Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces

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Abstract. The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems

\[-\text{div} \left( a(x, u, \nabla u) + \Phi(u) \right) + g(x, u, \nabla u) = f - \text{div} \, F,\]

in a bounded open set \(\Omega\) and \(u = 0\) on \(\partial \Omega\), in the framework of Orlicz-Sobolev spaces without any restriction on the \(M\)-function of the Orlicz spaces, where \(-\text{div} \left( a(x, u, \nabla u) \right)\) is a Leray-Lions operator defined from \(W^{1,0}_{\text{loc}}(\Omega)\) into its dual, \(\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)\). The function \(g(x, u, \nabla u)\) is a nonlinear lower order term with natural growth with respect to \(|\nabla u|\), satisfying the sign condition and the datum \(\mu\) is assumed to belong to \(L^1(\Omega) + W^{-1}E_M(\Omega)\).

Keywords: Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.


1. Introduction

Let \(\Omega\) be a bounded open set of \(\mathbb{R}^N, N \geq 2\), and let \(M\) be an \(N\)-function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem

\[
\begin{align*}
A(u) - \text{div} \, \Phi(u) + g(x, u, \nabla u) &= f - \text{div} \, F \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

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Here, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$, while the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The non everywhere defined nonlinear operator $A(u) = -\text{div} (a(x, u, \nabla u))$ acts from its domain $D(A) \subset W^{1,0}_0 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$. The function $a(x, u, \nabla u)$ is assumed to satisfy, among others, $a(x, u, \nabla u)$ nonstandard growth condition governed by the $N$-function $M$, and the source term $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$. $\overline{M}$ stands for the conjugate of $M$.

We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with $\Phi \equiv 0$, $g \equiv 0$ and $L^1(\Omega)$-data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when $f$ is a bounded Radon measure datum and $g$ grows at most like $|\nabla u|^{p-1}$ by Beta et al. in [9, 10, 11] with $\Phi \equiv 0$ and by Guibé and Mercaldo in [23, 24] when $\Phi(u)$ behaves at most like $|u|^{p-1}$. Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in $L^1(\Omega)$. Recall that to get both existence and uniqueness of a solution to problems with $L^1$-data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall’Aglio [14].

The authors in [5] have dealt with the equation (1.1) with $g = g(x, u)$ and $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, under the restriction that the $N$-function $M$ satisfies the $\Delta_2$-condition. This work was then extended in [2] for $N$-functions not satisfying necessarily the $\Delta_2$-condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the $N$-function $M$. Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, an existence result has been proved in [3]. Specific examples to which our results apply include the following:

$$- \text{div} \left(|\nabla u|^{p-2} \nabla u + |u|^s u\right) + u|\nabla u|^p = \mu \text{ in } \Omega,$$
$$- \text{div} \left(|\nabla u|^{p-2} \nabla u \log^\beta (1 + |\nabla u|) + |u|^s u\right) = \mu \text{ in } \Omega,$$
$$- \text{div} \left(\frac{M(|\nabla u)|\nabla u}{|\nabla u|^2} + |u|^s u\right) + M(|\nabla u|) = \mu \text{ in } \Omega,$$

where $p > 1$, $s > 0$, $\beta > 0$ and $\mu$ is a given Radon measure on $\Omega$.

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form $f - \text{div} F$ with $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, in the setting of Orlicz spaces without any restriction on the $N$-functions $M$. The approximate equations provide a $W^{1,0}_0 L_M(\Omega)$ bound for the corresponding solution $u_n$. This allows us to obtain
a function \( u \) as a limit of the sequence \( u_n \). Hence, appear two difficulties. The first one is how to give a sense to \( \Phi(u) \), the second difficulty lies in the need of the convergence almost everywhere of the gradients of \( u_n \) in \( \Omega \). This is done by using suitable test functions built upon \( u_n \) which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

2. Preliminaries

Let \( M: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be an \( N \)-function, i.e., \( M \) is continuous, increasing, convex, with \( M(t) > 0 \) for \( t > 0 \), \( \frac{M(t)}{t} \rightarrow 0 \) as \( t \rightarrow 0 \), and \( \frac{M(t)}{t} \rightarrow +\infty \) as \( t \rightarrow +\infty \). Equivalently, \( M \) admits the representation:

\[
M(t) = \int_0^t a(s) \, ds,
\]

where \( a: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is increasing, right continuous, with \( a(0) = 0 \), \( a(t) > 0 \) for \( t > 0 \) and \( a(t) \) tends to \( +\infty \) as \( t \rightarrow +\infty \).

The conjugate of \( M \) is also an \( N \)-function and it is defined by \( \overline{M} = \int_0^t \bar{a}(s) \, ds \), where \( \bar{a}: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is the function \( \bar{a}(t) = \sup\{s: a(s) \leq t\} \) (see [1]).

An \( N \)-function \( M \) is said to satisfy the \( \Delta_2 \)-condition if, for some \( k \),

\[
M(2t) \leq k M(t) \quad \forall t \geq 0,
\]

(2.1)

when (2.1) holds only for \( t > t_0 > 0 \) then \( M \) is said to satisfy the \( \Delta_2 \)-condition near infinity. Moreover, we have the following Young’s inequality

\[
st \leq M(t) + \overline{M}(s), \quad \forall s, t \geq 0.
\]

Given two \( N \)-functions, we write \( P \ll Q \) to indicate \( P \) grows essentially less rapidly than \( Q \); i.e. for each \( \epsilon > 0 \), \( \frac{P(t)}{Q(\epsilon t)} \rightarrow 0 \) as \( t \rightarrow +\infty \). This is the case if and only if

\[
\lim_{t \to \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.
\]

Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). The Orlicz class \( k_M(\Omega) \) (resp. the Orlicz space \( L_M(\Omega) \)) is defined as the set of (equivalence classes of) real valued measurable functions \( u \) on \( \Omega \) such that

\[
\int_{\Omega} M(|u(x)|) \, dx < +\infty \quad \text{(resp. } \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right) \, dx < +\infty \text{ for some } \lambda > 0\text{).}
\]
The set $L_M(\Omega)$ is a Banach space under the norm
\[
\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\},
\]
and $k_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\infty}(\Omega)$ by means of the pairing $\int_{\Omega} uv \, dx$, and the dual norm of $L_{\infty}(\Omega)$ is equivalent to $\|u\|_{M,\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1 L_M(\Omega)$ [resp. $W^1 E_M(\Omega)$] is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm
\[
\|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}.
\]

Thus, $W^1 L_M(\Omega)$ and $W^1 E_M(\Omega)$ can be identified with subspaces of product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_M)$ and $\sigma(\prod L_M, \prod L_{\infty})$.

The space $W^1_0 E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1 E_M(\Omega)$ and the space $W^1_0 L_M(\Omega)$ as the $\sigma(\prod L_M, \prod L_{\infty})$ closure of $D(\Omega)$ in $W^1 L_M(\Omega)$. We say that $u_n$ converges to $u$ for the modular convergence in $W^1 L_M(\Omega)$ if for some $\lambda > 0$, $\int_{\Omega} M \left( \frac{D^\alpha u_n - D^\alpha u}{\lambda} \right) \, dx \to 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\prod L_M, \prod L_{\infty})$. If $M$ satisfies the $\Delta_2$ condition on $\mathbb{R}^+$ (near infinity only when $\Omega$ has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1} L_{\infty}(\Omega)$ [resp. $W^{-1} E_{\infty}(\Omega)$] denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\infty}(\Omega)$ [resp. $E_{\infty}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain $\Omega$ has the segment property if for every $x \in \partial \Omega$ there exists an open set $G_x$ and a nonzero vector $y_x$ such that $x \in G_x$ and if $z \in \Omega \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$. The following lemmas can be found in [6].

**Lemma 2.1.** Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $M$ be an $N$-function and let $u \in W^1 L_M(\Omega)$ [resp. $W^1 E_M(\Omega)$]. Then $F(u) \in W^1 L_M(\Omega)$ [resp. $W^1 E_M(\Omega)$]. Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then
\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}
\]
Lemma 2.2. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of $F'$ is finite. Let $M$ be an $N$-function, then the mapping $F : W^{1}_M(\Omega) \to W^{1}L_{M}(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_{M}, \prod E^{\mathcal{M}})$. 

Lemma 2.3. ([21]) Let $\Omega$ have the segment property. Then for each $\nu \in W_{0}^{1}L_{M}(\Omega)$, there exists a sequence $\nu_{n} \in \mathcal{D}(\Omega)$ such that $\nu_{n}$ converges to $\nu$ for the modular convergence in $W_{0}^{1}L_{M}(\Omega)$. Furthermore, if $\nu \in W_{0}^{1}L_{M}(\Omega) \cap L^{\infty}(\Omega)$, then

$$
||\nu_{n}||_{L^{\infty}(\Omega)} \leq (N + 1)||\nu||_{L^{\infty}(\Omega)}.
$$

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.4. Let $\Omega$ be an open subset of $\mathbb{R}^{N}$ with finite measure. Let $M, P, Q$ be $N$-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:

$$
|f(x,s)| \leq c(x) + k_{1}P^{-1}M(k_{2}|s|),
$$

where $k_{1}, k_{2}$ are real constants and $c(x) \in E_{Q}(\Omega)$.

Then the Nemytskii operator $N_{f}$ defined by $N_{f}(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_{M}(\Omega), \frac{1}{\mathcal{P}}) = \{u \in L_{M}(\Omega) : d(u, E_{M}(\Omega)) < \frac{1}{\mathcal{P}}\}$ into $E_{Q}(\Omega)$.

We will also use the following technical lemma.

Lemma 2.5. ([26]) If $\{f_{n}\} \subset L^{1}(\Omega)$ with $f_{n} \to f \in L^{1}(\Omega)$ a.e. in $\Omega$, $f_{n}, f \geq 0$ a.e. in $\Omega$ and $\int_{\Omega} f_{n}(x) \, dx \to \int_{\Omega} f(x) \, dx$, then $f_{n} \to f$ in $L^{1}(\Omega)$.

3. Structural Assumptions and Main Result

Throughout the paper $\Omega$ will be a bounded subset of $\mathbb{R}^{N}$, $N \geq 2$, satisfying the segment property. Let $M$ and $P$ be two $N$-functions such that $P \ll M$. Let $A$ be the non everywhere defined operator defined from its domain $\mathcal{D}(\Omega) \subset W_{0}^{1}L_{M}(\Omega)$ into $W^{-1}L_{\mathcal{M}}(\Omega)$ given by

$$
A(u) := - \text{div} \, a(\cdot, u, \nabla u),
$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^{N} \to \mathbb{R}^{N}$ is a Carathéodory function. We assume that there exist a nonnegative function $c(x)$ in $E_{\mathcal{M}}(\Omega)$, $\alpha > 0$ and positive real constants $k_{1}, k_{2}, k_{3}$ and $k_{4}$, such that for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^{N}$, $\xi' \in \mathbb{R}^{N}$ ($\xi \neq \xi'$) and for almost every $x \in \Omega$

$$
|a(x, s, \xi)| \leq c(x) + k_{1}\mathcal{P}^{-1}M(k_{2}|s|) + k_{3}\mathcal{M}^{-1}M(k_{4}|\xi|),
$$

(3.1)
Here, \( g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is a Carathéodory function satisfying for almost every \( x \in \Omega \) and for all \( s \in \mathbb{R}, \xi \in \mathbb{R}^N \),

\[
\|g(x, s, \xi)\| \leq b(|s|) (d(x) + M(|\xi|)),
\]

(3.4)

\[
g(x, s, \xi) s \geq 0,
\]

(3.5)

where \( b : \mathbb{R} \to \mathbb{R}^+ \) is a continuous and increasing function while \( d \) is a given nonnegative function in \( L^1(\Omega) \).

The right-hand side of (1.1) and \( \Phi : \mathbb{R} \to \mathbb{R}^N \), are assumed to satisfy

\[
f \in L^1(\Omega) \text{ and } |F| \in E^{F^*}(\Omega),
\]

(3.6)

\[
\Phi \in C^0(\mathbb{R}, \mathbb{R}^N).
\]

(3.7)

Our aim in this paper is to give a meaning to a possible solution of (1.1).

In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution \( u \) of problem (1.1) is the Orlicz-Sobolev space \( W_0^1 L_M(\Omega) \). But when \( u \) is only in \( W_0^1 L_M(\Omega) \) there is no reason for \( \Phi(u) \) to be in \((L^1(\Omega))^N \) since no growth hypothesis is assumed on the function \( \Phi \). Thus, the term \( \text{div} (\Phi(u)) \) may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally through a pointwise multiplication of equation (1.1) by \( h(u) \) where \( h \) belongs to \( C_c^1(\mathbb{R}) \), the class of \( C^1(\mathbb{R}) \) functions with compact support.

**Definition 3.1.** A measurable function \( u : \Omega \to \mathbb{R} \) is called a renormalized solution of (1.1) if \( u \in W_0^1 L_M(\Omega) \), \( a(x, u, \nabla u) \in (L^{F^*}(\Omega))^N \),

\[
g(x, u, \nabla u) \in L^1(\Omega), \ g(x, u, \nabla u) u \in L^1(\Omega),
\]

and

\[
\lim_{m \to +\infty} \int_{\{x \in \Omega : m \leq |u(x)| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx = 0,
\]

and

\[
\begin{cases}
-\text{div} a(x, u, \nabla u) h(u) - \text{div} (\Phi(u) h(u)) + h'(u) \Phi(u) \nabla u \\
+ g(x, u, \nabla u) h(u) = f h(u) - \text{div} (F h(u)) + h'(u) F \nabla u \text{ in } D'(\Omega),
\end{cases}
\]

(3.8)

for every \( h \in C_c^1(\mathbb{R}) \).

**Remark 3.2.** Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for \( h \) in \( C_c^1(\mathbb{R}) \) and \( u \) in \( W_0^1 L_M(\Omega) \), \( h(u) \) belongs to \( W^1 L_M(\Omega) \) and for \( \varphi \) in \( D(\Omega) \) the function \( \varphi h(u) \) belongs to \( W_0^1 L_M(\Omega) \). Since \( (-\text{div} a(x, u, \nabla u)) \in W^{-1} L^{F^*}(\Omega) \), we also have

\[
\langle -\text{div} a(x, u, \nabla u) h(u), \varphi \rangle_{D'(\Omega), D(\Omega)} = \langle -\text{div} a(x, u, \nabla u), \varphi h(u) \rangle_{W^{-1} L^{F^*}(\Omega), W_0^1 L_M(\Omega)}
\]

\( \forall \varphi \in D(\Omega) \).
Finally, since $\Phi h$ and $\Phi h'$ belong to $(C^0_b(R))^N$, for any measurable function $u$ we have $\Phi(u)h(u)$ and $\Phi(u)h'(u) \in (L^\infty(\Omega))^N$ and then $\text{div} \ (\Phi(u)h(u)) \in W^{-1,\infty}(\Omega)$ and $\Phi(u)h'(u) \in L_M(\Omega)$.

Our main result is the following

**Theorem 3.3.** Suppose that assumptions (3.1)-(3.7) are fulfilled. Then, problem (1.1) has at least one renormalized solution.

**Remark 3.4.** The condition (3.4) can be replaced by the weaker one

$$|g(x,s,\xi)| \leq d(x) + b(|s|)M(|\xi|),$$

with $b : R \to R^+$ a continuous function belonging to $L^1(\Omega)$ and $d(x) \in L^1(\Omega)$.

Actually the original equation (1.1) will be recovered whenever $h(u) \equiv 1$, but unfortunately this cannot happen in general strong additional requirements on $u$. Therefore, (3.8) is to be viewed as a weaker form of (1.1).

4. **Proof of the Main Result**

From now on, we will use the standard truncation function $T_k$, $k > 0$, defined for all $s \in R$ by $T_k(s) = \max\{-k, \min\{k, s\}\}$.

**Step 1: Approximate problems.** Let $f_n$ be a sequence of $L^\infty(\Omega)$ functions that converge strongly to $f$ in $L^1(\Omega)$. For $n \in N$, $n \geq 1$, let us consider the following sequence of approximate equations

$$-\text{div} \ a(x,u_n,\nabla u_n) + \text{div} \ \Phi_n(u_n) + g_n(x,u_n,\nabla u_n) = f_n - F \text{ in } D'(\Omega),$$

(4.1)

where we have set $\Phi_n(s) = \Phi(T_n(s))$ and $g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{2}v g(x,s,\xi)}$. For fixed $n \geq 1$, it’s obvious to observe that

$$g_n(x,s,\xi) \geq 0, \ |g_n(x,s,\xi)| \leq |g(x,s,\xi)| \text{ and } |g_n(x,s,\xi)| \leq n.$$

Moreover, since $\Phi$ is continuous one has $|\Phi_n(t)| \leq \max_{|t| \leq n} |\Phi(t)|$. Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduces that there exists at least one solution $u_n$ of the approximate Dirichlet problem (4.1) in the sense

$$\begin{cases}
    u_n \in W^1_0 L_M(\Omega), a(x,u_n,\nabla u_n) \in (L^\infty(\Omega))^N \text{ and } \\
    \int_\Omega a(x,u_n,\nabla u_n)\nabla vdx + \int_\Omega \Phi_n(u_n)\nabla vdx \\
    + \int_\Omega g_n(x,u_n,\nabla u_n)vdx = \langle f_n, v \rangle + \int_\Omega F\nabla vdx, \text{ for every } v \in W^1_0 L_M(\Omega).
\end{cases}$$

(4.2)
Step 2: Estimation in $W^{1}_0L_M(\Omega)$. Taking $u_n$ as function test in problem (4.2), we obtain
\[
\int_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n \, dx + \int_{\Omega} \Phi_n(u_n)\nabla u_n \, dx
+ \int_{\Omega} g_n(x,u_n,\nabla u_n)u_n \, dx = (f_n,u_n) + \int_{\Omega} F\nabla u_n \, dx.
\] (4.3)

Define $\tilde{\Phi}_n \in (C^1(\mathbb{R}))^N$ as $\tilde{\Phi}_n(t) = \int_{0}^{t} \Phi_n(\tau) \, d\tau$. Then formally
\[
\text{div} (\tilde{\Phi}_n(u_n)) = \Phi_n(u_n)\nabla u_n, \quad u_n = 0 \text{ on } \partial \Omega, \quad \tilde{\Phi}_n(0) = 0 \quad \text{and by the Divergence theorem}
\int_{\Omega} \Phi_n(u_n)\nabla u_n \, dx = \int_{\Omega} \text{div} (\tilde{\Phi}_n(u_n)) \, dx = \int_{\partial \Omega} \tilde{\Phi}_n(u_n) \vec{n} \, ds = 0,
\]
where $\vec{n}$ is the outward pointing unit normal field of the boundary $\partial \Omega$ ($ds$ may be used as a shorthand for $\vec{n} \, ds$). Thus, by virtue of (3.5) and Young’s inequality, we get
\[
\int_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n \, dx \leq C_1 + \frac{\alpha}{2} \int_{\Omega} M(|\nabla u_n|) \, dx,
\] (4.4)
which, together with (3.3) give
\[
\int_{\Omega} M(|\nabla u_n|) \, dx \leq C_2.
\] (4.5)
Moreover, we also have
\[
\int_{\Omega} g_n(x,u_n,\nabla u_n)u_n \, dx \leq C_3.
\] (4.6)
As a consequence of (4.5) there exist a subsequence of $\{u_n\}_n$, still indexed by $n$, and a function $u \in W^{1}_0L_M(\Omega)$ such that
\[
u_n \to u \quad \text{weakly in } W^{1}_0L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega),\Pi E^M(\Omega)),
\]
\[
u_n \to u \quad \text{strongly in } E_M(\Omega) \text{ and } \text{a. e. in } \Omega.
\] (4.7)

Step 3: Boundedness of $(a(x,u_n,\nabla u_n))_n \in (L^\infty(\Omega))^N$. Let $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Thanks to (3.2), we can write
\[
\left( a(x,u_n,\nabla u_n) - (a(x,u_n,\frac{w}{k_4})) \left( \nabla u_n - \frac{w}{k_4} \right) \right) \geq 0,
\]
which implies
\[
\frac{1}{k_4} \int_{\Omega} a(x,u_n,\nabla u_n)w \, dx \leq \int_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n \, dx
+ \int_{\Omega} a\left(x,u_n,\frac{w}{k_4}\right)\left(\frac{w}{k_4} - \nabla u_n\right) \, dx.
\]
Thanks to (4.4) and (4.5), one has
\[
\int_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n \, dx \leq C_5.
\]
Define $\lambda = 1 + k_1 + k_3$. By the growth condition (3.1) and Young’s inequality, one can write

\[
\begin{align*}
\left| \int_{\Omega} a(x, u_n, \frac{u}{k_3}) \left( \frac{u}{k_4} - \nabla u_n \right) \, dx \right| \\
\leq \left( 1 + \frac{1}{k_4} \right) \left( \int_{\Omega} M(c(x)) \, dx + k_1 \int_{\Omega} M(k_2 |u_n|) \, dx \right) \\
+ k_3 \int_{\Omega} M(|u|) \, dx + \frac{\lambda}{k_4} \int_{\Omega} M(|w|) \, dx + \lambda \int_{\Omega} M(|\nabla u_n|) \, dx.
\end{align*}
\]

By virtue of [18] and Lemma 4.14 of [20], there exists an $N$-function $Q$ such that $M \ll Q$ and the space $W_0^1 L_M(\Omega)$ is continuously embedded into $L_Q(\Omega)$. Thus, by (4.5) there exists a constant $c_0 > 0$, not depending on $n$, satisfying $\|u_n\|_Q \leq c_0$. Since $M \ll Q$, we can write $M(k_2 t) \leq Q(t^{\frac{1}{c_0}})$, for $t > 0$ large enough. As $P \ll M$, we can find a constant $c_1$, not depending on $n$, such that

\[
\int_{\Omega} M^{-1}(k_2 |u_n|) \, dx \leq \int_{\Omega} Q\left( \frac{|u_n|}{c_0} \right) + c_1.
\]

Hence, we conclude that the quantity $\left| \int_{\Omega} a(x, u_n, \nabla u_n) \, dx \right|$ is bounded from above for all $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Using the Orlicz norm we deduce that

\[
\left( a(x, u_n, \nabla u_n) \right)_{n} \text{ is bounded in } (L_{M}(\Omega))^N.
\] (4.8)

**Step 4: Renormalization identity for the approximate solutions.** For any $m \geq 1$, define $\theta_m(r) = T_m(r) - T_{m+1}(r)$. Observe that by [19, Lemma 2] one has $\theta_m(u_n) \in W_0^1 L_M(\Omega)$. The use of $\theta_m(u_n)$ as test function in (4.2) yields

\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq \langle f_n, \theta_m(u_n) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n \, dx,
\]

By Hölder’s inequality and 4.5 we have

\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq \langle f_n, \theta_m(u_n) \rangle \\
+ C_0 \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx.
\]

It’s not hard to see that

\[
\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.
\]

So that by (4.5) and (4.7) one can deduce that

\[
\theta_m(u_n) \rightharpoonup \theta_m(u) \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{M}(\Omega)).
\]

Note that as $m$ goes to $\infty$, $\theta_m(u) \to 0$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M(\Omega), \Pi E_{M}(\Omega))$, and since $f_n$ converges strongly in $L^1(\Omega)$, by Lebesgue’s theorem we have

\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx = \lim_{m \to \infty} \lim_{n \to \infty} \langle f_n, \theta_m(u_n) \rangle = 0.
\]
By (3.3) we finally have
\[ \lim_{m \to \infty} \lim_{n \to \infty} \int_{\{ m \leq |u_n| \leq m+1 \}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \tag{4.9} \]

**Step 5: Almost everywhere convergence of the gradients.** Define
\[ \phi(s) = s e^{\lambda s^2} \text{ with } \lambda = \left( \frac{b(k)}{2\alpha} \right)^2. \]
One can easily verify that for all \( s \in \mathbb{R} \)
\[ \phi'(s) - \frac{b(k)}{\alpha} |\phi(s)| \geq \frac{1}{2}. \tag{4.10} \]

For \( m \geq k \), we define the function \( \psi_m \) by
\[
\begin{cases}
\psi_m(s) = 1 & \text{if } |s| \leq m, \\
\psi_m(s) = m + 1 - |s| & \text{if } m \leq |s| \leq m + 1, \\
\psi_m(s) = 0 & \text{if } |s| \geq m + 1.
\end{cases}
\]

By virtue of [21, Theorem 4] there exists a sequence \( \{v_j\}_j \subset D(\Omega) \) such that \( v_j \to u \) in \( W^1_0 L_M(\Omega) \) for the modular convergence and a.e. \( \Omega \). Let us define the following functions \( \theta^j_n = T_k(u_n) - T_k(v_j) \), \( \theta^j = T_k(u) - T_k(v_j) \) and \( z^j_{n,m} = \phi(\theta^j) \psi_m(u_n) \). Using \( z^j_{n,m} \in W^1_0 L_M(\Omega) \) as test function in (4.2) we get
\[
\begin{align*}
&\int_\Omega a(x, u_n, \nabla u_n) \nabla z^j_{n,m} dx + \int_\Omega \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx \\
&+ \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \nabla \psi_m(u_n) (T_k(u_n) - T_k(v_j)) dx \\
&+ \int_\Omega g_n(x, u_n, \nabla u_n) z^j_{n,m} dx = \int_\Omega f_n z^j_{n,m} dx + \int_\Omega F \nabla z^j_{n,m} dx.
\end{align*}
\tag{4.11}
\]

From now on we denote by \( \epsilon_i(n,j) \), \( i = 0, 1, 2, ... \), various sequences of real numbers which tend to zero, when \( n \) and \( j \to +\infty \), i.e.
\[ \lim_{j \to +\infty} \lim_{n \to +\infty} \epsilon_i(n,j) = 0. \]
In view of (4.7), we have \( z^j_{n,m} \to \phi(\theta^j) \psi_m(u) \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) as \( n \to +\infty \), which yields
\[ \lim_{n \to +\infty} \int_\Omega f_n z^j_{n,m} dx = \int_\Omega f \phi(\theta^j) \psi_m(u) dx, \]
and since \( \phi(\theta^j) \to 0 \) weakly in \( L^\infty(\Omega) \) for \( \sigma(L^\infty, L^1) \) as \( j \to +\infty \), we have
\[ \lim_{j \to +\infty} \int_\Omega f \phi(\theta^j) \psi_m(u) dx = 0. \]
Thus, we write
\[ \int_\Omega f_n z^j_{n,m} dx = \epsilon_0(n,j). \]
Thanks to (4.5) and (4.7), we have as \( n \to +\infty \)
\[ z^j_{n,m} \to \phi(\theta^j) \psi_m(u) \text{ in } W^1_0 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\mathcal{M}}(\Omega)), \]
which implies that
\[ \lim_{n \to +\infty} \int_{\Omega} F \nabla z_{n,m}^j \, dx = \int_{\Omega} F \nabla \theta^j \phi(\theta^j) \psi_m(u) \, dx + \int_{\Omega} F \nabla \theta^j \psi_m'(u) \, dx \]

On the one hand, by Lebesgue’s theorem we get
\[ \lim_{j \to +\infty} \int_{\Omega} F \nabla \theta^j \phi(\theta^j) \psi_m'(u) \, dx = 0, \]
on the other hand, we write
\[ \int_{\Omega} F \nabla \theta^j \phi(\theta^j) \psi_m(u) \, dx = \int_{\Omega} F \nabla T_k(u) \phi(\theta^j) \psi_m(u) \, dx - \int_{\Omega} F \nabla T_k(v_j) \phi(\theta^j) \psi_m(u) \, dx, \]
so that, by Lebesgue’s theorem one has
\[ \lim_{j \to +\infty} \int_{\Omega} F \nabla T_k(u) \phi(\theta^j) \psi_m(u) \, dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) \, dx. \]
Let \( \lambda > 0 \) such that \( M(\frac{\nabla v_j - \nabla u}{\lambda}) \to 0 \) strongly in \( L^1(\Omega) \) as \( j \to +\infty \) and \( M(\frac{\nabla u}{\lambda}) \in L^1(\Omega) \), the convexity of the \( N \)-function \( M \) allows us to have
\[ M\left( \frac{\nabla T_k(v_j) \phi(\theta^j) \psi_m(u) - \nabla T_k(u) \phi(\theta^j) \psi_m(u)}{\lambda} \right) \]
\[ = \frac{1}{4} M\left( \frac{\nabla v_j - \nabla u}{\lambda} \right) + \frac{1}{4} \left( 1 + \frac{1}{\sigma(2k)} \right) M\left( \frac{\nabla u}{\lambda} \right). \]
Then, by using the modular convergence of \( \{ \nabla v_j \} \) in \( (L_M(\Omega))^N \) and Vitali’s theorem, we obtain
\[ \nabla T_k(v_j) \phi(\theta^j) \psi_m(u) \to \nabla T_k(u) \psi_m(u) \text{ in } (L_M(\Omega))^N, \]
for the modular convergence, and then
\[ \lim_{j \to +\infty} \int_{\Omega} F \nabla T_k(u) \phi(\theta^j) \psi_m(u) \, dx = \int_{\Omega} F \nabla T_k(u) \psi_m(u) \, dx. \]
We have proved that
\[ \int_{\Omega} F \nabla z_{n,m}^j \, dx = \epsilon_1(n,j). \]
It’s easy to see that by the modular convergence of the sequence \( \{ v_j \} \), one has
\[ \lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\{ m \leq |u_n| \leq m+1 \}} \Phi_n(u_n) \nabla u_n \psi_m'(u_n) \phi(T_k(u_n) - T_k(v_j)) \, dx = 0, \]
while for the third term in the left-hand side of (4.11) we can write
\[ \int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) \, dx \]
\[ = \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) \, dx - \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx. \]
Firstly, we have
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) dx = 0.
\]
In view of (4.7), one has
\[
\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n) \to \Phi(u) \phi'(\theta^j) \psi_m(u),
\]
almost everywhere in \( \Omega \) as \( n \) tends to +\( \infty \). Furthermore, we can check that
\[
\|\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n)\|_{\overline{\mathcal{M}}} \leq M(\phi'(2k))|\Omega| + 1,
\]
where \( c_m = \max_{t \leq m+1} \Phi(t) \). Applying [27, Theorem 14.6] we get
\[
\lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx.
\]
Using the modular convergence of the sequence \( \{v_j\} \), we obtain
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx = \int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx.
\]
Then, using again the Divergence theorem we get
\[
\int_{\Omega} \Phi(u) \nabla T_k(u) \psi_m(u) dx = 0.
\]
Therefore, we write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx = \epsilon_2(n, j).
\]
Since \( g_n(x, u_n, \nabla u_n) z_{n,m}^j \geq 0 \) on the set \( \{| u_n | > k\} \) and \( \psi_m(u_n) = 1 \) on the set \( \{| u_n | \leq k\} \), from (4.11) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx + \int_{\{|u_n| > k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) dx \leq \epsilon_3(n, j). \tag{4.12}
\]
We now evaluate the first term of the left-hand side of (4.12) by writing
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx
\]
\[
= \int_{\Omega} a(x, u_n, \nabla u_n)(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) \psi_m(u_n) dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi'(\theta_n^j) \psi_m(u_n) dx
\]
\[
= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) dx
\]
\[
- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi'(\theta_n^j) \psi_m(u_n) dx.
\]
and then
\[
\begin{align*}
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j dx &= \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(v_j), \nabla T_k(v_j)) \right)
(\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^k) dx \\
&+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^k) dx \\
&- \int_{\partial \Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \nabla T_k(v_j) \phi'(\theta_n^k) dx \\
&- \int_{\{u_n > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^k) \psi_m(u_n) dx \\
&+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi'(\theta_n^k) \psi_m(u_n) dx,
\end{align*}
\]
(4.13)

where by $\chi_j^s$, $s > 0$, we denote the characteristic function of the subset
\[\Omega_j^s = \{ x \in \Omega : |\nabla T_k(v_j)| \leq s \} .\]

For fixed $m$ and $s$, we will pass to the limit in $n$ and then in $j$ in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^k) dx
\]
\[
\rightarrow \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j) dx,
\]
as $n \rightarrow +\infty$. Since by lemma (2.4) one has
\[a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^k) \rightarrow a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j),\]
strongly in $(E_{M}(\Omega))^N$ as $n \rightarrow \infty$, while by (4.5)
\[\nabla T_k(u_n) \rightarrow \nabla T_k(u),\]
weakly in $(L_M(\Omega))^N$. Let $\chi^s$ denote the characteristic function of the subset
\[\Omega_j^s = \{ x \in \Omega : |\nabla T_k(u)| \leq s \} .\]

As $\nabla T_k(v_j) \chi_j^s \rightarrow \nabla T_k(u) \chi^s$ strongly in $(E_M(\Omega))^N$ as $j \rightarrow +\infty$, one has
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta^j) dx \rightarrow 0,
\]
as $j \rightarrow \infty$. Then
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) \phi'(\theta_n^k) dx = \epsilon_4(n, j). \quad (4.14)
\]

We now estimate the third term of (4.13). It’s easy to see that by (3.3), $a(x, s, 0) = 0$ for almost everywhere $x \in \Omega$ and for all $s \in \mathbb{R}$. Thus, from (4.8) we have that $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ is bounded in $(L_M(\Omega))^N$ for all $k \geq 0$. 
Therefore, there exist a subsequence still indexed by $n$ and a function $l_k$ in $(L^\infty(\Omega))^N$ such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \to l_k$$

weakly in $(L^\infty(\Omega))^N$ for $\sigma(\Pi L^\infty, \Pi E)$. \hspace{1cm} (4.15)

Then, since $\nabla T_k(v_j)\chi_{\Omega\setminus \Omega_j^l} \in (E^\infty(\Omega))^N$, we obtain

$$\int_{\Omega\setminus \Omega_j^l} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta^i_n)dx \to \int_{\Omega\setminus \Omega_j^l} l_k \nabla T_k(v_j) \phi'(\theta^i)dx,$$

as $n \to +\infty$. The modular convergence of $\{v_j\}$ allows us to get

$$- \int_{\Omega\setminus \Omega_j^l} l_k \nabla T_k(v_j) \phi'(\theta^i)dx \to - \int_{\Omega\setminus \Omega} l_k \nabla T_k(u)dx,$$

as $j \to +\infty$. This proves

$$- \int_{\Omega\setminus \Omega_j^l} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta^i_n)\psi_m(u_n)dx = - \int_{\Omega\setminus \Omega} l_k \nabla T_k(u)dx + \epsilon_5(n, j). \hspace{1cm} (4.16)$$

As regards the fourth term, observe that $\psi_m(u_n) = 0$ on the subset $\{|u_n| \geq m + 1\}$, so we have

$$- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta^i_n)\psi_m(u_n)dx =$$

$$- \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta^i_n)\psi_m(u_n)dx.$$

Since

$$- \int_{\{|u_n| > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta^i_n)\psi_m(u_n)dx =$$

$$- \int_{\{|u| > k\}} l_{m+1} \nabla T_k(u)\psi_m(u)dx + \epsilon_5(n, j),$$

observing that $\nabla T_k(u) = 0$ on the subset $\{|u| > k\}$, one has

$$- \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta^i_n)\psi_m(u_n)dx = \epsilon_6(n, j). \hspace{1cm} (4.17)$$

For the last term of (4.13), we have

$$\left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^i_n)\psi_m'(u_n)dx \right|$$

$$\leq \left| \int_{\{|u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^i_n)\psi_m'(u_n)dx \right|$$

$$\leq \phi(2k) \int_{\{|u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx.$$
To estimate the last term of the previous inequality, we use \((T_1(u_n - T_m(u_n)) \in W_0^1 L_M(\Omega))\) as test function in (4.2), to get
\[
\int_{\left\{ |u_n| \leq m+1 \right\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx + \int_{\left\{ |u_n| \leq m+1 \right\}} \Phi_n(u_n) \nabla u_n \, dx \\
+ \int_{\left\{ |u_n| \geq m \right\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \, dx = \langle f_n, T_1(u_n - T_m(u_n)) \rangle \\
+ \int_{\left\{ |u_n| \leq m+1 \right\}} F \nabla u_n \, dx.
\]
By Divergence theorem, we have
\[
\int_{\left\{ |u_n| \leq m+1 \right\}} \Phi_n(u_n) \nabla u_n \, dx = 0.
\]
Using the fact that \(g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \geq 0\) on the subset \(\left\{ |u_n| \geq m \right\}\) and Young’s inequality, we get
\[
\int_{\left\{ |u_n| \leq m+1 \right\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \\
\leq \langle f_n, T_1(u_n - T_m(u_n)) \rangle + \int_{\left\{ |u_n| \leq m+1 \right\}} M(|F|) \, dx.
\]
It follows that
\[
\left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_j^n) \phi'(\theta_j^n) \, dx \right| \\
\leq 2\phi(2k) \left( \int_{\left\{ |u_n| \leq m+1 \right\}} |f_n| \, dx + \int_{\left\{ |u_n| \leq m+1 \right\}} M(|F|) \, dx \right).
\]
From (4.14), (4.16), (4.17) and (4.18) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m} \, dx \\
\geq \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_j^n) \right) \\
\quad \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^n \right) \phi(\theta_j^n) \, dx \\
\quad - \alpha \phi(2k) \left( \int_{\left\{ |u_n| \leq m+1 \right\}} |f_n| \, dx + \int_{\left\{ |u_n| \leq m+1 \right\}} M(|F|) \, dx \right) \\
\quad - \int_{\Omega \setminus \Omega^*} l_k \cdot \nabla T_k(u) \, dx + c_\gamma(n, j).
\]
(4.19)
Now, we turn to second term in the left-hand side of (4.12). We have
\[
\left| \int_{\left\{ |u_n| \leq k \right\}} g_n(x, u_n, \nabla u_n) \phi(\theta_j^n) \, dx \right| \\
= \left| \int_{\left\{ |u_n| \leq k \right\}} g_n(x, T_k(u_n), \nabla T_k(u_n)) \phi(\theta_j^n) \, dx \right| \\
\leq b(k) \int_{\Omega} M(|\nabla T_k(u_n)|) \phi(\theta_j^n) \, dx + b(k) \int_{\Omega} d(x) \phi(\theta_j^n) \, dx \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \phi(\theta_j^n) \, dx + c_\delta(n, j).
\]
Then
\[
\int_{\{|u_n| \leq k\}} g_n(x,u_n,\nabla u_n)\phi(\theta^j_n)dx \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} \left( a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(v_j),\nabla T_k(v_j)\chi^*_j) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j)\chi^*_j \right) \phi(\theta^j_n)|dx \\
+ \frac{b(k)}{\alpha} \int_{\Omega} a(x,T_k(u_n),\nabla T_k(v_j)\chi^*_j)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi^*_j)|\phi(\theta^j_n)|dx \\
+ \frac{b(k)}{\alpha} \int_{\Omega} a_n(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^*_j|\phi(\theta^j_n)|dx + \epsilon_9(n,j).
\]

(4.20)

We proceed as above to get
\[
\frac{b(k)}{\alpha} \int_{\Omega} a(x,T_k(u_n),\nabla T_k(v_j)\chi^*_j)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi^*_j)|\phi(\theta^j_n)|dx = \epsilon_9(n,j)
\]
and
\[
\frac{b(k)}{\alpha} \int_{\Omega} a_n(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^*_j|\phi(\theta^j_n)|dx = \epsilon_{10}(n,j).
\]

Hence, we have
\[
\int_{\{|u_n| \leq k\}} g_n(x,u_n,\nabla u_n)\phi(\theta^j_n)dx \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} \left( a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(v_j),\nabla T_k(v_j)\chi^*_j) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j)\chi^*_j \right) \phi(\theta^j_n)|dx + \epsilon_{11}(n,j).
\]

(4.21)

Combining (4.12), (4.19) and (4.21), we get
\[
\int_{\Omega} \left( a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(v_j),\nabla T_k(v_j)\chi^*_j) \right)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi^*_j) \\
\left( \phi(\theta^j_n) - \frac{b(k)}{\alpha}\phi(\theta^j_n) \right)|dx \\
\leq \int_{\Omega\setminus\Omega^*} l_k \nabla T_k(u)dx + a\phi(2k)(\int_{\{m \leq |u_n|\}} |f_n|dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|)dx) \\
+ \epsilon_{12}(n,j).
\]

By (4.10), we have
\[
\int_{\Omega} \left( a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(v_j),\nabla T_k(v_j)\chi^*_j) \right)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi^*_j)dx \\
\leq 2\int_{\Omega\setminus\Omega^*} l_k \nabla T_k(u)dx + 4a\phi(2k)(\int_{\{m \leq |u_n|\}} |f_n|dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|)dx) \\
+ \epsilon_{12}(n,j).
\]

(4.22)
On the other hand we can write
\[
\int_{\Omega} (a(x, T_k(u_n)), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s_n)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s_n) dx = \int_{\Omega} (a(x, T_k(u_n)) - a(x, T_k(v_j)) (\nabla T_k(u_n) - \nabla T_k(v_j)\chi^s_n) dx \\
+ \int_{\Omega} (a(x, T_k(u_n)) (\nabla T_k(v_j))\chi^s_n - \nabla T_k(u)\chi^s_n) dx \\
- \int_{\Omega} (a(x, T_k(u_n)) (\nabla T_k(u)\chi^s_n - \nabla T_k(v_j)\chi^s_n) dx \\
+ \int_{\Omega} (a(x, T_k(u_n)) (\nabla T_k(v_j)\chi^s_n - \nabla T_k(v_j)\chi^s_n) dx
\]

We shall pass to the limit in \(n\) and then in \(j\) in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain
\[
\int_{\Omega} a(x, T_k(u_n)) (\nabla T_k(u_n)) (\nabla T_k(v_j))\chi^s_n - \nabla T_k(u)\chi^s_n) dx = \epsilon_{13}(n, j), \\
\int_{\Omega} a(x, T_k(u_n)) (\nabla T_k(u)\chi^s_n - \nabla T_k(v_j)\chi^s_n) dx = \epsilon_{14}(n, j), \\
\int_{\Omega} a(x, T_k(u_n)) (\nabla T_k(v_j)\chi^s_n - \nabla T_k(v_j)\chi^s_n) dx = \epsilon_{15}(n, j).
\]

So that
\[
\int_{\Omega} (a(x, T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s_n)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s_n) dx \\
+ \epsilon_{16}(n, j),
\]

Let \(r \leq s\). Using (3.2), (4.22) and (4.24) we can write
\[
0 \leq \int_{\Omega^r} (a(x, T_k(u_n)), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\
\leq \int_{\Omega^r} (a(x, T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\
= \int_{\Omega} (a(x, T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s_n)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s_n) dx \\
\leq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s_n)) (\nabla T_k(u_n) - \nabla T_k(u)\chi^s_n) dx \\
+ \epsilon_{15}(n, j) \\
\leq 2 \int_{\Omega^r} l_k \nabla T_k(u) dx + 2a \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} |\mathcal{M}(F)| dx \right) \\
+ \epsilon_{17}(n, j).
\]
By passing to the superior limit over $n$ and then over $j$,

$$0 \leq \limsup_{n \to +\infty} \int_{\Omega^r} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right) \nabla T_k(u_n) - \nabla T_k(u) \right) dx$$

\[ \leq 2 \int_{\Omega \setminus \Omega^r} l_k \nabla T_k(u) dx + 4 \alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right).
\]

Letting $s \to +\infty$ and then $m \to +\infty$, taking into account that $l_k \nabla T_k(u) \in L^1(\Omega)$, $f \in L^1(\Omega)$, $|F| \in (E_{\Pi E M}(\Omega))^N$, $|\Omega \setminus \Omega^r| \to 0$, and $|[m \leq |u| \leq m+1]| \to 0$, one has

$$\int_{\Omega^r} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right) \nabla T_k(u_n) - \nabla T_k(u) dx,$$

\[ (4.25) \]

tends to 0 as $n \to +\infty$. As in [20], we deduce that there exists a subsequence of $\{u_n\}$ still indexed by $n$ such that

$$\nabla u_n \rightharpoonup \nabla u \text{ a.e. in } \Omega.$$

(4.26)

Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get

$$a(x, u, \nabla u) \in (L_{\Pi E M}(\Omega))^N$$

and

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\Pi E M}(\Omega))^N \text{ for } \sigma(\Pi L_{\Pi E M}, \Pi E M).$$

(4.27)

**Step 6: Modular convergence of the truncations.** Going back to equation (4.22), we can write

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx$$

\[ \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u_j) \chi_j^s dx 
+ \int_{\Omega} a(x, T_k(u), \nabla T_k(u_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(u_j) \chi_j^s) dx 
+ 2 \alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right) 
+ 2 \int_{\Omega \setminus \Omega^r} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{12}(n, j). \]

By (4.23) we get

$$\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx$$

\[ \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u_j) \chi_j^s dx 
+ 2 \alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) dx \right) 
+ 2 \int_{\Omega \setminus \Omega^r} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + \epsilon_{18}(n, j). \]
We now pass to the superior limit over \( n \) in both sides of this inequality using (4.27), to obtain
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + 2\alpha(2k) \left( \int_{\{m \leq |u|\}} |f| dx + \int_{\{m \leq |u| \leq m+1\}} \mathcal{M}(|F|) dx \right) \\
+ 2 \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
\]

We then pass to the limit in \( j \) to get
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx + 2\alpha(2k) \left( \int_{\{m \leq |u|\}} |f| dx + \int_{\{m \leq |u| \leq m+1\}} \mathcal{M}(|F|) dx \right) \\
+ 2 \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
\]

Letting \( s \) and then \( m \to +\infty \), one has
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
\]

On the other hand, by (3.3), (4.5), (4.26) and Fatou’s lemma, we have
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \leq \liminf_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx.
\]

It follows that
\[
\lim_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx = \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
\]

By Lemma 2.5 we conclude that for every \( k > 0 \)
\[
a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u),
\]
strongly in \( L^1(\Omega) \). The convexity of the \( N \)-function \( M \) and (3.3) allow us to have
\[
M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) \]
\[
\leq \frac{1}{2\alpha} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) + \frac{1}{2\alpha} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u).
\]

From Vitali’s theorem we deduce
\[
\limsup_{|E| \to 0} \sup_n \int_E M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) dx = 0.
\]

Thus, for every \( k > 0 \)
\[
T_k(u_n) \to T_k(u) \text{ in } W^1_0 L_M(\Omega),
\]
Step 7: Compactness of the nonlinearities. We need to prove that
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \] (4.29)

By virtue of (4.7) and (4.26) one has
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a. e. in } \Omega. \] (4.30)

Let \( E \) be measurable subset of \( \Omega \) and let \( m > 0 \). Using (3.3) and (3.4) we can write
\[
\int_E |g_n(x, u_n, \nabla u_n)| \, dx = \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| \, dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| \, dx 
\leq b(m) \int_E d(x) \, dx + b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx 
+ \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx.
\]

From (3.5) and (4.6), we deduce that
\[ 0 \leq \int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx \leq C_3. \]

So
\[ 0 \leq \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx \leq \frac{C_3}{m}. \]

Then
\[ \lim_{m \to +\infty} \frac{1}{m} \int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx = 0. \]

Thanks to (4.28) the sequence \( \{a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)\}_n \) is equi-integrable. This fact allows us to get
\[ \lim_{|E| \to 0} \sup_n \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) \, dx = 0. \]

This shows that \( g_n(x, u_n, \nabla u_n) \) is equi-integrable. Thus, Vitali’s theorem implies that \( g(x, u, \nabla u) \in L^1(\Omega) \) and
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]

Step 8: Renormalization identity for the solutions. In this step we prove that
\[ \lim_{m \to +\infty} \int_{\{|u| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx = 0. \] (4.31)
Indeed, for any $m \geq 0$ we can write
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \\
= \int_\Omega a(x, u_n, \nabla u_n) (\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) \, dx \\
= \int_\Omega a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) \, dx \\
- \int_\Omega a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx.
\]

In view of (4.28), we can pass to the limit as $n$ tends to $+\infty$ for fixed $m \geq 0$
\[
\lim_{n \to +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \\
= \int_\Omega a(x, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_{m+1}(u) \, dx \\
- \int_\Omega a(x, T_m(u), \nabla T_m(u)) \nabla T_m(u) \, dx \\
= \int_\Omega a(x, u, \nabla u) (\nabla T_{m+1}(u) - \nabla T_m(u)) \, dx \\
= \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx.
\]

Having in mind (4.9), we can pass to the limit as $m$ tends to $+\infty$ to obtain (4.31).

**Step 9: Passing to the limit.** Thanks to (4.28) and Lemma (2.5), we obtain
\[
a(x, u_n, \nabla u_n) \nabla u_n \to a(x, u, \nabla u) \nabla u \text{ strongly in } L^1(\Omega). \tag{4.32}
\]

Let $h \in C^1_c(\mathbb{R})$ and $\varphi \in \mathcal{D}(\Omega)$. Inserting $h(u_n)\varphi$ as test function in (4.2), we get
\[
\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi \, dx + \int_\Omega a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) \, dx \\
+ \int_\Omega \Phi_n(u_n) \nabla (h(u_n)) \varphi \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx \\
= (f_n, h(u_n)\varphi) + \int_\Omega F \nabla (h(u_n)) \varphi \, dx. \tag{4.33}
\]

We shall pass to the limit as $n \to +\infty$ in each term of the equality (4.33). Since $h$ and $h'$ have compact support on $\mathbb{R}$, there exists a real number $\nu > 0$, such that $\text{supp } h \subset [-\nu, \nu]$ and $\text{supp } h' \subset [-\nu, \nu]$. For $n > \nu$, we can write
\[
\Phi_n(t)h(t) = \Phi(T_{\nu}(t))h(t) \text{ and } \Phi_n(t)h'(t) = \Phi(T_{\nu}(t))h'(t).
\]

Moreover, the functions $\Phi h$ and $\Phi h'$ belong to $(C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N$. Observe first that the sequence $\{h(u_n)\varphi\}_n$ is bounded in $W^1_0 L^M(\Omega)$. Indeed, let $\rho > 0$. 

be a positive constant such that \( \| h(u_n) \nabla \varphi \|_{\infty} \leq \rho \) and \( \| h'(u_n) \varphi \|_{\infty} \leq \rho \). Using the convexity of the \( N \)-function \( M \) and taking into account (4.5) we have
\[
\int_{\Omega} M \left( \frac{\| h(u_n) \nabla \varphi \|}{2\rho} \right) dx \leq \frac{1}{2} M(1) |\Omega| + \frac{1}{2} \int_{\Omega} M(\| \nabla u_n \|) dx
\]
\[
\leq \frac{1}{2} M(1) |\Omega| + \frac{1}{2} C_2.
\]
This, together with (4.7), imply that
\[
h(u_n) \varphi \rightharpoonup h(u) \varphi \text{ strongly in } (\text{EM}(\Omega))^N.
\]
This enables us to get
\[
(f_n, h(u_n) \varphi) \rightharpoonup (f, h(u) \varphi).
\]
Let \( E \) be a measurable subset of \( \Omega \). Define \( c_\nu = \max_{|t| \leq \nu} \Phi(t) \). Let us denote by \( \| v \|_{M} \) the Orlicz norm of a function \( v \in L_M(\Omega) \). Using strengthened Hölder inequality with both Orlicz and Luxemburg norms, we get
\[
\| \Phi(T_\nu(u_n)) \chi_E \|_{\overline{\text{M}}(\Omega)} = \sup_{\| v \|_{M} \leq 1} |\int_{E} \Phi(T_\nu(u_n)) v dx|
\]
\[
\leq c_\nu \sup_{\| v \|_{M} \leq 1} \| \chi_E \|_{\overline{\text{M}}} \| v \|_{M}
\]
\[
\leq c_\nu |E|M^{-1}\left( \frac{1}{|E|} \right).
\]
Thus, we get
\[
\lim_{|E| \to 0} \sup_n \| \Phi(T_\nu(u_n)) \chi_E \|_{\overline{\text{M}}} = 0.
\]
Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain
\[
\Phi(T_\nu(u_n)) \to \Phi(T_\nu(u)) \text{ strongly in } (\text{EM}(\Omega))^N,
\]
which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have
\[
\int_{\Omega} \Phi(T_\nu(u_n)) \nabla (h(u_n) \varphi) dx \to \int_{\Omega} \Phi(T_\nu(u)) \nabla (h(u) \varphi) dx.
\]
We remark that
\[
|a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi| \leq \rho a(x, u_n, \nabla u_n) \nabla u_n.
\]
Consequently, using (4.32) and Vitali’s theorem, we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi dx \to \int_{\Omega} a(x, u, \nabla u) \nabla uh'(u) \varphi dx.
\]
and
\[
\int_{\Omega} F \nabla u_n h'(u_n) \varphi dx \to \int_{\Omega} F \nabla uh'(u) \varphi dx.
\]
For the second term of (4.33), as above we have
\[
h(u_n) \nabla \varphi \rightharpoonup h(u) \nabla \varphi \text{ strongly in } (\text{EM}(\Omega))^N.
\]
which together with (4.27) give
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi(h(u_n)) \, dx \to \int_{\Omega} a(x, u, \nabla u) \nabla \varphi(h(u)) \, dx
\]
and
\[
\int_{\Omega} F \nabla \varphi(h(u_n)) \, dx \to \int_{\Omega} F \nabla \varphi(h(u)) \, dx.
\]
The fact that \( h(u_n) \varphi \rightharpoonup h(u) \varphi \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get
\[
\int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx \to \int_{\Omega} g(x, u, \nabla u) h(u) \varphi \, dx.
\]
At this point we can pass to the limit in each term of (4.33) to get
\[
\int_{\Omega} a(x, u, \nabla u)(\nabla \varphi h(u) + h'(u) \varphi \nabla u) \, dx + \int_{\Omega} \Phi(u) h'(u) \varphi \nabla u \, dx
\]
\[
+ \int_{\Omega} \Phi(u) h(u) \nabla \varphi \, dx + \int_{\Omega} g(x, u, \nabla u) h(u) \varphi \, dx
\]
\[
= (f, h(u) \varphi) + \int_{\Omega} F(\nabla \varphi h(u) + h'(u) \varphi \nabla u) \, dx,
\]
for all \( h \in C^1_c(\mathbb{R}) \) and for all \( \varphi \in D(\Omega) \). Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou’s lemma to get \( g(x, u, \nabla u) u \in L^1(\Omega) \). By virtue of (4.7), (4.27), (4.29), (4.31), the function \( u \) is a renormalized solution of problem (1.1).

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REFERENCES


