|\varphi\| + \frac{1}{2} \int_\Omega M(|\nabla u_n|)dx \\
\leq \frac{1}{2} M(1)\|\varphi\| + \frac{1}{2} C_2.
\]

This, together with (4.7), imply that
\[
h(u_n)\varphi \rightharpoonup h(u)\varphi \text{ weakly in } W^1_0 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E^\Omega_M).
\]

This enables us to get
\[
(f_n, h(u_n)\varphi) \rightharpoonup (f, h(u)\varphi).
\]
Let \( E \) be a measurable subset of \( \Omega \). Define \( c_\nu = \max_{|\varphi| \leq \nu} \Phi(t) \). Let us denote by \( \|v\|_{\nu M} \) the Orlicz norm of a function \( v \in L_M(\Omega) \). Using strengthened Hölder inequality with both Orlicz and Luxemburg norms, we get
\[
\|\Phi(T_\nu(u_n))\chi_E\|_{M} = \sup_{\|v\|_{\nu M} \leq 1} \left| \int_E \Phi(T_\nu(u_n))vdx \right| \\
\leq c_\nu \sup_{\|v\|_{\nu M} \leq 1} \|\chi_E\|_{M} \|v\|_{\nu M} \\
\leq c_\nu |E| M^{-1}\left(\frac{1}{|E|}\right).
\]
Thus, we get
\[
\lim_{|E| \to 0} \sup_n \|\Phi(T_\nu(u_n))\chi_E\|_{M} = 0.
\]
Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain
\[
\Phi(T_\nu(u_n)) \to \Phi(T_\nu(u)) \text{ strongly in } (E^\Omega_M)^N,
\]
which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have
\[
\int_\Omega \Phi(T_\nu(u_n))\nabla (h(u_n)\varphi)dx \to \int_\Omega \Phi(T_\nu(u))\nabla (h(u)\varphi)dx.
\]
We remark that
\[
|a(x, u_n, \nabla u_n)\nabla u_n h'(u_n)\varphi| \leq \rho a(x, u_n, \nabla u_n)\nabla u_n.
\]
Consequently, using (4.32) and Vitali’s theorem, we obtain
\[
\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n h'(u_n)\varphi dx \to \int_\Omega a(x, u, \nabla u)\nabla uh'(u)\varphi dx.
\]
and
\[
\int_\Omega F\nabla u_n h'(u_n)\varphi dx \to \int_\Omega F\nabla uh'(u)\varphi dx.
\]
For the second term of (4.33), as above we have
\[
h(u_n)\nabla \varphi \rightharpoonup h(u)\nabla \varphi \text{ strongly in } (E_M(\Omega))^N,
\]
which together with (4.27) give
\[ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi^h(u_n) \, dx \to \int_{\Omega} a(x, u, \nabla u) \nabla \varphi^h(u) \, dx \]
and
\[ \int_{\Omega} F \nabla \varphi^h(u_n) \, dx \to \int_{\Omega} F \nabla \varphi^h(u) \, dx. \]
The fact that
\[ h(u_n) \varphi \rightharpoonup h(u) \varphi \text{ weakly in } L^\infty(\Omega) \]
for \( \sigma^*(L^\infty, L^1) \) and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get
\[ \int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx \to \int_{\Omega} g(x, u, \nabla u) h(u) \varphi \, dx. \]
At this point we can pass to the limit in each term of (4.33) to get
\[
\begin{align*}
\int_{\Omega} a(x, u, \nabla u)(\nabla \varphi^h(u) + h'(u) \varphi \nabla u) \, dx &+ \int_{\Omega} \Phi(u) h'(u) \varphi \, dx \\
&= (f, h(u) \varphi) + \int_{\Omega} F(\nabla \varphi^h(u) + h'(u) \varphi \nabla u) \, dx,
\end{align*}
\]
for all \( h \in C^1_c(\mathbb{R}) \) and for all \( \varphi \in D(\Omega) \). Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou’s lemma to get \( g(x, u, \nabla u)u \in L^1(\Omega) \). By virtue of (4.7), (4.27), (4.29), (4.31), the function \( u \) is a renormalized solution of problem (1.1).

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REFERENCES


