**Renormalized Solutions of Strongly Nonlinear Elliptic Problems with Lower Order Terms and Measure Data in Orlicz-Sobolev Spaces**

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**Abstract.** The purpose of this paper is to prove the existence of a renormalized solution of perturbed elliptic problems

\[-\text{div} \left( a(x,u,\nabla u) + \Phi(u) \right) + g(x,u,\nabla u) = f - \text{div} F,\]

in a bounded open set $\Omega$ and $u = 0$ on $\partial \Omega$, in the framework of Orlicz-Sobolev spaces without any restriction on the $M$-function of the Orlicz spaces, where $-\text{div} \left( a(x,u,\nabla u) \right)$ is a Leray-Lions operator defined from $W^{1,0}_{1}(\Omega)$ into its dual, $\Phi \in C^{0}(\mathbb{R},\mathbb{R}^{N})$. The function $g(x,u,\nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$, satisfying the sign condition and the datum $\mu$ is assumed to belong to $L^{1}(\Omega) + W^{-1}E_{M}(\Omega)$.

**Keywords:** Elliptic equation, Orlicz-Sobolev spaces, Renormalized solution.


1. **Introduction**

Let $\Omega$ be a bounded open set of $\mathbb{R}^{N}$, $N \geq 2$, and let $M$ be an $N$-function. In the present paper we prove an existence result of a renormalized solution of the following strongly nonlinear elliptic problem

\[
\begin{cases}
A(u) - \text{div} \Phi(u) + g(x,u,\nabla u) = f - \text{div} F & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(1.1)
Here, $\Phi \in C^0(\mathbb{R}, \mathbb{R}^N)$, while the function $g(x, u, \nabla u)$ is a non linear lower order term with natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The non everywhere defined nonlinear operator $A(u) = -\text{div} \ (a(x, u, \nabla u))$ acts from its domain $D(A) \subset W_0^1 L_M(\Omega)$ into $W^{-1} L_{\overline{M}}(\Omega)$. The function $a(x, u, \nabla u)$ is assumed to satisfy, among others, $a(x, u, \nabla u)$ nonstandard growth condition governed by the $N$-function $M$, and the source term $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, $\overline{M}$ stands for the conjugate of $M$.

We use here the notion of renormalized solutions, which was introduced by R.J. DiPerna and P.-L. Lions in their papers [16, 15] where the authors investigate the existence of solutions of the Boltzmann equation, by introducing the idea of renormalized solution. This concept of solution was then adapted to study (1.1) with $\Phi \equiv 0$, $g \equiv 0$ and $L^1(\Omega)$-data by F. Murat in [29, 28], by G. Dal Maso et al. in [13] with general measure data and then when $f$ is a bounded Radon measure datum and $g$ grows at most like $|\nabla u|^{p-1}$ by Beta et al. in [9, 10, 11] with $\Phi \equiv 0$ and by Guibé and Mercaldo in [23, 24] when $\Phi(u)$ behaves at most like $|u|^{p-1}$. Renormalization idea was then used in [12] for variational equations and in [30] when the source term is in $L^1(\Omega)$. Recall that to get both existence and uniqueness of a solution to problems with $L^1$-data, two notions of solution equivalent to the notion of renormalized solution were introduced, the first is the entropy solution by Bénilan et al. [4] and then the second is the SOLA by Dall’Aglio [14].

The authors in [5] have dealt with the equation (1.1) with $g = g(x, u)$ and $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, under the restriction that the $N$-function $M$ satisfies the $\Delta_2$-condition. This work was then extended in [2] for $N$-functions not satisfying necessarily the $\Delta_2$-condition. Our goal here is to extend the result in [2] solving the problem (1.1) without any restriction on the $N$-function $M$. Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $\mu \in W^{-1} E_{\overline{M}}(\Omega)$, an existence result has been proved in [3], Specific examples to which our results apply include the following:

\[-\text{div} \left( |\nabla u|^{p-2} \nabla u + |u|^s u \right) + u|\nabla u|^p = \mu \text{ in } \Omega,\]
\[-\text{div} \left( |\nabla u|^{p-2} \nabla u \log(1 + |\nabla u|) + |u|^s u \right) = \mu \text{ in } \Omega,\]
\[-\text{div} \left( \frac{M(|\nabla u|) \nabla u}{|\nabla u|^2} + |u|^s u \right) + M(|\nabla u|) = \mu \text{ in } \Omega,\]

where $p > 1$, $s > 0$, $\beta > 0$ and $\mu$ is a given Radon measure on $\Omega$.

It is our purpose in this paper, to prove the existence of a renormalized solution for the problem (1.1) when the source term has the form $f - \text{div} F$ with $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, in the setting of Orlicz spaces without any restriction on the $N$-functions $M$. The approximate equations provide a $W_0^1 L_M(\Omega)$ bound for the corresponding solution $u_n$. This allows us to obtain...
a function $u$ as a limit of the sequence $u_n$. Hence, appear two difficulties. The first one is how to give a sense to $\Phi(u)$, the second difficulty lies in the need of the convergence almost everywhere of the gradients of $u_n$ in $\Omega$. This is done by using suitable test functions built upon $u_n$ which make licit the use of the divergence theorem for Orlicz functions. We note that the techniques we used in the proof are different from those used in [2, 5, 12, 17, 25].

Let us briefly summarize the contents of the paper. The Section 2 is devoted to developing the necessary preliminaries, we introduce some technical lemmas. Section 3 contains the basic assumptions, the definition of renormalized solution and the main result, while the Section 4 is devoted to the proof of the main result.

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an $N$-function, i.e., $M$ is continuous, increasing, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$, and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$. Equivalently, $M$ admits the representation:

$$M(t) = \int_0^t a(s) \, ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to $+\infty$ as $t \to +\infty$.

The conjugate of $M$ is also an $N$-function and it is defined by $\overline{M} = \int_0^t \bar{a}(s) \, ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is the function $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1]).

An $N$-function $M$ is said to satisfy the $\Delta_2$-condition if, for some $k$,

$$M(2t) \leq kM(t) \quad \forall t \geq 0,$$

(2.1)

when (2.1) holds only for $t \geq t_0 > 0$ then $M$ is said to satisfy the $\Delta_2$-condition near infinity. Moreover, we have the following Young’s inequality

$$st \leq M(t) + \overline{M}(s), \quad \forall s, t \geq 0.$$

Given two $N$-functions, we write $P \ll Q$ to indicate $P$ grows essentially less rapidly than $Q$; i.e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if

$$\lim_{t \to \infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $k_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions $u$ on $\Omega$ such that

$$\int_{\Omega} M(|u(x)|) \, dx < +\infty \quad (\text{resp.} \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right) \, dx < +\infty \text{ for some } \lambda > 0).$$
The set $L_M(\Omega)$ is a Banach space under the norm
\[ \|u\|_{M,\Omega} = \inf\{ \lambda > 0 : \int_\Omega M(\frac{|u(x)|}{\lambda}) \, dx \leq 1 \}, \]
and $k_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\text{M}}(\Omega)$ by means of the pairing $\int_\Omega uv \, dx$, and the dual norm of $L_{\text{M}}(\Omega)$ is equivalent to $\|\cdot\|_{M,\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm
\[ \|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}. \]

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\text{M}})$ and $\sigma(\prod L_M, \prod L_{\text{M}})$.

The space $W^1_0E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W^1_0L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\text{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that $u_n$ converges to $u$ for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) \, dx \to 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\prod L_M, \prod L_{\text{M}})$. If $M$ satisfies the $\Delta_2$ condition on $\mathbb{R}^+$ (near infinity only when $\Omega$ has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}_{\text{M}}(\Omega)$ [resp. $W^{-1}E_{\text{M}}(\Omega)$] denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\text{M}}(\Omega)$ [resp. $E_{\text{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [1]).

A domain $\Omega$ has the segment property if for every $x \in \partial\Omega$ there exists an open set $G_x$ and a nonzero vector $y_x$ such that $x \in G_x$ and if $z \in \overline{G_x} \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$. The following lemmas can be found in [6].

**Lemma 2.1.** Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $M$ be an $N$-function and let $u \in W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$]. Then $F(u) \in W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$]. Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then
\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}
\]
Lemma 2.2. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of $F'$ is finite. Let $M$ be an $N$-function, then the mapping $F : W^{1,1}_M(\Omega) \to W^{1,1}_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\mathcal{M}})$.

Lemma 2.3. ([21]) Let $\Omega$ have the segment property. Then for each $\nu \in W^{1,1}_0(\Omega)$, there exists a sequence $\nu_n \in \mathcal{D}(\Omega)$ such that $\nu_n$ converges to $\nu$ for the modular convergence in $W^{1,1}_0(\Omega)$. Furthermore, if $\nu \in W^{1,1}_0(\Omega)$, then
\[
\|\nu_n\|_{L^\infty(\Omega)} \leq (N + 1)\|\nu\|_{L^\infty(\Omega)}.
\]

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [8]).

Lemma 2.4. Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure. Let $M, P, Q$ be $N$-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:
\[
|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),
\]
where $k_1, k_2$ are real constants and $c(x) \in E_Q(\Omega)$.

Then the Nemytskii operator $N_f$ defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), \frac{1}{M}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{M}\}$ into $E_Q(\Omega)$.

We will also use the following technical lemma.

Lemma 2.5. ([26]) If $\{f_n\} \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in $\Omega$, then $f_n \to f$ a.e. in $\Omega$ and $\int_{\Omega} f_n(x) \, dx \to \int_{\Omega} f(x) \, dx$, then $f_n \to f$ in $L^1(\Omega)$.

3. Structural Assumptions and Main Result

Throughout the paper $\Omega$ will be a bounded subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property. Let $M$ and $P$ be two $N$-functions such that $P \ll M$. Let $A$ be the non everywhere defined operator defined from its domain $\mathcal{D}(\Omega) \subset W^{1,1}_0(\Omega)$ into $W^{-1}_M(\Omega)$ given by
\[
A(u) := - \text{div} a(\cdot, u, \nabla u),
\]
where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. We assume that there exist a nonnegative function $c(x)$ in $E_{\mathcal{M}}(\Omega)$, $\alpha > 0$ and positive real constants $k_1, k_2, k_3$ and $k_4$, such that for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, $\xi' \in \mathbb{R}^N$ ($\xi \neq \xi'$) and for almost every $x \in \Omega$
\[
|a(x, s, \xi)| \leq c(x) + k_1 P^{-1} M(k_2|s|) + k_3 M^{-1} M(k_4|\xi|),
\]
Here, \( g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is a Carathéodory function satisfying for almost every \( x \in \Omega \) and for all \( s \in \mathbb{R}, \xi \in \mathbb{R}^N \),
\[
|g(x, s, \xi)| \leq b(|s|) (d(x) + M(|\xi|)),
\]
where \( b : \mathbb{R} \rightarrow \mathbb{R}^+ \) is a continuous and increasing function while \( d \) is a given nonnegative function in \( L^1(\Omega) \).

The right-hand side of (1.1) and \( \Phi : \mathbb{R} \rightarrow \mathbb{R}^N \), are assumed to satisfy
\[
f \in L^1(\Omega) \text{ and } |F| \in E_{\mathcal{M}}(\Omega),
\]
and
\[
\Phi \in C^0(\mathbb{R}, \mathbb{R}^N).
\]

Our aim in this paper is to give a meaning to a possible solution of (1.1). In view of assumptions (3.1), (3.2), (3.3) and (3.6), the natural space in which one can seek for a solution \( u \) of problem (1.1) is the Orlicz-Sobolev space \( W^1_0L_M(\Omega) \). But when \( u \) is only in \( W^1_0L_M(\Omega) \) there is no reason for \( \Phi(u) \) to be in \( (L^1(\Omega))^N \) since no growth hypothesis is assumed on the function \( \Phi \). Thus, the term \( \text{div} (\Phi(u)) \) may be ill-defined even as a distribution. This hindrance is bypassed by solving some weaker problem obtained formally through a pointwise multiplication of equation (1.1) by \( h(u) \) where \( h \) belongs to \( C^1_c(\mathbb{R}) \), the class of \( C^1(\mathbb{R}) \) functions with compact support.

**Definition 3.1.** A measurable function \( u : \Omega \rightarrow \mathbb{R} \) is called a renormalized solution of (1.1) if \( u \in W^1_0L_M(\Omega) \), \( a(x, u, \nabla u) \in (L^1(\Omega))^N \), \( g(x, u, \nabla u) \in L^1(\Omega) \), \( g(x, u, \nabla u)u \in L^1(\Omega) \), and
\[
\lim_{m \rightarrow +\infty} \int_{\{x \in \Omega : m \leq |u(x)| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx = 0,
\]
and
\[
\left\{ \begin{array}{l}
-\text{div} a(x, u, \nabla u)h(u) - \text{div} (\Phi(u)h(u)) + h'(u)\Phi(u)\nabla u \\
+ g(x, u, \nabla u)h(u) = fh(u) - \text{div} (Fh(u)) + h'(u)F\nabla u \text{ in } \mathcal{D}'(\Omega),
\end{array} \right.
\]
for every \( h \in C^1_c(\mathbb{R}) \).

**Remark 3.2.** Every term in the problem (3.8) is meaningful in the distributional sense. Indeed, for \( h \) in \( C^1_c(\mathbb{R}) \) and \( u \) in \( W^1_0L_M(\Omega) \), \( h(u) \) belongs to \( W^1L_M(\Omega) \) and for \( \varphi \) in \( \mathcal{D}(\Omega) \) the function \( \varphi h(u) \) belongs to \( W^1_0L_M(\Omega) \). Since \( -\text{div} a(x, u, \nabla u) \) belongs to \( W^{-1}L^1(\Omega) \), we also have
\[
\langle -\text{div} a(x, u, \nabla u)h(u), \varphi \rangle_{\mathcal{D}'(\Omega) \cap \mathcal{D}(\Omega)} = \langle -\text{div} a(x, u, \nabla u), \varphi h(u) \rangle_{W^{-1}L^1(\Omega), W^1_0L_M(\Omega)} \quad \forall \varphi \in \mathcal{D}(\Omega).
\]
Finally, since $\Phi_h$ and $\Phi'_h \in (C^0(\mathbb{R}))^N$, for any measurable function $u$ we have $\Phi(u)h(u)$ and $\Phi(u)h'(u) \in (L^\infty(\Omega))^N$ and then $\text{div} (\Phi(u)h(u)) \in W^{-1,\infty}(\Omega)$ and $\Phi(u)h'(u) \in L_M(\Omega)$.

Our main result is the following

**Theorem 3.3.** Suppose that assumptions (3.1)-(3.7) are fulfilled. Then, problem (1.1) has at least one renormalized solution.

**Remark 3.4.** The condition (3.4) can be replaced by the weaker one

$$|g(x, s, \xi)| \leq d(x) + b(|s|)M(|\xi|),$$

with $b : \mathbb{R} \to \mathbb{R}^+$ a continuous function belonging to $L^1(\mathbb{R})$ and $d(x) \in L^1(\Omega)$.

Actually the original equation (1.1) will be recovered whenever $h(u) \equiv 1$, but unfortunately this cannot happen in general strong additional requirements on $u$. Therefore, (3.8) is to be viewed as a weaker form of (1.1).

4. **Proof of the Main Result**

From now on, we will use the standard truncation function $T_k$, $k > 0$, defined for all $s \in \mathbb{R}$ by $T_k(s) = \max\{-k, \min\{k, s\}\}$.

**Step 1: Approximate problems.** Let $f_n$ be a sequence of $L^\infty(\Omega)$ functions that converge strongly to $f$ in $L^1(\Omega)$. For $n \in \mathbb{N}$, $n \geq 1$, let us consider the following sequence of approximate equations

$$-\text{div} a(x, u_n, \nabla u_n) + \text{div} \Phi_n(u_n) + g_n(x, u_n, \nabla u_n) = f_n - \text{div} F \text{ in } \mathcal{D}'(\Omega),$$

(4.1)

where we have set $\Phi_n(s) = \Phi(T_n(s))$ and $g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{n} |g(x, s, \xi)|}$. For fixed $n > 0$, it’s obvious to observe that

$$g_n(x, s, \xi) \geq 0, \quad |g_n(x, s, \xi)| \leq |g(x, s, \xi)| \text{ and } |g_n(x, s, \xi)| \leq n.$$

Moreover, since $\Phi$ is continuous one has $|\Phi_n(t)| \leq \max_{|t| \leq n} |\Phi(t)|$. Therefore, applying both Proposition 1, Proposition 5 and Remark 2 of [22] one can deduces that there exists at least one solution $u_n$ of the approximate Dirichlet problem (4.1) in the sense

$$\begin{align*}
  u_n &\in W^1_0 L_M(\Omega), a(x, u_n, \nabla u_n) \in (L^\infty(\Omega))^N \quad \text{and} \\
  \int_\Omega a(x, u_n, \nabla u_n) \nabla v dx + \int_\Omega \Phi_n(u_n) \nabla v dx \\
  + \int_\Omega g_n(x, u_n, \nabla u_n) v dx = \langle f_n, v \rangle + \int_\Omega F \nabla v dx, \quad \text{for every } v \in W^1_0 L_M(\Omega).
\end{align*}
$$

(4.2)
Step 2: Estimation in $W^1_0 L_M(\Omega)$. Taking $u_n$ as function test in problem (4.2), we obtain

$$
\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \, dx + \int_\Omega \Phi_n(u_n) \nabla u_n \, dx
+ \int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx = (f_n, u_n) + \int_\Omega F \nabla u_n \, dx.
$$

(4.3)

Define $\tilde{\Phi}_n \in (C^1(\mathbb{R}))^N$ as $\tilde{\Phi}_n(t) = \int_0^t \Phi_n(\tau) \, d\tau$. Then formally

$$
\text{div}(\tilde{\Phi}_n(u_n)) = \Phi_n(u_n) \nabla u_n, \quad u_n = 0 \text{ on } \partial \Omega,
\tilde{\Phi}_n(0) = 0 \text{ and by the Divergence theorem}
$$

$$
\int_\Omega \Phi_n(u_n) \nabla u_n \, dx = \int_\Omega \text{div}(\tilde{\Phi}_n(u_n)) \, dx = \int_{\partial \Omega} \tilde{\Phi}_n(u_n) \, \vec{n} \, ds = 0,
$$

where $\vec{n}$ is the outward pointing unit normal field of the boundary $\partial \Omega$ ($ds$ may be used as a shorthand for $\vec{n} \, ds$). Thus, by virtue of (3.5) and Young’s inequality, we get

$$
\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq C_1 + \frac{\alpha}{2} \int_\Omega M(|\nabla u_n|) \, dx,
$$

(4.4)

which, together with (3.3) give

$$
\int_\Omega M(|\nabla u_n|) \, dx \leq C_2.
$$

(4.5)

Moreover, we also have

$$
\int_\Omega g_n(x, u_n, \nabla u_n) u_n \, dx \leq C_3.
$$

(4.6)

As a consequence of (4.5) there exist a subsequence of $\{u_n\}_n$, still indexed by $n$, and a function $u \in W^1_0 L_M(\Omega)$ such that

$$
u_n \rightharpoonup u \text{ weakly in } W^1_0 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_M(\Omega)),
u_n \rightarrow u \text{ strongly in } E_M(\Omega) \text{ and a.e. in } \Omega.
$$

(4.7)

Step 3: Boundedness of $(a(x, u_n, \nabla u_n))_n$ in $(L_M(\Omega))^N$. Let $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Thanks to (3.2), we can write

$$
(a(x, u_n, \nabla u_n) - (a(x, u_n, \frac{w}{k_4}))(\nabla u_n - \frac{w}{k_4}) \geq 0,
$$

which implies

$$
\frac{1}{k_4} \int_\Omega a(x, u_n, \nabla u_n) w \, dx \leq \int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \, dx
+ \int_\Omega a \left( x, u_n, \frac{w}{k_4} \right) \left( \frac{w}{k_4} - \nabla u_n \right) \, dx.
$$

Thanks to (4.4) and (4.5), one has

$$
\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq C_5.
$$
Define $\lambda = 1 + k_1 + k_3$. By the growth condition (3.1) and Young’s inequality, one can write

$$
\left| \int_{\Omega} a(x, u_n, \frac{w}{k_4}) \left( \frac{w}{k_4} \nabla u_n \right) dx \right| \\
\leq \left( 1 + \frac{1}{k_4} \right) \left( \int_{\Omega} M(c(x)) dx + k_1 \int_{\Omega} M^{-1}(k_2 |u_n|) dx \\
+ k_3 \int_{\Omega} M(|w|) dx \right) + \frac{\lambda}{k_4} \int_{\Omega} M(|w|) dx + \lambda \int_{\Omega} M(|\nabla u_n|) dx.
$$

By virtue of [18] and Lemma 4.14 of [20], there exists an $N$-function $Q$ such that $M \ll Q$ and the space $W_0^1 L_M(\Omega)$ is continuously embedded into $L_Q(\Omega)$. Thus, by (4.5) there exists a constant $c_0 > 0$, not depending on $n$, satisfying $\|u_n\|_Q \leq c_0$. Since $M \ll Q$, we can write $M(k_2 t) \leq Q \left( \frac{t}{c_0} \right)$, for $t > 0$ large enough. As $P \ll M$, we can find a constant $c_1$, not depending on $n$, such that

$$
\int_{\Omega} M^{-1}(k_2 |u_n|) dx \leq \int_{\Omega} Q \left( \frac{|u_n|}{c_0} \right) + c_1.
$$

Hence, we conclude that the quantity $\left| \int_{\Omega} a(x, u_n, \nabla u_n w) dx \right|$ is bounded from above for all $w \in (E_M(\Omega))^N$ with $\|w\|_M \leq 1$. Using the Orlicz norm we deduce that

$$
\left( a(x, u_n, \nabla u_n) \right)_n \text{ is bounded in } (L_{\overline{M}}(\Omega))^N.
$$

Step 4: Renormalization identity for the approximate solutions. For any $m \geq 1$, define $\theta_n(m) = e^{-1} \theta_m(r) - T_m(r)$. Observe that by [19, Lemma 2] one has $\theta_m(u_n) \in W_0^1 L_M(\Omega)$. The use of $\theta_m(u_n)$ as test function in (4.2) yields

$$
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n dx.
$$

By Hölder’s inequality and 4.5 we have

$$
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx \leq \langle f_n, \theta_m(u_n) \rangle \\
+ C_0 \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx.
$$

It’s not hard to see that

$$
\|\nabla \theta_m(u_n)\|_M \leq \|\nabla u_n\|_M.
$$

So that by (4.5) and (4.7) one can deduce that

$$
\theta_m(u_n) \rightharpoonup \theta_m(u) \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega)).
$$

Note that as $m$ goes to $\infty$, $\theta_m(u) \to 0$ weakly in $W_0^1 L_M(\Omega)$ for $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$, and since $f_n$ converges strongly in $L^1(\Omega)$, by Lebesgue’s theorem we have

$$
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx = \lim_{m \to \infty} \lim_{n \to \infty} \langle f_n, \theta_m(u_n) \rangle = 0.
$$
By (3.3) we finally have
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx = 0. \tag{4.9}
\]

**Step 5: Almost everywhere convergence of the gradients.** Define
\[
\phi(s) = se^{\lambda s^2} \text{ with } \lambda = \left(\frac{b(k)}{2\alpha}\right)^2. \quad \text{One can easily verify that for all } s \in \mathbb{R} \quad \phi'(s) - \frac{b(k)}{\alpha} |\phi(s)| \geq \frac{1}{2}. \tag{4.10}
\]

For \( m \geq k \), we define the function \( \psi_m \) by
\[
\psi_m(s) = \begin{cases} 1 & \text{if } |s| \leq m, \\ m + 1 - |s| & \text{if } m \leq |s| \leq m + 1, \\ 0 & \text{if } |s| \geq m + 1. \end{cases}
\]

By virtue of [21, Theorem 4] there exists a sequence \( \{v_j\}_j \subset D(\Omega) \) such that \( v_j \to u \) in \( W^{1}_0L_M(\Omega) \) for the modular convergence and a.e. in \( \Omega \). Let us define the following functions \( \theta^j_n = T_k(u_n) - T_k(v_j), \theta^j = T_k(u) - T_k(v_j) \) and \( z^j_{n,m} = \phi(\theta^j_n)\psi_m(u_n) \). Using \( z^j_{n,m} \in W^{1}_0L_M(\Omega) \) as test function in (4.2) we get
\[
\begin{align*}
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z^j_{n,m} \, dx + & \int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) \, dx \\
+ & \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \psi'_m(u_n) \phi(T_k(u_n) - T_k(v_j)) \, dx \\
+ & \int_{\Omega} g_n(x, u_n, \nabla u_n) z^j_{n,m} \, dx = \int_{\Omega} f_n z^j_{n,m} \, dx + \int_{\Omega} F \nabla z^j_{n,m} \, dx. \tag{4.11}
\end{align*}
\]

From now on we denote by \( \epsilon_i(n,j), i = 0, 1, 2, ..., \) various sequences of real numbers which tend to zero, when \( n \) and \( j \to +\infty \), i.e.
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \epsilon_i(n,j) = 0.
\]

In view of (4.7), we have \( z^j_{n,m} \to \phi(\theta^j)\psi_m(u) \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) as \( n \to +\infty \), which yields
\[
\lim_{n \to +\infty} \int_{\Omega} f_n z^j_{n,m} \, dx = \int_{\Omega} f \phi(\theta^j) \psi_m(u) \, dx,
\]
and since \( \phi(\theta^j) \to 0 \) weakly in \( L^\infty(\Omega) \) for \( \sigma(L^\infty, L^1) \) as \( j \to +\infty \), we have
\[
\lim_{j \to +\infty} \int_{\Omega} f \phi(\theta^j) \psi_m(u) \, dx = 0.
\]
Thus, we write
\[
\int_{\Omega} f_n z^j_{n,m} \, dx = \epsilon_0(n,j).
\]

Thanks to (4.5) and (4.7), we have as \( n \to +\infty \),
\[
z^j_{n,m} \to \phi(\theta^j)\psi_m(u) \text{ in } W^{1}_0L_M(\Omega) \text{ for } \sigma(\Pi L_M(\Omega), \Pi E_M(\Omega)).
\]
which implies that
\[
\lim_{n \to +\infty} \int_\Omega F \nabla z_{n,m}^j dx = \int_\Omega F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx + \int_\Omega F \nabla u \phi'(\theta^j) \psi_m'(u) dx
\]
On the one hand, by Lebesgue’s theorem we get
\[
\lim_{j \to +\infty} \int_\Omega F \nabla u \phi'(\theta^j) \psi_m'(u) dx = 0,
\]
on the other hand, we write
\[
\int_\Omega F \nabla \theta^j \phi'(\theta^j) \psi_m(u) dx = \int_\Omega F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx - \int_\Omega F \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) dx,
\]
so that, by Lebesgue’s theorem one has
\[
\lim_{j \to +\infty} \int_\Omega F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx = \int_\Omega F \nabla T_k(u) \psi_m(u) dx.
\]
Let \( \lambda > 0 \) such that \( M\left(\frac{|\nabla v_j - \nabla u|}{\lambda}\right) \to 0 \) strongly in \( L^1(\Omega) \) as \( j \to +\infty \) and \( M\left(\frac{|\nabla u|}{\lambda}\right) \in L^1(\Omega) \), the convexity of the \( N \)-function \( M \) allows us to have
\[
M\left(\frac{|\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) - \nabla T_k(u) \psi_m(u)|}{1 + \sigma(2k)}\right) = M\frac{1}{\lambda} M\left(\frac{|\nabla v_j - \nabla u|}{\lambda}\right) + \frac{1}{4} M\left(\frac{|\nabla u|}{\lambda}\right).
\]
Then, by using the modular convergence of \( \{\nabla v_j\} \) in \( (L_M(\Omega))^N \) and Vitali’s theorem, we obtain
\[
\nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) \to \nabla T_k(u) \psi_m(u) \text{ in } (L_M(\Omega))^N,
\]
for the modular convergence, and then
\[
\lim_{j \to +\infty} \int_\Omega F \nabla T_k(u) \phi'(\theta^j) \psi_m(u) dx = \int_\Omega F \nabla T_k(u) \psi_m(u) dx.
\]
We have proved that
\[
\int_\Omega F \nabla z_{n,m}^j dx = \epsilon_1(n,j).
\]
It’s easy to see that by the modular convergence of the sequence \( \{v_j\} \), one has
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \psi_m'(u_n) \phi(T_k(u_n) - T_k(v_j)) dx = 0,
\]
while for the third term in the left-hand side of (4.11) we can write
\[
\int_\Omega \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) dx
\]
\[
= \int_\Omega \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) dx - \int_\Omega \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) dx.
\]
Firstly, we have
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(u_n) \phi'(\theta_n^j) \psi_m(u_n) \, dx = 0.
\]
In view of (4.7), one has
\[
\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n) \to \Phi(u) \phi'(\theta^j) \psi_m(u)
\]
almost everywhere in \( \Omega \) as \( n \) tends to \(+\infty\). Furthermore, we can check that
\[
\|\Phi_n(u_n) \phi'(\theta_n^j) \psi_m(u_n)\|_{TV} \leq M \epsilon_m \phi'(2k) |\Omega| + 1,
\]
where \( c_m = \max_{t \leq m+1} \Phi(t) \). Applying [27, Theorem 14.6] we get
\[
\lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx = \int_{\Omega} \Phi(u) \nabla T_k(v_j) \phi'(\theta^j) \psi_m(u) \, dx.
\]
Using the modular convergence of the sequence \( \{v_j\}_j \), we obtain
\[
\lim_{j \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx = \int_{\Omega} \Phi(u) \nabla T_k(u) \phi'(\theta^j) \psi_m(u) \, dx.
\]
Then, using again the Divergence theorem we get
\[
\int_{\Omega} \Phi(u) \nabla T_k(u) \phi'(\theta^j) \psi_m(u) \, dx = 0.
\]
Therefore, we write
\[
\int_{\Omega} \Phi_n(u_n) \nabla \phi(T_k(u_n) - T_k(v_j)) \psi_m(u_n) \, dx = \epsilon_2(n, j).
\]
Since \( g_n(x, u_n, \nabla u_n) z_{n,m}^j \geq 0 \) on the set \( \{ u_n > k \} \) and \( \psi_m(u_n) = 1 \) on the set \( \{ u_n \leq k \} \), from (4.11) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j \, dx + \int_{\{ u_n \leq k \}} g_n(x, u_n, \nabla u_n) \phi(\theta_n^j) \, dx \leq \epsilon_3(n, j) \quad (4.12)
\]
We now evaluate the first term of the left-hand side of (4.12) by writing
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j \, dx
\]
\[
= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) \psi_m(u_n) \, dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) \, dx
\]
\[
= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)) \phi'(\theta_n^j) \psi_m(u_n) \, dx
\]
\[
- \int_{\{ u_n > k \}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta_n^j) \psi_m(u_n) \, dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi_m(u_n) \, dx.
\]
and then
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m}^j \, dx = \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j^s)) \right)
\]
\[
\quad \left( \nabla T_k(u_n) - \nabla T_k(v_j^s) \right) \phi'(\theta_n^j) \, dx
\]
\[
+ \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(v_j^s)) \left( \nabla T_k(u_n) - \nabla T_k(v_j^s) \right) \phi'(\theta_n^j) \right) \, dx
\]
\[
- \int_{\Omega \setminus \Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j^s) \phi'(\theta_n^j) \, dx
\]
\[
- \int_{\{u_n > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j^s) \phi'(\theta_n^j) \psi_m(u_n) \, dx
\]
\[
+ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n^j) \psi'_m(u_n) \, dx,
\]
(4.13)

where by \( \chi_j^s, s > 0 \), we denote the characteristic function of the subset
\[
\Omega_j^s = \{ x \in \Omega : |\nabla T_k(v_j^s)| \leq s \}.
\]

For fixed \( m \) and \( s \), we will pass to the limit in \( n \) and then in \( j \) in the second, third, fourth and fifth terms in the right side of (4.13). Starting with the second term, we have
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j^s)) \left( \nabla T_k(u_n) - \nabla T_k(v_j^s) \right) \phi'(\theta_n^j) \, dx
\]
\[
\rightarrow \int_{\Omega} a(x, T_k(u), \nabla T_k(v_j^s)) \left( \nabla T_k(u) - \nabla T_k(v_j^s) \right) \phi'(\theta^j) \, dx,
\]
as \( n \rightarrow +\infty \). Since by lemma (2.4) one has
\[
a(x, T_k(u_n), \nabla T_k(v_j^s) \phi'(\theta_n^j) \rightarrow a(x, T_k(u), \nabla T_k(v_j^s) \phi'(\theta^j),
\]
strongly in \((E_{\infty}(\Omega))^N\) as \( n \rightarrow \infty \), while by (4.5)
\[
\nabla T_k(u_n) \rightarrow \nabla T_k(u),
\]
weakly in \((L_M(\Omega))^N\). Let \( \chi^s \) denote the characteristic function of the subset
\[
\Omega^s = \{ x \in \Omega : |\nabla T_k(u)| \leq s \}.
\]
As \( \nabla T_k(v_j^s) \rightarrow \nabla T_k(u) \chi^s \) strongly in \((E_M(\Omega))^N\) as \( j \rightarrow +\infty \), one has
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(v_j^s)) \cdot \left( \nabla T_k(u) - \nabla T_k(v_j^s) \phi'(\theta^j) \right) \, dx \rightarrow 0,
\]
as \( j \rightarrow \infty \). Then
\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j^s)) \left( \nabla T_k(u_n) - \nabla T_k(v_j^s) \phi'(\theta_n^j) \right) \, dx = \epsilon_4(n, j). \quad (4.14)
\]
We now estimate the third term of (4.13). It’s easy to see that by (3.3),
\[
a(x, s, 0) = 0 \quad \text{for almost everywhere} \ x \in \Omega \quad \text{and for all} \ s \in \mathbb{R}.
\]
Thus, from (4.8) we have that \( \left( a(x, T_k(u_n), \nabla T_k(u_n)) \right)_n \) is bounded in \((E_{\infty}(\Omega))^N\) for all \( k \geq 0 \).
Therefore, there exist a subsequence still indexed by $n$ and a function $l_k$ in $(L^M(\Omega))^N$ such that
\[ a(x, T_k(u_n), \nabla T_k(u_n)) \to l_k \text{ weakly in } (L^M(\Omega))^N \text{ for } \sigma(HL^M, HE_M). \] (4.15)

Then, since $\nabla T_k(v_j) \chi_{\Omega \setminus \Omega^j} \in (E^M(\Omega))^N$, we obtain
\[ \int_{\Omega \setminus \Omega^j} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi'(\theta^j_0) dx \to \int_{\Omega \setminus \Omega^j} l_k \nabla T_k(v_j) \phi'(\theta^j) dx, \]
as $n \to +\infty$. The modular convergence of $\{v_j\}$ allows us to get
\[ -\int_{\Omega \setminus \Omega^j} l_k \nabla T_k(v_j) \phi'(\theta^j) dx \to -\int_{\Omega \setminus \Omega^j} l_k \nabla T_k(u) dx, \]
as $j \to +\infty$. This proves (4.14).

As regards the fourth term, observe that $\psi_m(u_n) = 0$ on the subset $\{|u_n| \geq m + 1\}$, so we have
\[ -\int_{\{u_n > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \psi_m(u_n) dx = -\int_{\{u_n > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta^j_n) \psi_m(u_n) dx. \]

Since
\[ -\int_{\{u_n > k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_k(v_j) \phi'(\theta^j_n) \psi_m(u_n) dx = -\int_{\{u > k\}} l_{m+1} \nabla T_k(u) \psi_m(u) dx + \epsilon_5(n, j), \]
observing that $\nabla T_k(u) = 0$ on the subset $\{|u| > k\}$, one has
\[ -\int_{\{u_n > k\}} a(x, u_n, \nabla u_n) \nabla T_k(v_j) \phi'(\theta^j_n) \psi_m(u_n) dx = \epsilon_6(n, j). \] (4.17)

For the last term of (4.13), we have
\[
\left| \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^j_n) \psi'_m(u_n) dx \right| \\
= \left| \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta^j_n) \psi'_m(u_n) dx \right| \\
\leq \phi(2k) \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx.
\]
To estimate the last term of the previous inequality, we use
\((T_1(u_n - T_m(u_n)) \in W_0^1 L^1(\Omega))\) as test function in (4.2), to get
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
+ \int_{\{u_n \geq m\}} g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \, dx = \langle f_n, T_1(u_n - T_m(u_n)) \rangle
+ \int_{\{m \leq |u_n| \leq m+1\}} F \nabla u_n \, dx.
\]
By Divergence theorem, we have
\[
\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n \, dx = 0.
\]
Using the fact that \(g_n(x, u_n, \nabla u_n) T_1(u_n - T_m(u_n)) \geq 0\) on the subset \(\{u_n \geq m\}\) and Young's inequality, we get
\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
\leq \langle f_n, T_1(u_n - T_m(u_n)) \rangle + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx.
\]
It follows that
\[
\left| \int_{\{u_n \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n \phi(\theta_n') \phi'(\theta_n') \, dx \right|
\leq 2\phi(2k) \left( \int_{\{m \leq |u_n| \leq m+1\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right). \tag{4.18}
\]
From (4.14), (4.16), (4.17) and (4.18) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla z_{n,m} \, dx
\geq \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi^j_n) \right)
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi^j_n \phi(\theta_n') \phi'(\theta_n') \, dx \right)
- \alpha \phi(2k) \left( \int_{\{m \leq |u_n| \leq m+1\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right)
- \int_{\Omega \setminus \Omega^*} l_k \cdot \nabla T_k(u) \, dx + c_\gamma(n, j). \tag{4.19}
\]
Now, we turn to second term in the left-hand side of (4.12). We have
\[
\left| \int_{\{u_n \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_n') \, dx \right|
= \left| \int_{\{u_n \leq k\}} g_n(x, T_k(u_n), \nabla T_k(u_n)) \phi(\theta_n') \, dx \right|
\leq b(k) \int_{\Omega} M(|\nabla T_k(u_n)|) \phi(\theta_n') \, dx + b(k) \int_{\Omega} d(x) \phi(\theta_n') \, dx
\leq \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \phi(\theta_n') \, dx + c_\delta(n, j).
\]
Then
\[
\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_j^i) dx \right| \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(v_j), \nabla T_k(v_j)) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \phi(\theta_j^i) \right) dx \\
+ \frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j) \phi(\theta_j^i)) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \phi(\theta_j^i) \right) dx \\
+ \frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi(\theta_j^i)) dx + \epsilon_9(n, j).
\]

(4.22)

We proceed as above to get
\[
\frac{b(k)}{\alpha} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \phi(\theta_j^i)) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \phi(\theta_j^i) \right) dx = \epsilon_9(n, j)
\]
and
\[
\frac{b(k)}{\alpha} \int_{\Omega} a_n(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \phi(\theta_j^i)) dx = \epsilon_{10}(n, j).
\]

Hence, we have
\[
\left| \int_{\{|u_n| \leq k\}} g_n(x, u_n, \nabla u_n) \phi(\theta_j^i) dx \right| \\
\leq \frac{b(k)}{\alpha} \int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(v_j), \nabla T_k(v_j)) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \phi(\theta_j^i) \right) dx + \epsilon_{11}(n, j).
\]

(4.21)

Combining (4.12), (4.19) and (4.21), we get
\[
\int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(v_j), \nabla T_k(v_j)) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \phi(\theta_j^i) \right) dx \\
\leq \int_{\Omega} l_k \nabla T_k(u) dx + a\phi(2k) \left( \int_{\{|u_n| \leq k\}} |f_n| dx + \int_{\{|m \leq |u_n| \leq m+1\}} M(|F|) dx \right) \\
+ \epsilon_{12}(n, j).
\]

By (4.10), we have
\[
\int_{\Omega} \left( a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(v_j), \nabla T_k(v_j)) \right) \\
\left( \nabla T_k(u_n) - \nabla T_k(v_j) \phi(\theta_j^i) \right) dx \\
\leq 2 \int_{\Omega} l_k \nabla T_k(u) dx + 4a\phi(2k) \left( \int_{\{|u_n| \leq k\}} |f_n| dx + \int_{\{|m \leq |u_n| \leq m+1\}} M(|F|) dx \right) \\
+ \epsilon_{12}(n, j).
\]

(4.22)
On the other hand we can write

\[
\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) \left( \nabla T_k(u_n) - \nabla T_k(u)\chi^s \right) \, dx \\
= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi^s)) \left( \nabla T_k(u_n) - \nabla T_k(v_j)\chi^s \right) \, dx \\
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \chi^s \left( \nabla T_k(u_n) - \nabla T_k(u)\chi^s \right) \, dx \\
- \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) \left( \nabla T_k(u_n) - \nabla T_k(u)\chi^s \right) \, dx \\
+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j)\chi^s \right) \, dx
\]

We shall pass to the limit in \( n \) and then in \( j \) in the last three terms of the right hand side of the above equality. In a similar way as done in (4.13) and (4.20), we obtain

\[
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \chi^s - \nabla T_k(u)\chi^s \, dx = \epsilon_{13}(n,j), \\
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)\chi^s) \left( \nabla T_k(u_n) - \nabla T_k(u)\chi^s \right) \, dx = \epsilon_{14}(n,j), \\
\int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)\chi^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j)\chi^s \right) \, dx = \epsilon_{15}(n,j).
\]

So that

\[
\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) \left( \nabla T_k(u_n) - \nabla T_k(u)\chi^s \right) \, dx \\
= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi^s)) \left( \nabla T_k(u_n) - \nabla T_k(v_j)\chi^s \right) \, dx \\
+ \epsilon_{16}(n,j).
\]

Let \( r \leq s \). Using (3.2), (4.22) and (4.24) we can write

\[
0 \leq \int_{\Omega}^{r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) \, dx \\
\leq \int_{\Omega}^{r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \nabla T_k(u_n) - \nabla T_k(u) \, dx \\
= \int_{\Omega}^{r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) \nabla T_k(u_n) - \nabla T_k(u) \, dx \\
\leq \int_{\Omega}^{r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s)) \nabla T_k(u_n) - \nabla T_k(u) \chi^s \, dx \\
= \int_{\Omega}^{r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi^s)) \nabla T_k(u_n) - \nabla T_k(v_j) \chi^s \, dx \\
+ \epsilon_{15}(n,j) \\
\leq 2 \int_{\Omega} l_k \nabla T_k(u) \, dx + 2\alpha\phi(2k) \left( \int_{m \leq |u_n|} |f_n| \, dx + \int_{m \leq |u_n| \leq m + 1} M(|F|) \, dx \right) \\
+ \epsilon_{17}(n,j).
\]
By passing to the superior limit over $n$ and then over $j$,

$$0 \leq \limsup_{n \to +\infty} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \nabla T_k(u_n) - \nabla T_k(u)) \, dx$$

$$\leq 2 \int_{\Omega} l_k \nabla T_k(u) \, dx + 4\alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right).$$

Letting $s \to +\infty$ and then $m \to +\infty$, taking into account that $l_k \nabla T_k(u) \in L^1(\Omega)$, $f \in L^1(\Omega)$, $|F| \in (E_{\text{TV}}(\Omega))^N$, $|\Omega \setminus \Omega^s| \to 0$, and $\{|m \leq |u| \leq m+1|\} \to 0$, one has

$$\int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \nabla T_k(u_n) - \nabla T_k(u)) \, dx,$$

(4.25)
tends to 0 as $n \to +\infty$. As in [20], we deduce that there exists a subsequence of $\{u_n\}$ still indexed by $n$ such that

$$\nabla u_n \to \nabla u \text{ a. e. in } \Omega.$$  (4.26)

Therefore, having in mind (4.8) and (4.7), we can apply [27, Theorem 14.6] to get

$$a(x, u, \nabla u) \in (L_{\text{TV}}(\Omega))^N$$

and

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L_{\text{TV}}(\Omega))^N \text{ for } \sigma(\Pi L_{\text{TV}}(\Pi E_M)).$$  (4.27)

**Step 6: Modular convergence of the truncations.** Going back to equation (4.22), we can write

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx$$

$$\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^* \, dx$$

$$+ \int_{\Omega} a(x, T_k(u_n), \nabla T_k(v_j)) \chi_j^* ((\nabla T_k(u_n) - \nabla T_k(v_j)) \chi_j^* \, dx$$

$$+ 2\alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right)$$

$$+ 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx + \epsilon_{12}(n, j).$$

By (4.23) we get

$$\int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx$$

$$\leq \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^* \, dx$$

$$+ 2\alpha \phi(2k) \left( \int_{\{m \leq |u_n|\}} |f_n| \, dx + \int_{\{m \leq |u_n| \leq m+1\}} M(|F|) \, dx \right)$$

$$+ 2 \int_{\Omega \setminus \Omega^s} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx + \epsilon_{18}(n, j).$$
We now pass to the superior limit over \( n \) in both sides of this inequality using (4.27), to obtain
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \\
+ 2\alpha \phi(2k) \left( \int_{\{m \leq |u| \}} |f| dx + \int_{\{m \leq |u| \leq m+1 \}} \mathcal{M}(|F|) dx \right) \\
+ 2 \int_{\Omega \setminus \Omega'} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
\]
We then pass to the limit in \( j \) to get
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \\
\leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \\
+ 2\alpha \phi(2k) \left( \int_{\{m \leq |u| \}} |f| dx + \int_{\{m \leq |u| \leq m+1 \}} \mathcal{M}(|F|) dx \right) \\
+ 2 \int_{\Omega \setminus \Omega'} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
\]
Letting \( s \) and then \( m \to +\infty \), one has
\[
\limsup_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
\]
On the other hand, by (3.3), (4.5), (4.26) and Fatou’s lemma, we have
\[
\int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx \leq \liminf_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx.
\]
It follows that
\[
\lim_{n \to +\infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx = \int_{\Omega} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) dx.
\]
By Lemma 2.5 we conclude that for every \( k > 0 \)
\[
a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u),
\]
strongly in \( L^1(\Omega) \). The convexity of the \( N \)-function \( M \) and (3.3) allow us to have
\[
M \left( \frac{\nabla T_k(u_n) - \nabla T_k(u)}{2} \right) \\
\leq \frac{1}{2m} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) + \frac{1}{2m} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u).
\]
From Vitali’s theorem we deduce
\[
\limsup_{|E| \to 0} \int_E M \left( \frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2} \right) dx = 0.
\]
Thus, for every \( k > 0 \)
\[
T_k(u_n) \to T_k(u) \text{ in } W^1_0 L_M(\Omega),
\]
Step 7: Compactness of the nonlinearities. We need to prove that
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \] (4.29)

By virtue of (4.7) and (4.26) one has
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a.e. in } \Omega. \] (4.30)

Let \( E \) be measurable subset of \( \Omega \) and let \( m > 0 \). Using (3.3) and (3.4) we can write
\[
\int_E |g_n(x, u_n, \nabla u_n)| dx \\
= \int_{E \cap \{|u_n| \leq m\}} |g_n(x, u_n, \nabla u_n)| dx + \int_{E \cap \{|u_n| > m\}} |g_n(x, u_n, \nabla u_n)| dx \\
\leq b(m) \int_E d(x) dx + b(m) \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) dx \\
+ \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx.
\]

From (3.5) and (4.6), we deduce that
\[ 0 \leq \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx \leq C_3. \]

So
\[ 0 \leq \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx \leq \frac{C_3}{m}. \]

Then
\[ \lim_{m \to +\infty} \frac{1}{m} \int_{\Omega} g_n(x, u_n, \nabla u_n) u_n dx = 0. \]

Thanks to (4.28) the sequence \( \{a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n)\}_n \) is equi-integrable. This fact allows us to get
\[ \lim_{|E| \to 0} \sup_n \int_E a(x, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) dx = 0. \]

This shows that \( g_n(x, u_n, \nabla u_n) \) is equi-integrable. Thus, Vitali’s theorem implies that \( g(x, u, \nabla u) \in L^1(\Omega) \) and
\[ g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \]

Step 8: Renormalization identity for the solutions. In this step we prove that
\[ \lim_{m \to +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u dx = 0. \] (4.31)
Indeed, for any \( m \geq 0 \) we can write

\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
= \int_{\Omega} a(x, u_n, \nabla u_n) (\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) \, dx
= \int_{\Omega} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_m(u_n) \, dx
- \int_{\Omega} a(x, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) \, dx.
\]

In view of (4.28), we can pass to the limit as \( n \) tends to \(+\infty\) for fixed \( m \geq 0 \)

\[
\lim_{n \to +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx
= \int_{\Omega} a(x, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_m(u) \, dx
- \int_{\Omega} a(x, T_m(u), \nabla T_m(u)) \nabla T_m(u) \, dx
= \int_{\Omega} a(x, u, \nabla u)(\nabla T_{m+1}(u) - \nabla T_m(u)) \, dx
= \int_{\{m \leq |u| \leq m+1\}} a(x, u, \nabla u) \nabla u \, dx.
\]

Having in mind (4.9), we can pass to the limit as \( m \) tends to \(+\infty\) to obtain (4.31).

**Step 9: Passing to the limit.** Thanks to (4.28) and Lemma (2.5), we obtain

\[
a(x, u_n, \nabla u_n) \nabla u_n \to a(x, u, \nabla u) \nabla u \text{ strongly in } L^1(\Omega).
\]

Let \( h \in C^1_c(\mathbb{R}) \) and \( \varphi \in D(\Omega) \). Inserting \( h(u_n)\varphi \) as test function in (4.2), we get

\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi \, dx + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) \, dx
+ \int_{\Omega} \Phi_n(u_n) \nabla (h(u_n) \varphi) \, dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx
= \langle f_n, h(u_n) \varphi \rangle + \int_{\Omega} F \nabla (h(u_n) \varphi) \, dx.
\]

We shall pass to the limit as \( n \to +\infty \) in each term of the equality (4.33). Since \( h \) and \( h' \) have compact support on \( \mathbb{R} \), there exists a real number \( \nu > 0 \), such that \( \text{supp } h \subset [-\nu, \nu] \) and \( \text{supp } h' \subset [-\nu, \nu] \). For \( n > \nu \), we can write

\[
\Phi_n(t) h(t) = \Phi(T_{\nu}(t)) h(t) \text{ and } \Phi_n(t) h'(t) = \Phi(T_{\nu}(t)) h'(t).
\]

Moreover, the functions \( \Phi h \) and \( \Phi h' \) belong to \((C^0(\mathbb{R}) \cap L^\infty(\mathbb{R}))^N\). Observe first that the sequence \( \{h(u_n)\varphi\}_n \) is bounded in \( W^1_0 L_M(\Omega) \). Indeed, let \( \rho > 0 \)
be a positive constant such that \( \| h(u_n) \nabla \varphi \|_{\infty} \leq \rho \) and \( \| h'(u_n) \varphi \|_{\infty} \leq \rho \). Using the convexity of the \( N \)-function \( M \) and taking into account (4.5) we have

\[
\int_\Omega M \left( \frac{|\nabla (h(u_n)\varphi)|}{2\rho} \right) dx \leq \int_\Omega M \left( \frac{|h(u_n)\nabla \varphi| + |h'(u_n)\varphi||\nabla u_n|}{2\rho} \right) dx
\]

\[
\leq \frac{1}{2} M(1)|\Omega| + \frac{1}{2} \int_\Omega M(|\nabla u_n|) dx
\]

\[
\leq \frac{1}{2} M(1)|\Omega| + \frac{1}{2} C_2.
\]

This, together with (4.7), imply that

\[
h(u_n)\varphi \rightharpoonup h(u)\varphi \text{ weakly in } W^{1}_0 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_M).
\]

This enables us to get

\[
\langle f, h(u_n)\varphi \rangle \rightharpoonup \langle f, h(u)\varphi \rangle.
\]

Let \( E \) be a measurable subset of \( \Omega \). Define \( c_\nu = \max_{|\nu| \leq \varphi} \Phi(t) \). Let us denote by \( \| v \|_M \) the Orlicz norm of a function \( v \in L_M(\Omega) \). Using strengthened Hölder inequality with both Orlicz and Luxemborg norms, we get

\[
\| \Phi(T_\nu(u_n)) \chi_E \|_{(\Pi E)} = \sup_{\| v \|_M \leq 1} \left| \int_E \Phi(T_\nu(u_n)) v dx \right|
\]

\[
\leq c_\nu \sup_{\| v \|_M \leq 1} \| \chi_E \|_{(\Pi E)} \| v \|_M
\]

\[
\leq c_\nu |E|M^{-1} \left( \frac{1}{|E|} \right).
\]

Thus, we get

\[
\lim_{|E| \to 0} \sup_n \| \Phi(T_\nu(u_n)) \chi_E \|_{(\Pi E)} = 0.
\]

Therefore, thanks to (4.7) by applying [27, Lemma 11.2] we obtain

\[
\Phi(T_\nu(u_n)) \rightharpoonup \Phi(T_\nu(u)) \text{ strongly in } (E_M)^N,
\]

which jointly with (4.34) allow us to pass to the limit in the third term of (4.33) to have

\[
\int_\Omega \Phi(T_\nu(u_n)) \nabla(h(u_n)\varphi) dx \to \int_\Omega \Phi(T_\nu(u)) \nabla(h(u)\varphi) dx.
\]

We remark that

\[
|a(x, u_n, \nabla u_n) \nabla u_n h'(u_n)\varphi| \leq \rho a(x, u_n, \nabla u_n) \nabla u_n.
\]

Consequently, using (4.32) and Vitali’s theorem, we obtain

\[
\int_\Omega a(x, u_n, \nabla u_n) \nabla u_n h'(u_n) \varphi dx \to \int_\Omega a(x, u, \nabla u) \nabla uh'(u) \varphi dx.
\]

and

\[
\int_\Omega F \nabla u_n h'(u_n) \varphi dx \to \int_\Omega F \nabla uh'(u) \varphi dx.
\]

For the second term of (4.33), as above we have

\[
h(u_n) \nabla \varphi \rightharpoonup h(u) \nabla \varphi \text{ strongly in } (E_M(\Omega))^N,
\]
which together with (4.27) give
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla \varphi h(u_n) \, dx \to \int_{\Omega} a(x, u, \nabla u) \nabla \varphi h(u) \, dx
\]
and
\[
\int_{\Omega} F \nabla \varphi h(u_n) \, dx \to \int_{\Omega} F \nabla \varphi h(u) \, dx.
\]
The fact that \( h(u_n) \varphi \rightharpoonup h(u) \varphi \) weakly in \( L^\infty(\Omega) \) for \( \sigma^*(L^\infty, L^1) \) and (4.29) enable us to pass to the limit in the fourth term of (4.33) to get
\[
\int_{\Omega} g_n(x, u_n, \nabla u_n) h(u_n) \varphi \, dx \to \int_{\Omega} g(x, u, \nabla u) h(u) \varphi \, dx.
\]
At this point we can pass to the limit in each term of (4.33) to get
\[
\int_{\Omega} a(x, u, \nabla u) (\nabla \varphi h(u) + h'(u) \varphi \nabla u) \, dx + \int_{\Omega} \Phi(u) h'(u) \varphi \, dx
\]
\[
+ \int_{\Omega} \Phi(u) h(u) \nabla \varphi \, dx + \int_{\Omega} g(x, u, \nabla u) h(u) \varphi \, dx
\]
\[
= (f, h(u) \varphi) + \int_{\Omega} F(\nabla \varphi h(u) + h'(u) \varphi \nabla u) \, dx,
\]
for all \( h \in C^1_c(\mathbb{R}) \) and for all \( \varphi \in \mathcal{D}(\Omega) \). Moreover, as we have (3.5), (4.6) and (4.30) we can use Fatou’s lemma to get \( g(x, u, \nabla u) u \in L^1(\Omega) \). By virtue of (4.7), (4.27), (4.29), (4.31), the function \( u \) is a renormalized solution of problem (1.1).

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