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Extended Jacobi and Laguerre Functions and Their Applications

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Abstract. The aim of this paper is to introduce two new extensions of the Jacobi and Laguerre polynomials as the eigenfunctions of two non-classical Sturm-Liouville problems. We prove some important properties of these operators such as: These sets of functions are orthogonal with respect to a positive definite inner product defined over the compact intervals $[-1, 1]$ and $[0, \infty)$, respectively and also these sequences form two new orthogonal bases for the corresponding Hilbert spaces. Finally, the spectral and Rayleigh-Ritz methods are carry out using these basis functions to solve some examples. Our numerical results are compared with other existing results to confirm the efficiency and accuracy of our method.

Keywords: Sturm-Liouville theory, Orthogonal polynomials, Ordinary differential equations, Non-classical Sturm-Liouville problems, Spectral method, Collocation method, Galerkin method, Rayleigh-Ritz method.

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1. Introduction

Sturm-Liouville theory is a study of the linear differential equations under appropriate boundary conditions whose solutions are complete orthogonal sets of functions in L^2. According to the Bochner's theorem [4], it was believed that among the sequences of orthogonal polynomials, only the classical orthogonal polynomials are the solutions of second order differential equations, however in 2009 exceptional orthogonal polynomials ($X_{i\ (i\geq 1)}$) was presented by Gomez-Ullate et al. [10] in the framework of Sturm-Liouville theory. Although these sequences of orthogonal polynomials start with a polynomial of degree one instead of degree zero, avoiding the restrictions of the Bochner's theorem, they satisfy a second order differential equations. They explicitly obtained the X_1-Jacobi and X_1-Laguerre polynomials. In 2009 Odak and Sasaki [17] derived two sets of infinitely exceptional (X_i) Laguerre and Jacobi polynomials which satisfy a second order differential equations in the framework of quantum mechanics and shape invariant potentials. Authors of [18], obtained another set of X_i-Laguerre polynomials. We refer the interested readers to [18] for detailed information. In the present paper, we introduce orthogonal functions that are solutions of a non-classical Sturm-Liouville problem. So, the paper shows that not only there exist some sequences of orthogonal polynomials which satisfy the Sturm-Liouville problems but also there are a lot of sequences of orthogonal functions which satisfy such problems. Also we provide some applications of extended functions by solving some problems such as Lane-Emden type equation [21], Thomas-Fermi equation [15], weakly singular integral equation, an ordinary differential equation [30] and a fractional calculus of variation problem [29].

This paper is organized as follows: Section 2 consists of a brief review of some useful definitions and theorems. In Section 3 we introduce extended Jacobi and Laguerre functions which depend on a function $g(x)$. In Section 5 we briefly discuss about the effect of the function $g(x)$ on our numerical results. For this purpose, we provide some ordinary and fractional examples.

2. Preliminaries

In this section, we provide some important definitions and theorems which will be used in the next sections.

2.1. Self-adjoint operators.

Definition 2.1. [11] Let X be a linear space with an inner product $\langle .,. \rangle$, then X is called an inner product space.

Definition 2.2. [1] Let A be a linear operator on the inner product space X, the operator A', if it exists, is adjoint to A if

$$\langle Ax, y \rangle = \langle x, A'y \rangle \,, \ \ \forall x, y \in X.$$

Then A is a self-adjoint operator if $A' = A$.

Theorem 2.3. [1] *Let* $L : \mathcal{L}^2((a,b),\omega) \cap C^2(a,b) \to \mathcal{L}^2((a,b),\omega)$ *be a linear differential operator defined by*

$$Lu = p(x)y'' + q(x)y' + r(x)y, \quad x \in (a,b),$$

where $p \in C^2(a,b)$, $q \in C^1(a,b)$, *and* $r \in C(a,b)$. *Then*

- L *is self-adjoint, if the coefficients* p, q, *and* r *are real functions and* $q(x) = p'(x)$, *for all* $x \in (a,b)$, *and*

$$p(f'\overline{g} - f\overline{g}')|_a^b = 0,$$

  for all $f, g \in C^2(a,b) \cap \mathcal{L}^2((a,b),\omega)$. *The term* $f(x)|_a^b$ *is to be defined as* $f(b) - f(a)$.
- *If* L *is a self-adjoint operator, then the eigenvalues of the equation* $Ly + \lambda y = 0$ *are all real and any pair of eigenfunctions associated with distinct eigenvalues are orthogonal in* $\mathcal{L}^2((a,b),\omega)$.

2.2. **Spectral methods.** Spectral methods belong to the class of weighted residual methods [24]. Consider the following initial-boundary value problem:

$$\begin{cases} LU = f, & x \in I, \\ BU = 0, & x \in \{a,b\}, \end{cases} \tag{2.1}$$

where L is a differential operator and B is a linear boundary operator and $f(x,t)$ is given function. Generally, in the weighted residual methods the approximate solution is considered as follows:

$$u(x) \simeq u_N(x) = \sum_{i=0}^{N} u_{N,i}\phi_i(x), \tag{2.2}$$

where the trial functions $\phi_i(x)$, $0 \le i \le N$, are linearly independent. We define the residual function $R_N(x)$ as:

$$R_N(x) = Lu_N(x) - f(x) \neq 0, \quad x \in I.$$

Since the method attempts to minimize $R_N(x)$, coefficients $\{u_{N,i}\}_{i=0}^N$ must be obtained by solving the following system:

$$(R_N, \psi_j)_\omega := \int_I R_N(x)\psi_j(x)\omega(x)dx = 0, \quad 0 \le j \le N, \tag{2.3}$$

where $\{\psi_j\}$ are the test functions, and ω is a positive weight function.

Galerkin and collocation methods belong to the class of the spectral methods. The main difference between Galerkin and collocation methods is the choice of the test functions. In the Galerkin method the test functions and trial functions are the same as each other. In the collocation method the test functions are defined by:

$$\psi_j(x) = \delta(x - x_j), \quad 0 \le j \le N,$$

where $\delta(x)$ is Dirac delta function. The points $\{x_j\}_{j=0}^{N}$ are called the collocation points.

3. Extended Jacobi and Laguerre Functions

This section is devoted to our main results. Two new classes of orthogonal functions in both $[-1,1]$ and $[0,\infty)$ are introduced and some important properties of them are provided. To do so and for the reader's convenience we split this section into two subsections.

3.1. **Extended Jacobi functions.** Consider the functions

$$u_0 = g(x), u_i = g(x)x^i, \quad i \geq 1, \tag{3.1}$$

where $g(x) \neq 0$ for all $x \in [-1,1]$. We define the following measure

$$d\hat{\mu}_{\alpha,\beta} = \hat{\omega}_{\alpha,\beta}dx, \quad \alpha, \beta \geq -1,$$

$$\hat{\omega}_{\alpha,\beta} = \frac{(1-x)^{\alpha}(1+x)^{\beta}}{(g(x))^2},$$

and observe that $\hat{\omega}_{\alpha,\beta} > 0$ for $x \in (-1,1)$ so the scalar product on $\mathcal{L}^2\left([-1,1], \hat{\omega}_{\alpha,\beta}\right)$

$$(f,h)_{\alpha,\beta} := \int_{-1}^{1} f(x)h(x)d\hat{\mu}_{\alpha,\beta}, \tag{3.2}$$

is positive definite.

Definition 3.1. The new orthogonal functions which are obtained by Gram-Schmidt orthogonalization from $\{u_i\}_{i=1}^{\infty}$ in Eq. (3.1) is named the extended Jacobi functions (EJFs) and to be denoted by $\{\hat{P}_i^{\alpha,\beta}(x)\}_{i=0}^{\infty}$.

Consider $I = (-1,1)$ and the differential equation

$$\hat{T}_{\alpha,\beta}(y) = \hat{p}(x)y'' + \hat{q}(x)y' + \hat{r}(x)y, \tag{3.3}$$

which

$$\begin{aligned} \hat{q}(x) &= \left(\frac{q(x)g(x) - 2\hat{p}(x)g'(x)}{g(x)}\right), \\ \hat{r}(x) &= \left(\hat{p}(x)\left(\frac{2(g'(x))^2 - g''(x)g(x)}{g^2(x)}\right) - \frac{q(x)g'(x)}{g(x)}\right). \end{aligned} \tag{3.4}$$

Here, the functions $\hat{p}(x)$ and $q(x)$ are defined as:

$$\hat{p}(x) = x^2 - 1, \; q(x) = (\alpha + \beta + 2)x + \alpha - \beta,$$

where $\hat{p}(x) \in C^2(I)$ does not vanish on the interval I, and we consider the function $g(x) \neq 0$ such that $\hat{q}(x) \in C^1(I)$ and $\hat{r}(x) \in C(I)$.

$\hat{T}_{\alpha,\beta}$ is not a self-adjoint operator however it can be transferred to a self-adjoin operator. For this purpose, without loss of generality, suppose that

$\hat{p}(x) > 0$ for all $x \in I$. Multiplying both sides of (3.3) by $\rho \in C^2(I)$ yields the following equation

$$\tilde{T}_{\alpha,\beta} = \rho \hat{T}_{\alpha,\beta} = \rho \hat{p} \frac{d^2}{dx^2} + \rho \hat{q} \frac{d}{dx} + \rho \hat{r}.$$

According to Theorem 2.3, $\tilde{T}_{\alpha,\beta}$ will be self-adjoint if

1. $\rho \hat{q} = (\rho \hat{p})' = \rho' \hat{p} + \rho \hat{p}'$,
2. $\rho \hat{p}(f'\bar{h} - f\bar{h}')|_{-1}^{1} = 0, \ \ \forall f, h \in C^2(-1,1) \cap \mathcal{L}^2\left((-1,1), \rho\right)$.

First item is a first-order differential equation in ρ, whose solution is

$$\rho(x) = \frac{c}{\hat{p}(x)} exp \left(\int_{-1}^{x} \frac{\hat{q}(t)}{\hat{p}(t)} dt \right) = \frac{(1-x)^{\alpha}(1+x)^{\beta}}{g(x)^2},$$

where c is a constant. Note that $\hat{p}\rho \in C^2[-1,1]$ is strictly positive on I. $\tilde{T}_{\alpha,\beta}$ is satisfied in the second item due to the fact that:

$$\lim_{x \to -1^+} \frac{(1-x)^{\alpha+1}(x+1)^{\beta+1}}{g(x)^2} (f'(x)h(x) - f(x)h'(x)) = 0, \tag{3.5}$$

$$\lim_{x \to 1^-} \frac{(1-x)^{\alpha+1}(x+1)^{\beta+1}}{g(x)^2} (f'(x)h(x) - f(x)h'(x)) = 0. \tag{3.6}$$

Now, we can conclude that $\hat{T}_{\alpha,\beta}$ is a self-adjoint operator. So according to Theorem 2.3 the eigenvalues of the eigenvalue problem

$$\tilde{T}_{\alpha,\beta} y + \lambda \rho y = 0, \tag{3.7}$$

are all real and any pair of eigenfunctions associated with distinct eigenvalues are orthogonal in $\mathcal{L}^2\left((-1,1), \rho\right)$.

Definition 3.2. The extended Jacobi boundary value problem is a differential equation of the following form

$$\hat{T}_{\alpha,\beta}(y) = \lambda y,$$

where $y = y(x)$ is a twice differentiable function on $(-1,1)$ and also subject to the boundary conditions (3.5) and (3.6).

According to the above discussion we can present the next theorem:

Theorem 3.3. *$\tilde{T}_{\alpha,\beta}$ is a self-adjoint operator and the eigenvalues of the equation* (3.7) *are all real and also any pair of eigenfunctions associated with distinct eigenvalues are orthogonal in $\mathcal{L}^2\left((-1,1), \rho\right)$.*

Proof. The proof is discussed in detail. □

3.2. **Extended Laguerre functions.** Let $k > 0$ be a real parameter. Consider the functions

$$v_0 = g(x), v_i = g(x)x^i, i \geq 1, \tag{3.8}$$

where $g(x) \neq 0$ for all $x \in (0, \infty)$. We define the following measure on the interval $(0, \infty)$:

$$d\hat{\mu}_k = \hat{\omega}_k dx, \ \ x \in (0, \infty),$$
$$\hat{\omega}_k = \frac{x^k e^{-x}}{g(x)^2}, \ k > -1,$$

and observe that $\hat{\omega}_k > 0$ for all $x \in (0, \infty)$ so the following scalar product is positive definite

$$(f, h)_k := \int_0^\infty f(x)h(x)d\hat{\mu}_k, \ \ \forall f, g \in \mathcal{L}^2\left([0, \infty), \hat{\omega}_k\right). \tag{3.9}$$

Definition 3.4. Extended Laguerre functions $\{\hat{L}_i\}_{i=0}^{\infty}$ (ELFs) are obtained by Gram-Schmidt orthogonalization from $\{\nu_i\}$ in Eq. (3.8) with the scalar product (3.9).

Consider the following differential equation

$$\hat{T}_k(y) = \hat{p}(x)y'' + \hat{q}(x)y' + \hat{r}(x)y,$$

whose coefficients $\hat{q}(x)$ and $\hat{r}(x)$ are defined in Eq. (3.4) and we define $\hat{p}(x)$ and $q(x)$ as

$$\hat{p}(x) = x, \ q(x) = (\alpha + 1 - x). \tag{3.10}$$

Definition 3.5. Extended Laguerre boundary value problem is defined as the differential equation

$$\hat{T}_k(y) = \lambda y,$$

where $y = y(x)$ is a twice-differentiable function on $(0, \infty)$ and subject to the following boundary conditions:

$$\lim_{x \to 0^+} \frac{x^k e^{-x}}{g(x)^2}(f'(x)h(x) - f(x)h'(x)) = 0, \tag{3.11}$$

$$\lim_{x \to \infty} \frac{x^k e^{-x}}{g(x)^2}(f'(x)h(x) - f(x)h'(x)) = 0, \tag{3.12}$$

where are satisfied for all $f, h \in C^2(0, \infty) \cap \mathcal{L}^2\left((0, \infty), \hat{\omega}_k\right)$. Also $g(x)$ have to be chosen such that (3.11) and (3.12) are satisfied.

Theorem 3.6. *Let*

$$\tilde{T}_k(y) = \rho\hat{p}(x)y'' + \rho\hat{q}(x)y' + \rho\hat{r}(x)y(x), \ \ \rho = \frac{x^k e^{-x}}{g(x)^2},$$

where $\hat{p}$, $\hat{q}$, and $\hat{r}$ are defined in Eqs. (3.4) *and* (3.10)*. Then*

- *$\tilde{T}_k$ is self-adjoint operator.*
- *The eigenvalues of $\tilde{T}_k$ are all real and any pair of eigenfunctions associated with distinct eigenvalues are orthogonal in $\mathcal{L}^2\left((0, \infty), \rho\right)$.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.3. □

Remark. EJFs and ELFs are two sets of orthogonal functions because of the fact that we have infinite choices for function $g(x)$. For example one can see the special cases of the EJFs with $g(x) = 1$, $e^{\frac{-x}{2}}$, e^{-x}, x^{α}, and $x^{\alpha}e^{\frac{-x}{2}}$ in [24], [24, 15, 2], [13], [13] and [9, 3, 16, 28], respectively, as well as the ELFs with $g(x) = 1$ and x^{α} in [24] and [7], respectively.

4. Some Properties of EJFs and ELFs

In this section, some of the important properties of EJFs and ELFs are listed.

4.1. Properties of the EJFs.

- **Completeness:**

  In this section we establish the completeness of the EJFs in their corresponding Hilbert spaces.

  Theorem 4.1. *A EJF series forms a complete set in* $\mathcal{L}^2([-1,1], \hat{\omega}_{\alpha,\beta})$.

  Proof. Let $\mathcal{P}_N$ be the set of polynomials of degree less than or equal to $N \in \mathbb{N}$. We need to show that for an arbitrary $f \in C[-1,1]$ and any $\epsilon > 0$, there exists a function $\tilde{p} \in \tilde{\mathcal{P}}_N$ where

  $$\tilde{\mathcal{P}}_N = \left\{ \tilde{p} | \tilde{p} = \sum_{i=0}^{N} a_i \hat{P}_i^{(\alpha,\beta)}, a_i \in \mathbb{R} \right\},$$

  such that

  $$|f(x) - \tilde{p}(x)| < \epsilon, \ \ \forall x \in [-1,1].$$

  Consider the function

  $$h(x) = \frac{f(x)}{g(x)} \in C[-1,1],$$

  By the Weierstrass approximation theorem, there exists a polynomial $p \in \mathcal{P}_N$ such that

  $$|h(x) - p(x)| < \frac{\epsilon}{\alpha}, \ \ \forall x \in [-1,1], \ \ \alpha = \max_{x \in [-1,1]} |g(x)|.$$

  Since the function $\tilde{p} = g(x)p$ belong to $\tilde{\mathcal{P}_N}$, we have

  $$|f(x) - \tilde{p}| = |g(x)||h(x) - p(x)| < \epsilon, \ \ \forall x \in [-1,1].$$

  So $\tilde{\mathcal{P}_N}$ is dense in $C[-1,1]$ with respect to the supremum norm, therefore it is dense in $\mathcal{L}^2([-1,1], \hat{\omega}_{\alpha,\beta})$. □

- **Orthogonality relation:**

$$\int_{-1}^{1} \hat{P}_m^{(\alpha,\beta)}(x)\hat{P}_n^{(\alpha,\beta)}(x)\hat{\omega}_{\alpha,\beta}(x)dx =$$
$$\frac{\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)n!}{\Gamma(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+2)}2^{(2n+\alpha+\beta+1)}\delta_{mn}, \quad m,n=0,1,2,\cdots.$$

- **Recurrence relations:**

The EJFs are satisfied in the following recurrence relation

$$\hat{P}_{n+1}^{(\alpha,\beta)}(x) = (x-A_n)\hat{P}_n^{(\alpha,\beta)}(x) - B_n\hat{P}_{n-1}^{(\alpha,\beta)}(x), \quad n=1,2,3,\cdots,$$

where

$$A_n = -\frac{(\alpha-\beta)(\alpha+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$B_n = -\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},$$

and

$$\hat{P}_0^{\alpha,\beta}(x) = g(x), \quad \hat{P}_1^{\alpha,\beta}(x) = g(x)\left(x - \frac{\alpha-\beta}{\alpha+\beta+2}\right).$$

- **Rodrigues formula:**

$$\hat{P}_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{g(x)\hat{\omega}_{\alpha,\beta}(x)2^n n!}\frac{d^n}{dx^n}\left(g(x)^2(1-x^2)^n\hat{\omega}_{\alpha,\beta}(x)\right),$$

for $n=0,1,2,\cdots$. In here $(m)_n$ is the Pochhammer symbol, which is defined by

$$(m)_n = \begin{cases} 1 & n=0, \\ (m)(m+1)\cdots(m+n-1) & n=1,2,3,\ldots. \end{cases}$$

- **Hypergeometric representation:**

$$\hat{P}_n^{(\alpha,\beta)}(x) = g(x)\frac{-2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\,{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{x+1}{2}\right), \quad n=1,2,3,\cdots,$$

or

$$\hat{P}_n^{(\alpha,\beta)}(x) = g(x)\frac{(2)^n(\beta+1)_n}{(n+\alpha+\beta+1)_n}\,{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix}; \frac{x-1}{2}\right), \quad n=1,2,3,\cdots.$$

4.2. **Properties of the ELFs.**

- **Completeness:**

It is worthwhile to note that the proof of the fact that the set $\{\hat{L}_i^{(k)}\}_{i=0}^{\infty}$ is an orthogonal basis of $\mathcal{L}^2([0,\infty),\hat{\omega}_k)$ is quite different from that proved in Theorem 4.1. Because it can use only for the subsets of the compact intervals. To this end, we state and prove the following theorem:

Theorem 4.2. *a ELF series $\{\hat{L}_i^{(k)}\}_{i=0}^{\infty}$ is an orthogonal basis of $\mathcal{L}^2([0,\infty),\hat{\omega}_k)$.*

Proof. A series of ELFs is orthogonal by construction because of it is defined by Gram-Schmidt orthogonalization from $\{v_i(x)\}_{i=0}^{\infty}$. Now, we aim to prove that $E = span\,\{v_i\}_{i=1}^{\infty}$ is dense in $\mathcal{L}^2([0,\infty),\hat{\omega}_k)$. Thus, it suffices to show that for all function $f \in \mathcal{L}^2([0,\infty),\hat{\omega}_k)$ and $\epsilon > 0$ there exists a function such $\tilde{p} \in E$ such that

$$|f(x) - \tilde{p}| < \epsilon, \quad \text{for all } x \in [0,\infty).$$

Notic that

$$E_N = span\left\{g(x)x^i\right\}_{i=0}^{N} = g(x)\mathcal{P}_N.$$

We define $\tilde{f}$ as:

$$\tilde{f} = \frac{f(x)}{g(x)}, x \geq 0,$$

clear that $\tilde{f} \in \mathcal{L}^2([0,\infty),\hat{\omega}_k)$. Since the associated Laguerre polynomial series is dense in $\mathcal{L}^2([0,\infty),x^k e^{-x})$ [27], there exists a polynomial $p \in \mathcal{P}_N$ such that

$$\int_0^{\infty} |\tilde{f}(x) - p(x)|^2 x^k e^{-x} dx < \epsilon,$$

therefore

$$\int_0^{\infty} |f(x) - g(x)p(x)|^2 \hat{\omega}_k(x) dx < \epsilon.$$

Since $g(x)p(x) \in E$, the proof is completed. □

- **Orthogonality relation:**

$$\int_0^{\infty} \hat{L}_n^{(k)}(x)\hat{L}_m^{(k)}\hat{\omega}_k(x)dx = \frac{\Gamma(n+k+1)e^{2a\varepsilon}n!}{(-2\varepsilon)^{2n+k+1}}\delta_{mn}, \quad m,n = 0,1,2,\cdots$$

- **Recurrence relation:**

$$\hat{L}_{n+1}^{(k)}(x) = \left(\frac{2n+1+k-x}{n+1}\right)\hat{L}_n^{(k)}(x) - \left(\frac{n+k}{n+1}\right)\hat{L}_{n-1}^{(k)}(x), \quad n = 1,2,3,\cdots,$$

where

$$\hat{L}_0^{(k)} = 1, \quad \hat{L}_1^{(k)} = 1 + k - x.$$

- **Rodrigues formula:**

$$\hat{L}_n^{(k)}(x) = \frac{1}{g(x)\hat{\omega}_k(x)n!}\frac{d^n}{dx^n}\left(g(x)^2 x^n \hat{\omega}_k(x)\right), \quad n = 0,1,2,\cdots.$$

- **Hypergeometric representation:**

$$\hat{L}_n^{(k)}(x) = \frac{(-1)^n g(x)}{n!}\,{}_1F_1\left(\begin{matrix} -n \\ k+1 \end{matrix}\ ;x\right), \quad n = 0,1,2,\cdots.$$

5. Numerical Examples

This section is devoted to test the EJFs and ELFs with some examples numerically. To doing so, we first would like to note that the good choice of the unknown function $g(x)$ is crucial from the numerical point of view. Generally, the selection of $g(x)$ is closely dependent on the structure of problem. This means, in fact, that we may select the unknown function $g(x)$ in such a way that either the initial (or boundary) conditions of the prescribed problem are satisfied automatically or the singularity (or singularities) of the solution is (or are) removed. Now, we are ready to state some examples. To make a good comparison, we denote:

$$E(N) = \max_{x \in I} |y(x) - y_N(x)|, \quad N \in \mathbb{N},$$

and

$$E(N, x) = y(x) - y_N(x), \; x \in I,$$

where $y(x)$ and $y_N(x)$ are the exact solution and the numerical solution of the considered problem on interval I, respectively.

Example 5.1. For the first example we consider the following non-linear differential equation:

$$y'' + \frac{2}{x}y' + y^5 = 0, \;\; 0 < x \leq 5, \; y(0) = 1, y'(0) = 0. \tag{5.1}$$

It is worthy to note that equation (5.1) is so-called as the Lane-Emden differential equation which is one of the most interesting problems in mathematical physics. It is easy to check that the exact solution of this problem is as follows [6]:

$$y(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}} \;\; x \geq 0.$$

To solve this problem we first set $Y(x) = y(x) - 1$. Clearly $Y(x)$ satisfy the following differential equation

$$Y'' + \frac{2}{x}Y' + (Y + 1)^5 = 0, \;\; 0 < x \leq 5, \tag{5.2}$$

subject to the homogeneous initial conditions $Y(0) = Y'(0) = 0$. Now we use the collocation method to solve the problem for the following cases of function $g(x)$. To do so, we can consider the following functions as test functions

(1) $\phi_i = P_i^{\alpha,\beta}(\frac{2}{5}x - 1), \;\; 0 \leq i \leq N, \;\; \alpha, \beta = 0, \; g(x) = 1,$
(2) $\phi_i = g(x)P_i^{\alpha,\beta}(\frac{2}{5}x - 1), \;\; 0 \leq i \leq N \;\; \alpha, \beta = 0, \;\; g(x) = x^2,$
(3) $\phi_i = g(x)L_i(x), \;\; 0 \leq i \leq N, \;\; g(x) = x^2,$

We also note that for the cases 2 and 3, the approximate solutions satisfy the zero initial conditions. Collocation method based on the roots of the N-th Legendre polynomials is carried out. Maximum absolute error of the methods presented in [8, 21] are compared with the new method and Then the results are shown in Table 1. Moreover, we also plot $E(N,x)$ for some values of N using the basis function presented in case 2 at Fig. 1.

Table 1. Maximum absolute error of methods of [8, 21] and the presented method for Example 5.1.

	ELF with $g(x)=x^2$	EJF with $g(x)=x^2$	Jacobi polynomial	Method of [8]	Method of [21]
N	$E(N)$	$E(N)$	$E(N)$	$E(N)$	$E(N)$
4	8.79×10^{-3}	8.79×10^{-3}	3.26×10^{-1}	3.12×10^{-2}	4.49×10^{-2}
6	1.05×10^{-3}	1.05×10^{-3}	4.44×10^{-1}	5.00×10^{-3}	3.14×10^{-3}
8	1.33×10^{-4}	1.33×10^{-4}	5.04×10^{-2}	4.86×10^{-4}	5.36×10^{-4}
10	1.81×10^{-5}	1.81×10^{-5}	2.87×10^{-2}	6.23×10^{-5}	4.93×10^{-5}

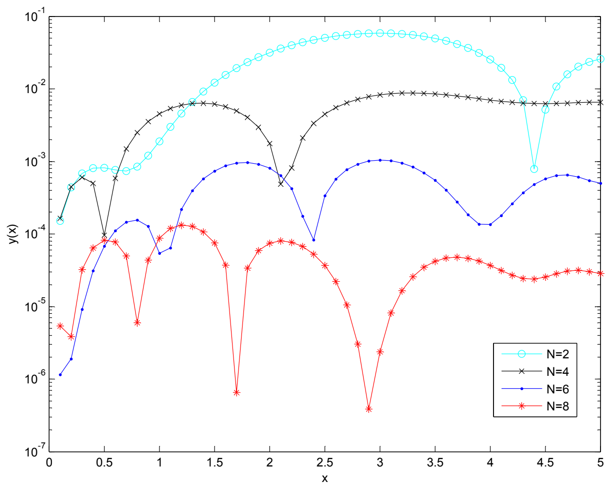

Figure 1. Error between the exact- and numerical solutions of our method for $g(x)=x^2$ and various of N, in Example 5.1.

Example 5.2. For the second example consider the following linear differential equation [19]

$$y''(x)+y'(x)-\epsilon\cos(x)=0;\ \ 0<x\le 10,\ \ y(0)=1,\ y'(0)=0. \tag{5.3}$$

It is easy to verify that the exact solution of this problem is: $y(x)=\cos(x)+\frac{1}{2}\epsilon x\sin(x)$.

First we transform this problem into the following problem with homogeneous initial conditions:

$$Y''(x)+Y'(x)-\epsilon\cos(x)=0;\ \ 0<x\le 10,\ \ Y(0)=0,\ Y'(0)=0. \tag{5.4}$$

where $Y(x) = y(x) - 1$. Spectral Gakerkin method is performed to solve the above problem (5.4). To employ the method we consider the approximate solution in the following form:

$$Y_N(x) = \sum_{i=0}^{N} Y_{N,i}\phi_i(x),$$

where the trial functions $\phi_i,\ i = 0, 1, \cdots, N$ must satisfy the boundary conditions. To do so, we consider the following test functions:

(1) For the first case we set:

$$\phi_i(x) = P_i^{\alpha,\beta}(0.2x-1)+s_1 P_{i+1}^{\alpha,\beta}(0.2x-1)+s_2 P_i^{\alpha,\beta}(0.2x-1),\ \ 0 < x \leq 10,\ \ 0 \leq i \leq N, \tag{5.5}$$

where $P_i^{\alpha,\beta}(x)$ is the i-th Jacobi polynomials with parameter α, β. The constant coefficients $s_1,\ s_2$ should be determined in such a way that $\phi_i(x)$ satisfy the initial conditions [24].

(2) For the second case we consider:

$$\phi_i(x) = g(x)P_i^{\alpha,\beta}(0.2x-1),\ \ 0 < x \leq 10,\ \ 0 \leq i \leq N. \tag{5.6}$$

We are interested to note that if we set $g(x) = x^2$, the function $\phi_i(x)$ satisfy the initial conditions.

Now, our aim is to obtain the unknown coefficients $Y_{N,0},\ Y_{N,1}, \cdots,\ Y_{N,N}$ by solving the following linear system of equation:

$$\sum_{j=0}^{N}\left(\sum_{i=0}^{N} Y_{N,i}\phi_i^{''}(x_j) + \sum_{i=0}^{N} Y_{N,i}\phi_i^{'}(x_j) - \epsilon\cos(x_j)\right)\phi_k(x_j)\omega_j = 0,\ \ 0 \leq k \leq N,$$

where $\{x_j, \omega_j\}_{j=0}^{N}$ be the set of Legendre-Gauss-Lobatto quadrature nodes and weights, respectively. $E(N, 10)$ for various values of N for these two cases of the trial functions (5.5) and (5.6) with parameters $\alpha = \beta = 0$, $\epsilon = 0.1$ together with the three stages generalized variable coefficient Runge-Kutt method (VCRK) [19] are listed in Table 2.

Table 2. $E(N, 10)$ for some values of N obtained by the presented method with parameters $\alpha = \beta = 0$, $\epsilon = 0.1$ for the trial functions (5.5) and (5.6) and VCRK method [19] for Example 5.2.

N	Using trial functions (5.5)	Using trial functions (5.6)	h	VCRK method [19]
5	3.61×10^{-2}	8.01×10^{-2}	2^{-3}	7.41×10^{-7}
10	4.31×10^{-7}	4.32×10^{-7}	2^{-4}	4.63×10^{-8}
12	1.89×10^{-7}	8.25×10^{-10}	2^{-5}	2.90×10^{-9}
14	2.56×10^{-6}	2.62×10^{-12}	2^{-6}	1.81×10^{-10}
16	4.16×10^{-6}	2.03×10^{-12}	2^{-7}	1.13×10^{-11}
18	5.27×10^{-3}	7.16×10^{-13}	2^{-8}	7.82×10^{-13}
20	2.24×10^{-1}	2.42×10^{-12}	2^{-9}	2.52×10^{-13}

Example 5.3. Consider the following weakly singular Volterra integral equation as the third example:

$$y(x) = \int_0^x \frac{s^{\mu-1}}{x^\mu} \frac{1}{\sin^2(s)} y(s)ds + g(x), \quad 0 \le x \le \pi, \quad \mu > 0, \tag{5.7}$$

with

$$g(t) = \sin^2 x - \frac{1}{\mu}.$$

The exact solution of equation (5.7) is $y(x) = \sin^2 x$.

This problem is solved via the collocation method using the basis EJFs with $g(x) = 1,\ \sin^2 x$. We also note that we choose $g(x) = \sin^2(x)$ to remove the singularity of the equation (5.7). It is also interesting to note that, the Legendre-Gauss nodes are used as the collocation points. The maximum absolute errors, $E(N)$, obtained by the collocation method for some values of N are reported in Table 3.

Table 3. Maximum absolute error of the presented method for the cases cases $g(x) = 1, \sin^2 x$ with $\mu = 1.1$ and some values of N for Example 5.3.

	EJF with $g(x) = x^2$	EJF with $g(x) = \sin^2 x$
N	$E(N)$	$E(N)$
5	7.80×10^{-1}	8.10×10^{-16}
10	7.09×10^{-5}	2.74×10^{-16}
15	1.30×10^{-8}	3.24×10^{-16}
20	6.92×10^{-12}	1.03×10^{-15}
25	5.34×10^{-12}	2.20×10^{-16}
30	5.57×10^{-12}	1.73×10^{-15}

Example 5.4. As the fourth example, consider one of the most popular differential equation [15]:

$$\sqrt{x} y''(x) - y^{\frac{3}{2}}(x) = 0, \quad 0 < x < \infty,$$

subject to the $y(0) = 1$ and $\lim_{x\to\infty} y(x) = 0$.

This differential equation is so-called as Thomas-Fermi equation and was solved by many researchers (For instance, Liao [14], Khan and Xu [12], Zhu et al. [31], Parand et al. [20] and etc). In order to make a comprehensive comparison, we briefly review how solve the Thomas-Fermi equation by the method of Liu and Zhu [15]. At first, we represent the solution of the problem (5.4) as sum of two parts because of its singularity at the origin (e.g., $\lim_{x\to\infty} y''(x) = \infty$). Then we split y as $y = \hat{y} + q$ such that the function $q : [0, \infty) \to \mathbb{R}$ satisfies the boundary conditions:

$$q(0) = 1, \ \lim_{x\to\infty} q(x) = 0. \tag{5.8}$$

Then the problem (5.4) is transformed to

$$\hat{y}'' - \frac{1}{\sqrt{x}}(\hat{y}+q)^{\frac{3}{2}} + q'', \quad 0 < x < \infty, \tag{5.9}$$

subject to $\hat{y}(0) = 0$ and $\hat{y}(\infty) = 0$. In this problem y has the series expansion at the origin [5]

$$y = 1 + \lambda x + \frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}\lambda x^{\frac{5}{2}} + \frac{1}{3}x^3 \cdots, \tag{5.10}$$

where λ is the value of the first derivative of $y(x)$ at the origin. We obtain

$$y'' = x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}} + 2x + \cdots,$$

which means that only the third term in (5.10) creates a singularity in the solution. We truncate the series (5.10) to the third term and denote

$$p := 1 + \lambda x + \frac{4}{3}x^{\frac{3}{2}}. \tag{5.11}$$

We take $q(x,\lambda) = e^{-\frac{x}{2}}p(x,\lambda)$, which satisfies (5.8). The following smooth problem is obtained from $q(x,\lambda)$ and (5.10)

$$\begin{cases} \hat{y}'' - \frac{1}{\sqrt{x}}\left(\hat{y} + e^{-\frac{x}{2}}p\right)^{\frac{3}{2}} + e^{-\frac{x}{2}}\left(\frac{1}{4}p - p' + p''\right) = 0, \quad 0 < x < \infty, \\ \hat{y}(0) = 0, \quad \hat{y}(\infty) = 0, \quad \hat{y}'(0) = \frac{1}{2}. \end{cases} \tag{5.12}$$

We also note that more accurate solution for (5.12) can be obtained by using a proper scaling factor b [15]. Let $\xi = bx$ and $u(\xi) = \hat{y}(x)$. Then problem (5.12) reduces to the following problem:

$$\begin{cases} u'' - \frac{1}{\sqrt{b^3\xi}}\left(u + e^{-\frac{\xi}{2b}}p\left(\frac{\xi}{b},\lambda\right)\right)^{\frac{3}{2}} + \frac{e^{-\frac{\xi}{2b}}}{b^2}\left(\frac{1}{4}p - p' + p''\right)\left(\frac{\xi}{b},\lambda\right) = 0, \\ u(0) = 0, \quad u(\infty) = 0, \quad u'(0) = \frac{1}{2b}, \quad 0 < \xi < \infty, \end{cases} \tag{5.13}$$

Now, the obtained problem can be soled by the well-known Laguerre pseudospectral method. Let $\{\xi_i\}_{i=0}^{N} (N \in \mathbb{N})$ be the Laguerre-Gauss interpolation points (i.e., zeros of $L_N(x)$) together with $\xi_0 = 0$. Therefore, the numerical solution can be expanded as the following nodal expansion:

$$u_N(\xi) = \sum_{j=0}^{N} u_N(\xi_j)\hat{F}(\xi),$$

where

$$\hat{F}(x) = e^{\frac{-(\xi-\xi_j)}{2}} \prod_{m=0, m\neq j}^{N} \frac{\xi - \xi_m}{\xi_j - \xi_m} = e^{-\frac{(\xi-\xi_j)}{2}} \frac{\xi L_N(\xi)}{(\xi L_N(\xi))'(\xi_j)(\xi-\xi_j)}, \quad 0 \le j \le N. \tag{5.14}$$

The Laguerre pseudospectral method for (5.13) is to find u_N such that

$$\begin{cases} u_N'' - \frac{1}{\sqrt{b^3\xi_i}}\left(u_N + e^{-\frac{\xi_i}{2b}} p\left(\frac{\xi_i}{b}, \lambda\right)\right)^{\frac{3}{2}} + \frac{e^{-\frac{\xi_i}{2b}}}{b^2}\left(\frac{1}{4}p - p' + p''\right)\left(\frac{\xi_i}{b}, \lambda\right) = 0, \\ u_N(0) = 0, \quad u_N(\infty) = 0, \quad u_N'(0) = \frac{1}{2b}, \end{cases} \tag{5.15}$$

We solve the Thomas-Fermi equation using the mentioned method and the obtained results are reported in Tables 4 (for the detailed information see [15]).

Moreover, without loss of the generality, we can assume $q = e^{-kx}$ $(k > 0)$ and use the ELFs with $g(x) = e^{-kx}$ to solve the nonlinear system (5.13). We obtain the numerical solution by performing the presented method but unfortunately, for large values of N, the condition number of the Jacobian matrix in the Newton's method may decrease very fast and therefore the numerical solution can not be obtained accurately. So, we report our numerical results with $N = 8$ and $g(x) = e^{-kx}$ with $k = 0.3, 0.4, 0.5, 0.6, 0.7$ in Table 5. Table 5 indicates that the numerical solution with $k = 0.5$ is more accurate than other values of k.

Table 4. Numerical solution of Thomas-Fermi by the method [15] (with $N = 64$, $b = 3$), and the methods of [14], [31] and [12] for Example 5.4.

x	Method of [14] $y(x)$	Method of [15] $y(x)$	Method of [31] $y(x)$	Method of [12] $y(x)$
1	0.424008000	0.424008068	0.424304901	0.423772000
2	0.243009000	0.243008476	0.243264008	0.242718000
4	0.108404000	0.108404284	0.108556966	0.109632000
6	0.059423000	0.059423000	0.059513053	0.063816200
8	0.036587300	0.036587209	0.036601372	0.043285900
10	0.024314300	0.024314333	0.024285183	0.032208100
50	0.000632255	0.000616135	0.000431660	0.000473089

Example 5.5. For the last example, we consider the following fractional calculus of variation problem [23]

$$minimize\ J[y] = \frac{1}{2}\int_0^1 ({}_0^C D_x^{0.5} y(x) - f(x))^4 dx, \tag{5.16}$$

subject to

$$y_0 = y(0) = 0, \ \ y_1 = y(1) = 1,$$

where $f(x)$ is given

$$f(x) = \frac{16\Gamma(6)}{\Gamma(5.5)} x^{4.5} - \frac{20\Gamma(4)}{\Gamma(3.5)} x^{2.5} + \frac{5}{\Gamma(1.5)} x^{0.5}.$$

In this case the exact solution is $y(x) = 16x^5 - 20x^3 + 5x$.

TABLE 5. Numerical solution of Thomas-Fermi and $y'(0)$ by presented method with $N = 8,\ b = 1$ and $g(x) = e^{-kx}$, with $k = 0.3, 0.4, 0.5, 0.6, 0.7$ for Example 5.4.

	$k = 0.3$	$k = 0.4$	$k = 0.5$	$k = 0.6$	$k = 0.7$
x	$y(x)$	$y(x)$	$y(x)$	$y(x)$	$y(x)$
1	0.4238955910	0.4238485308	0.4238817746	0.4239055735	0.4223208979
2	0.2431121234	0.2428464683	0.2426141529	0.2423483510	0.2400348781
4	0.1082333044	0.1085142007	0.1087684640	0.1090038390	0.1116903257
6	0.0594062477	0.0587492235	0.0586546125	0.0583863783	0.0526742642
8	0.0361676051	0.0371093606	0.0371957723	0.0372725644	0.0405964467
10	0.0235398204	0.0243205340	0.0248064919	0.0249378871	0.0283913379
50	−0.0277919185	−0.0014940678	−0.0000082724	0.0000004078	0.0000000602
$y'(0)$	−1.584032	−1.586677	−1.589387	−1.592191	−1.595777

We apply the Rayleigh-Ritz method to solve the problem (5.16) [7]. To do so, we expand the approximate solution $y_N(x)$ by a linear combination of certain linearly independent functions as follow:

$$y(x) \simeq y_N(x) = \phi_0(x) + \sum_{k=1}^{N} c_k \phi_k(x), \tag{5.17}$$

where

$$\phi_0(x) = y_0 + (y_1 - y_0)g(x), \quad \phi_k(x) = \left(P_k^{(0,0.5)}(2x-1) - P_k^{(0,0.5)}(1)\right) g(x). \tag{5.18}$$

For this example we set $g(x) = x^{0.5}$ and x, for these choices $y_N(x)$ satisfies the boundary conditions of equation (5.16). Now, it remains evaluate ${}_0^C D_x^{0.5} y_N(x)$ with $g(x) = x^{0.5}$. The following theorem states an interesting formula for the Caputo fractional derivatives of $y_N(x)$. For more detailed information about the definition of the Caputo fractional derivative and a proof of the next theorem see [7].

Theorem 5.6. [7] *Let $\alpha > 0$ be a real number and $x \in [0,1]$. Then*

$${}_0^C D_x^{\alpha} \left(x^{\alpha} P_k^{(0,\alpha)}(2x-1)\right) = \frac{\Gamma(k+\alpha+1)}{k!} P_k^{(0,\alpha)}(2x-1).$$

Substituting ${}_0^C D_x^{0.5} y_N(x)$ and (5.17) into (5.16) we obtain

$$J[y_N] = \frac{1}{2}\int_0^1 ({}_0^C D_x^{0.5} y_N(x) - f(x))^4 dx,$$

With the help of the Gauss-Legendre quadrature rule we get

$$J[y_N] = \frac{1}{2}\sum_{k=0}^{N-1} \omega_k ({}_0^C D_{x_k}^{0.5} y_N(x_k) - f(x_k))^4. \tag{5.19}$$

Our next aim is to determine the unknown coefficients $c_1, c_2, \cdots, c_N$ such that the functional J is minimized. So, we use the following well-known procedure:

$$\frac{\partial J}{\partial c_1} = 0, \cdots, \frac{\partial J}{\partial c_N} = 0.$$

The unknown coefficients $c_1, c_2, \cdots, c_N$ can be determined by solving the previous nonlinear system of equations. To compare our numerical results with $g(x) = x^{0.5}, x$ by the methods of [22, 23]. Moreover, these results are reported in Table 6. Furthermore, we plot $E(N, x)$ for some values of N using the basis function (5.18) in Fig. 2.

TABLE 6. Maximum absolute error of methods of [22, 23] and the method of [7] for $g(x) = x,\ x^{0.5}$ for Example 5.5.

	EJF with $g(x) = x^{0.5}$	EJF with $g(x) = x$	Method of [23]	Method of [22]
N	$E(N)$	$E(N)$	$E(N)$	$E(N)$
4	3.98×10^{-2}	8.56×10^{-3}	1.21	1.55
8	9.48×10^{-3}	1.54×10^{-2}	5.38×10^{-1}	7.38×10^{-1}
12	5.11×10^{-3}	1.48×10^{-2}	3.39×10^{-1}	4.82×10^{-1}
16	3.50×10^{-3}	1.15×10^{-2}	2.45×10^{-1}	3.58×10^{-1}
20	4.40×10^{-3}	7.98×10^{-3}	1.91×10^{-1}	8.97×10^{-2}

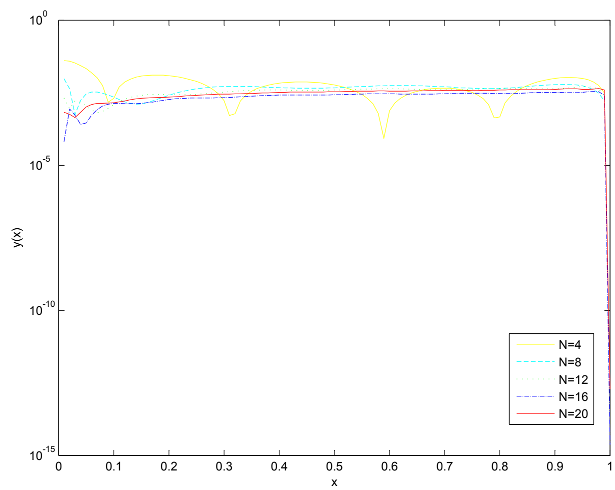

FIGURE 2. Error between the exact- and numerical- solutions of our method with $g(x) = x^{0.5}$ for Example 5.5.

6. Conclusion

This article presents two new extended- Jacobi and Laguerre basis functions which are the solution of two non-classical Sturm-Liouville eigenvalue problems. Some theoretical results concerning the new basis functions are proved in detail. Finally several examples are provided to verify these basis functions. The numerical results show that the new basis functions work well for various linear and nonlinear differential equation, integral equation and calculus of variation problem. The authors believe that the proposed method can be developed for other problems in engineering and science (see for instance [25, 26, 24, 3, 6] and references therein).

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References

1. M. Al-Gwaiz, *Sturm-Liouville Theory and its Applications*, Springer, 2007.
2. M. Aristidou, M. Davidson, G. Olafsson, Laguerre functions on symmetric cones and recursion relations in the real case, *Journal of computational and applied mathematics*, **199**(1), (2007), 95–112.
3. W. Bao, J. Shen, A generalized-Laguerre-Hermite pseudospectral method for computing symmetric and central vortex states in Bose-Einstein condensates, *Journal of Computational Physics*, **227**(23), (2008), 9778–9793.
4. S. Bochner, Über Sturm-Liouvillesche Polynomsysteme, *Mathematische Zeitschrift*, **29**(1), (1929), 730–736.
5. J. P. Boyd, Rational Chebyshev series for the Thomas-Fermi function: Endpointsin gularities and spectral methods, *Journal of computational and applied mathematics*, **244**, (2013), 90–101.
6. A. R. Davies, A. Karageorghis, T. N. Phillips, Spectral Galerkin methods for the primary two point boundary value problem in modelling viscoelastic flows, *International Journal for Numerical Methods in Engineering*, **26**(3), (1988), 647–662.
7. M. Dehghan, E. A. Hamedi, H. Khosravian-Arab, A numerical scheme for the solution of a class of fractional variational and optimal control problems using the modified Jacobi polynomials, *Journal of Vibration and Control*, (2014), 1077546314543727.
8. E. H. Doha, W. M. Abd-Elhameed, On the Coefficients of Integrated Expansions and Integrals of Chebyshev Polynomials of Third And Fourth Kinds, Bulletin of the Malaysian Mathematical Sciences Society, 2014.
9. G. A. Frolov, M. Y. Kitaev, A problem of numerical inversion of implicitly defined Laplace transforms, *Computers & Mathematics with Applications*, **36**(5), (1998), 35–44. erential Equations, Springer Verlag, New York, 1992.
10. D. Gómez-Ullate, N. Kamran, R. Milson, An extended class of orthogonal polynomials defined by a Sturm-Liouville problem, *Journal of Mathematical Analysis and Applications*, **359**(1), (2009), 352–367.
11. K. Hoffman, R. Kunze, *Linear Algebra*, Englewood Cliffs, New Jersey, 1971.

12. H. Khan, H. Xu, Series solution to the Thomas-Fermi equation, *Physics Letters A*, **365**, (2007), 111–115.
13. H. Khosravian-Arab, M. Dehghan, M. R. Eslahchi, Fractional SturmLiouville boundary value problems in unbounded domains: Theory and applications, *Journal of Computational Physics*, 299, (2015), 526–560.
14. S. Liao, Beyond perturbation: introduction to the homotopy analysis method, CRC press, 2003.
15. C. Liu, S. Zhu, Laguerre pseudospectral approximation to the Thomas-Fermi equation, Journal of Computational and Applied Mathematics, 282 (2015) 251–261.
16. M. Lorente, Continuous vs. discrete models for the quantum harmonic oscillator and the hydrogen atom, **Physics Letters A**, **285**(3), (2001), 119–126.
17. S. Odak, R. Sasaki, Infinitely many shape-invariant potentials and cubic identities of the Laguerre and Jacobi polynomials, *Journal of Mathematical Physics*, **51**(5), (2010), 053513.
18. S. Odak, R. Sasaki, Another set of infinitely many exceptional (X_l)-Laguerre polynomials, Physics Letters B, 684.2 (2010) 173–176.
19. K. Ozawa, A functional fitting Runge-Kutta method with variable coefficients, *Japan journal of industrial and applied mathematics*, **18**(1), (2001), 107–130.
20. K. Parand, M. Dehghan, A. Pirkhedri, The Sinc-collocation method for solving the Thomas-Fermi equation, *Journal of Computational and Applied Mathematics*, **237**(1), (2013), 244–252.
21. K. R. Pandey, N. Kumar, A. Bhardwaj, G. Dutta, Solution of LaneEmden type equations using Legendre operational matrix of differentiation, *Applied Mathematics and Computation*, **218**(14), (2012), 7629–7637.
22. S. Pooseh, R. Almeida, D. F. Torres, Discrete direct methods in the fractional calculus of variations, *Computers & Mathematics with Applications*, **66**(5), (2013), 668–676.
23. S. Pooseh, R. Almeida, D. F. Torres, A discrete time method to the first variation of fractional order variational functionals, *Open Physics*, **11**(10), (2013), 1262–1267.
24. J. Shen, T. Tang, L. L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer Science & Business Media, Vol. 41, 2011.
25. A. Azizi, J. Saeidian, S. Abdi, Applying Legendre Wavelet Method with Regularization for a Class of Singular Boundary Value Problems, *Iranian Journal of Mathematical Sciences and Informatics*, **11**(2), 57–69.
26. M. Jahanshahi, A. Ahmadkhanlu, The Wave Equation in Non-classic Cases: Non-self Adjoint with Non-local and Non-periodic Boundary Conditions, *Iranian Journal of Mathematical Sciences and Informatics*, **9**(1), 1–12.
27. G. Szegó, *Orthogonal Polynomials*, American Mathematical Soc, 1939.
28. Z. Q. Wang, B. Y. Guo, W. Zhang, Mixed spectral method for three-dimensional exterior problems using spherical harmonic and generalized Laguerre functions, *Journal of Computational and Applied Mathematics*, **217**(1), (2008), 277–298.
29. D. Wang, A. Xiao, Fractional variational integrators for fractional Euler-Lagrange equations with holonomic constraints, *Communications in Nonlinear Science and Numerical Simulation*, **18**(4), (2013), 905–914.
30. J. Wu, H. Tian, Functionally-fitted block methods for ordinary differential equations, *Journal of Computational and Applied Mathematics*, **271**, (2014), 356–368.
31. S. Zhu, H. Zhu, Q. Wu,Y. Khan, An adaptive algorithm for the Thomas-Fermi equation, *Numerical Algorithms*, **59**(3), (2012), 359–372.