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A topology on BCK-algebras via left and right stabilizers

T. Roudabri* and L. Torkzadeh

Department of Mathematics, Islamic Azad University of Kerman, Kerman, Iran

E-mail: T.Roodbarylor@yahoo.com
E-mail: lTorkzadeh@yahoo.com

Abstract. In this paper, we use the left(right) stabilizers of a BCK-algebra $(X, *, 0)$ and produce two basis for two different topologies. Then we show that the generated topological spaces by these basis are Bair, connected, locally connected and separable. Also we study the other properties of these topological spaces.

Keywords: BCK-algebra , basis topological, stabilizers, Bair space, connected space, locally connected and separable space.

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1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. In 1997, Y. Hung and Z. Chen introduced the notions of right and left stabilizers of every subset of a BCK-algebra. In this note, we by considering the left(right) stabilizers of a BCK-algebra $(X, *, 0)$, construct two basis for two topologies on $(X, *, 0)$. Then we obtain some results as mentioned in the abstract. M. M. Zahedi defined hyper $K(BCK)$-algebras. Also, T. Roudbari and M. M. Zahedi defined simple hyper $K(BCK)$-algebras [2,4,7]. Similar to

*Corresponding Author

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them, we can define two topologies via left and right hyper $K(BCK)$-stabilizers in hyper $K(BCK)$-algebras.

2. Preliminaries

We give herein the basic notions on BCK-algebras. For further information, we refer to the book [5]. By a BCK-algebra we mean an algebra $(X,*,0)$ of type (2,0) satisfying the following axioms: for every $x,y,z \in X$,

(i) $((x*y)*(x*z))*(z*y)=0$,
(ii) $(x*(x*y))*y=0$,
(iii) $x*x=0$,
(iv) $x*y=y*x=0 \Rightarrow x=y$,
(v) $0*x=0$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x*y=0$. In a BCK-algebra X, the following hold: for all $x,y,z \in X$,

(a) $x*0=x$,
(b) $x*y \leq x$,
(c) $(x*y)*z=(x*z)*y$,
(d) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$,
(e) $x*(x*(x*y))=x*y$.

A nonempty subset A of X is called an *ideal* of X if it satisfies
(i) $0 \in A$,
(ii) $(\forall x \in X)(\forall y \in A)$ $(x*y \in A \Rightarrow x \in A)$.
A *subalgebra* of X is a nonempty subset A of X such that $x*y \in A$, for all $x,y \in A$.
If there is an element 1 of X satisfying $x \leq 1$, for all $x \in X$, then the element 1 is called *unit* of X. A BCK-algebra with unit is called *bounded* .

Definition 2.1. [3] Let X be a BCK-algebra and A be a nonempty subset of X. Then the sets

$$A_l^* = \{x \in X| \;\; a*x=a, \forall a \in A\}$$

and

$$A_r^* = \{x \in X| \;\; x*a=x, \forall a \in A\}$$

are called the left and right stabilizers of A, respectively and the set $A^* = A_l^* \cap A_r^*$ is called the stabilizer of A.

Theorem 2.2. [3] Let A be a nonempty subset of a BCK-algebra X. Then

(i) A_l^* is an ideal of X.
(ii) A_r^* is a subalgebra of X.

Definition 2.3. [1] Consider A as a nonempty set, a mapping $\phi : P(A) \to P(A)$ is called a closure operator on A, if for all $X, Y \in P(A)$ the following holds:
(1) $X \subseteq \phi(X)$,
(2) $\phi^2(X) = \phi(X)$,
(3) $X \subseteq Y$ implies $\phi(X) \subseteq \phi(Y)$.

Note that all definitions and notations on a given topological space (X, τ) are stated from [6].

3. Closure operator on BCK-algebras

In the sequel X is a BCK-algebra.

Theorem 3.1. Let A and B be two nonempty subsets of X. Then
(i) $0 \in A_l^* \cap A_r^*$,
(ii) $A \subseteq (A_l^*)_r^* \cap (A_r^*)_l^*$,
(iii) If $A \subseteq B$, then $B_l^* \subseteq A_l^*$ and $B_r^* \subseteq A_r^*$,
(iv) $A_l^* = ((A_l^*)_r^*)_l^*$ and $A_r^* = ((A_r^*)_l^*)_r^*$,
(v)$(\bigcup_{j\in J} A_j)_l^* = \bigcap_{j\in J} (A_j)_l^*$.

Proof. (i) Since $0 * x = 0$ and $x * 0 = x$, for all $x \in X$, then $0 \in A_l^* \cap A_r^*$.
(ii) Let $a \in A$. Then $x * a = x$, $\forall x \in A_r^*$ and $a * y = a$, $\forall y \in A_l^*$. So $a \in (A_r^*)_l^* \cap (A_l^*)_r^*$.
(iii) Let $x \in B_l^*$. Then $b * x = b$, $\forall b \in B$. Since $A \subseteq B$ and $b * x = b$, $\forall b \in A$. So $x \in A_l^*$. Similarly $B_r^* \subseteq A_r^*$.
(iv) By (ii) we get that $A_l^* \subseteq ((A_l^*)_r^*)_l^*$ and $A_r^* \subseteq ((A_r^*)_l^*)_r^*$. Also by (ii) and (iii) we have $((A_r^*)_l^*)_r^* \subseteq A_r^*$ and $((A_l^*)_r^*)_l^* \subseteq A_l^*$. Therefore $A_l^* = ((A_l^*)_r^*)_l^*$ and $A_r^* = ((A_r^*)_l^*)_r^*$.
(iv) The proof is easy.

Note that we define $\emptyset_l^* = \emptyset$ and $\emptyset_r^* = \emptyset$.

Theorem 3.2. The function $\alpha : P(X) \to P(X)$, where $\alpha(D) = (D_l^*)_r^*$ is a closure operator on X.

Proof. By Theorem 3.1(ii), $D \subseteq \alpha(D)$, for all $D \in P(X)$. Also by Theorem 3.1(iv), $\alpha(D) = (D_l^*)_r^* = (((D_l^*)_r^*)_l^*)_r^* = \alpha^2(D)$, for all $D \in P(X)$. Let

$A \subseteq B$. Then by Theorem 3.1(iii), $\alpha(A) \subseteq \alpha(B)$. Therefore α is a closure operator on X.

Theorem 3.3. The function $\gamma : P(X) \to P(X)$, where $\gamma(D) = (D_r^*)_l^*$ is a closure operator on X.

Proof. The proof is similar to the proof of Theorem 3.2.

Theorem 3.4. Consider the function α given in Theorem 3.2. Then we can obtain that $\beta_\alpha = \{A \in P(X)|\alpha(A) = A\}$ is a basis for a topology on X.

Proof. It is easy to see that $X_l^* = \{0\}$ and also $\{0\}_r^* = X$. Then $\alpha(X) = X$ and so $X \in \beta_\alpha$. Thus for all $x \in X$ there is at least one element of β_α containing x. Let $x \in A \cap B$, for $A, B \in \beta_\alpha$. Since α is a closure operator, then we can obtain that $\alpha(A \cap B) = A \cap B$, i.e. $A \cap B \in \beta_\alpha$ containing x. Therefore β_α is a basis topology on X.

Theorem 3.5. Consider the function γ given in Theorem 3.3. Then $\beta_\gamma = \{A \in P(X)|\gamma(A) = A\}$ is a basis for a topology on X.

Proof. The proof is similar to the proof of Theorem 3.4.

Note that by Theorem 2.2 elements of β_α are subalgebras of X and elements of β_γ are ideals of X.

We define the topologies τ_α and τ_γ generated by basis β_α and β_γ, respectively.

Example 3.6. Let $X = \{0, a, b, c\}$ and $*$ operation be given by the following table

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then $(X, *, 0)$ is a BCK-algebra. We see that $0 \in \alpha(A)(0 \in \gamma(A))$, for all nonempty sub sets A of X, so if $0 \notin A \subseteq X$, we have $A \notin \beta_\alpha(\beta_\gamma))$. By some manipulations we get that $\beta_\alpha = \{\emptyset, X, \{0, b\}, \{0, c\}, \{0\}\}$ and $\beta_\gamma = \{\emptyset, X, \{0, a\}, \{0, c\}, \{0, a, c\}, \{0, a, b\}, \{0\}\}$. Thus $\tau_\alpha = \{\emptyset, X, \{0\}, \{0, b\}, \{0, c\}, \{0, b, c\}\}$ and $\tau_\gamma = \{\emptyset, X, \{0\}, \{0, a\}, \{0, c\}, \{0, a, c\}, \{0, a, b\}\}$. We see that in this example $\{0, a, b\} \notin \beta_\alpha$, because $\{0, a, b\}_l^* = \{0, b\}$ and $\{0, b\}_r^* = \{0, c\}$ and so $\{0,c\} = (\{0,a,b\}_l^*)_r^*) \neq \{0,a,b\}$. Also since $\tau_\alpha \nsubseteq \tau_\gamma$ and $\tau_\gamma \nsubseteq \tau_\alpha$, then τ_α is not finer than τ_γ and also τ_γ is not finer than τ_α.

Theorem 3.7. $(X,\tau_\alpha)((X,\tau_\gamma))$ is a Hausdorff space if and only if $X = \{0\}$.

Proof. Since for any $U \in \tau_\alpha$, we have $0 \in U$, so for any two arbitrary elements U, V of τ_α, we have $U \bigcap V \neq \emptyset$. Thus $(X,\tau_\alpha)((X,\tau_\gamma))$ is not Housdorff. Conversely, let $X = \{0\}$. Then $\tau_\alpha = \{\emptyset, X\}$. Thus it is clear that (X,τ_α) is a Hausdorff space.

Theorem 3.8. $(X,\tau_\alpha)((X,\tau_\gamma))$ is connected.

Proof. Since $0 \in U$, for any nonempty open set of X, then there are not nonempty open subsets U and V of X such that $X = U \cup V$ and $U \cap V = \emptyset$. Thus (X,τ_α) is connected space.

Corollary 3.9. Let U be a nonempty open subset of $(X,\tau_\alpha)((X,\tau_\gamma))$. Then U is a connected set of X.

Proof. It is similar to the proof of Theorem 3.8.

Corollary 3.10. Let U be a nonempty non-connected subset of $(X,\tau_\alpha)((X,\tau_\gamma))$. Then $0 \in U$.

Proof. It is straightforward.

Corollary 3.11. Let $A \neq X$ and $A \neq \emptyset$ be a closed subset of $(X,\tau_\alpha)((X,\tau_\gamma))$. Then A is a connected set of X.

Proof. Since $A \neq X$ is a closed set of $(X,\tau_\alpha)((X,\tau_\gamma))$, then $\emptyset \neq X - A$ is an open set of $(X,\tau_\alpha)((X,\tau_\gamma))$. By Theorem 3.1 we have $0 \in X - A$, therefore $0 \notin A$. Thus by Corollary 3.10, we get that A is a connected set of X.

Note that Corollaries 3.9 and 3.11 imply that all proper subsets of $(X,\tau_\alpha)((X,\tau_\gamma))$ are connected, whenever they are closed or open.

Theorem 3.12. Let A be a subset of topological space $(X,\tau_\alpha)((X,\tau_\gamma))$ and $0 \in A$. Then $\overline{A} = X$.

Proof. Let $x \in X$. If $x = 0$, then $0 \in \overline{A}$. Let $x \neq 0$, since $0 \in U$, for any nonempty open subset of X, then $U \cap A \neq \emptyset$, for any open set containing x. Therefore $x \in \overline{A}$.

By the above theorem we can get that the following corollary.

Corollary 3.13. Let U be a nonempty open subset of X. Then $\overline{U} = X$.

Proof. It is similar to a proof of Theorem 3.12.

Open problem. Is there any $A \subseteq (X, \tau_\alpha)((X, \tau_\gamma))$ such that $\overline{A} = X$, but $0 \notin A$.

Theorem 3.14. Let A be a nonempty subset of the topological space (X, τ_α) $((X, \tau_\gamma))$. Then $0 \in \overline{A}$ if and only if $\overline{A} = X$.

Proof. Let $0 \in \overline{A}$. Then $0 \in C$, for all closed subset C of X containing A. Since 0 is in any nonempty open subset of the topological space X, then the only closed subset of X containing 0 and A is X. So $\overline{A} = X$. The proof of the converse is clear.

Lemma 3.15. $\{0\}$ is an open subset of the topological space $(X, \tau_\alpha)((X, \tau_\gamma))$.

Proof. By Definition 3.1 we can get that $\{0\}^*_l = X$ and $X^*_r = \{0\}$, then $\{0\} \in \beta_\alpha$. Thus $\{0\}$ is an open set of the topological space $(X, \tau_\alpha)((X, \tau_\gamma))$.

Theorem 3.16. $(X, \tau_\alpha)((X, \tau_\gamma))$ is separable.

Proof. By Theorem 3.15 and Lemma 3.16 we get that $\overline{\{0\}} = X$. Then $(X, \tau_\alpha)((X, \tau_\gamma))$ is separable.

Theorem 3.17. $(X, \tau_\alpha)((X, \tau_\gamma))$ is locally connected.

Proof. Let x be an arbitrary element of X and U be an open set containing x. By Theorem 3.8, we get that U is connected and also containing x. Therefore (X, τ_α) is locally connected.

Open problem. How are the exact characterization of the compact sets in these topological spaces?

Convention 3.18. Let $(X, *, 0)$ be a totally ordered BCK-algebra and let β_o be the all sets of the following types:
(i) All open intervals $(a, b) = \{x \in X | \ a < x < b\}$,
(ii) All intervals $[0, b) = \{x \in X | \ 0 \leq x < b\}$.
(ii) All intervals $(0, 1] = \{x \in X | \ 0 < x \leq 1\}$, where 1 is unit of X

As we can see similar to [4] β_o is a basic for a topology on X, which is called the order topology. The topology induced by β_o is denoted by τ_o.

Theorem 3.19. Let $(X, *, 0)$ be a totally ordered BCK-algebra. Then (X, τ_γ) is finer than (X, τ_o).

Proof. Let $x \in A$ and $A \in \beta_\gamma$. Then there is $a \in A$ such that $x * a = 0$. We show that $[0, a) \subseteq A = (A_r^*)_l^*$. Let $b \in [0, a)$. Then $b * a = 0$ and so $b \in A$, by Theorem 2.2(i). Hence $a \in A$ implies that $b \in A$. Thus $[0, a) \subseteq A$. Also $x \in [0, a) \in \beta_o$. Therefore (X, τ_γ) is finer than (X, τ_o).

The following example shows that the condition " totally order " in Convention 3.18 is necessary.

Example 3.20. Let $X = \{0, 1, 2, 3\}$ and $*$ operation be given by the table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	3	3	0

Then $(X, *, 0)$ is not a totally ordered BCK-algebra. We see that β_o is not a basis for a topology on X, because $1 \in [0, 2) \bigcap [0, 3)$., but there is not a $M \in \tau_o$ such that $1 \in [0, 2) \bigcap [0, 3)$.

The following example shows that (X, τ_o) may not finer than (X, τ_γ).

Example 3.21. Let $X = \{0, 1, 2, ...\}$. Define " $*$ " on X by

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{if } y \leq x, y \neq 0, \\ x & \text{if } y \leq x, y = 0 \end{cases}$$

Then $(X, *, 0)$ is a non bounded BCK-algebra. By some manipulations we get that $\beta_\gamma = \{\emptyset, X, \{0\}\}$. Consider $A = [0, 2)$. We see that $1 \in A$, but there is not any $M \in \beta_o$ such that $1 \in M \subseteq A$.

The following example shows that (X, τ_o) may not finer than (X, τ_α).

Example 3.22. Let $X = [0\ ,\ \ 1]$. Define " $*$ " on X by

$$x * y = \begin{cases} 0 & \text{if } x \leq y, \\ x & \textit{otherwise} \end{cases}$$

Then $(X, *, 0)$ is a bounded BCK-algebra. By some manipulations we get that $\beta_\alpha = \{\emptyset, X, \{0\}\}$. Consider $A = [0, 1/2)$. We see that $1/3 \in A$, but there is not any $M \in \beta_o$ such that $1/3 \in M \subseteq A$.

Open problem. Is there a BCK-algebra $(X, *, 0)$ such that (X, τ_α) does not be finer than (X, τ_o)?

Conclusion. The paper has shown that $\{0\}$ is an open subset of the topological spaces (X, τ_α) and $((X, \tau_\gamma))$. The authors have proved that topological spaces (X, τ_α) and $((X, \tau_\gamma))$ are Bair, locally connected and separable. Finally they have shown that all proper subsets of (X, τ_α) and $((X, \tau_\gamma))$ are connected, whenever they are closed or open.

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