Abstract. Let $X$ be a topological space and $R$ be a subring of $\mathbb{R}^X$. By determining some special topologies on $X$ associated with the subring $R$, characterizations of maximal fixed and maximal $g$-ideals in $R$ of the form $M_x(R)$ are given. Moreover, the classes of $z_R$-ideals and $z^R$-ideals are introduced in $R$ which are topological generalizations of $z$-ideals and $z^0$-ideals of $C(X)$, respectively. Various characterizations of these ideals are established. Also, coincidence of $z_R$-ideals with $z$-ideals and $z^R$-ideals with $z^0$-ideals in $R$ are investigated. It turns out that some fundamental statements in the context of $C(X)$ are extended to the subrings of $\mathbb{R}^X$.

Keywords: $Z(R)$-topology, $Coz(R)$-topology, $g$-ideal, $z_R$-ideal, $z^R$-ideal, invertible subring.


1. Introduction

For a topological space $X$, $\mathbb{R}^X$ denotes the algebra of all real-valued functions and $C(X)$ (resp., $C^*(X)$) denotes the subalgebra of $\mathbb{R}^X$ consisting of all continuous functions (resp., bounded continuous functions). Moreover, we use $R$ to denote a unital subring of $\mathbb{R}^X$. Note that topological spaces which are considered in this paper are not necessarily Tychonoff. For each $f \in \mathbb{R}^X$, 

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$Z(f) = \{ x \in X : f(x) = 0 \}$ denotes the zero-set of $f$ and $Coz(f)$ denotes the complement of $Z(f)$ with respect to $X$. We denote by $Z(R)$ the collection of all the zero-sets of elements of $R$, we use $Z(X)$ instead of $Z(C(X))$. We denote by $M_\mu(R)$ the set $\{ f \in R : x \in Z(f) \}$, $M_\alpha(C(X))$ is denoted by $M_\alpha$. The subring $R$ is called invertible, if $f \in R$ and $Z(f) = \emptyset$ implies that $f$ is invertible in $R$. Moreover, $R$ is called a lattice-ordered subring if it is a sublattice of $\mathbb{R}^X$ (i.e., $f \land g$ and $f \lor g$ are in $R$ for each $f, g \in R$). It is clear that $C(X)$ is an invertible lattice-ordered subring of $\mathbb{R}^X$. However, the same statement does not hold for $C^*(X)$. A proper ideal $I$ of $R$ is called a growing ideal, briefly, a $g$-ideal, if contains no invertible element of $\mathbb{R}^X$, i.e., $Z(f) \neq \emptyset$ for each $f \in I$. It is evident that a subring $R$ is invertible if and only if every ideal of $R$ is a $g$-ideal. Clearly, $M^{p*}$, for each $p \in \beta X \setminus \nu X$, is not a $g$-ideal of $C^*(X)$. An ideal $I$ of $R$ is called fixed if $\bigcap_{f \in I} Z(f) \neq \emptyset$, otherwise, it is called free. By a maximal fixed ideal of $R$, we mean a fixed ideal which is maximal in the set of all fixed ideals of $R$. An ideal $I$ in a commutative ring $S$ is called a $z$-ideal (resp., $z^*$-ideal) if $M_a(S) \subseteq I$ (resp., $P_a(S) \subseteq I$), for each $a \in I$, where $M_a(S)$ (resp., $P_a(S)$) denotes the intersection of all the maximal (resp., minimal prime) ideals of $S$ containing $a$. It is well-known that in $C(X)$ an ideal $I$ is a $z$-ideal (resp., $z^*$-ideal) if and only if whenever $Z(f) \subseteq Z(g)$ (resp., $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$), $f \in I$ and $g \in C(X)$, then $g \in I$.

This paper consists of 4 sections. Section 1, as we have already noticed, is the introduction, in which we determine two special topologies on $X$ which the subring $R$ generate, namely, $Z(R)$-topology and $Coz(R)$-topology. Comparison and coincidence of these topologies are studied. Section 2 deals with maximal ideals in $R$, specially, maximal fixed and maximal $g$-ideals. Using the $Z(R)$-topology, characterizations of maximal fixed ideals of $R$, which are of the form $M_\mu(R)$, are given. Moreover, relations between mapping “$x \longrightarrow M_\mu(R)$” and the separation properties of the topological space $(X, \tau_{Z(R)})$ will be found. In section 3, we introduce the notion of $z_R$-ideal in a subring $R$ as a natural topological generalization of the notion of $z$-ideal in $C(X)$. Various characterizations of these ideals via $Z(R)$-topology are given and relations between $z_R$-ideals and $z$-ideals in $R$ (by their algebraic descriptions) are discussed. Section 4 deals with $z_R^*$-ideals of $R$ which are natural topological generalizations of $z^*$-ideals of $C(X)$. Using $Coz(R)$-topology, coincidence of $z_R^*$-ideals with $z^*$-ideals of $R$ (by their algebraic descriptions) are established.

**Definition 1.1.** For each subring $R$ of $\mathbb{R}^X$, clearly, $Z(R)$ and $Coz(R)$ constitute bases for some topologies on $X$. The induced topologies are called $Z(R)$-topology and $Coz(R)$-topology, respectively, and are denoted by $\tau_{Z(R)}$ and $\tau_{Coz(R)}$, respectively.

In the next three statements we compare these topologies. Note that two subsets $S_1, S_2$ of $\mathbb{R}^X$ are called zero-set equivalent, if $Z(S_1) = Z(S_2)$.
Proposition 1.2. Let $R$ be a subring of $\mathbb{R}^X$, if $S$ and $C(R)$ are zero-set equivalent subsets of $\mathbb{R}$ and $gof \in R$ for each $f \in R$ and each $g \in S$, then $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$ and the equality does not hold, in general.

Proof. We are to show that $Coz(R) \subseteq \tau_{Z(R)}$. If $x \not\in Z(f)$ where $f \in R$, then there is a $g$ in $S$ such that $f(x) \in Z(g)$ and $f^{-1}(Z(g)) \cap Z(f) = \emptyset$. Therefore, $gof \in R$, $x \in Z(gof)$ and $Z(gof) \cap Z(f) = \emptyset$ which proves the inclusion. Now, we show that the inclusion may be proper. Let $(X, \tau_X)$ be a Tychonoff space which has at least one non-open zero-set $Z$. Set $R = C(X)$, then $\tau_{Coz(R)} = \tau_X$, whereas $Z \not\subseteq \tau_X$ and hence, $\tau_{Coz(R)} \not\subseteq \tau_{Z(R)}$. □

Proof of the following proposition is standard.

Proposition 1.3. The following statements are equivalent.

1. $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$.
2. Every $Z \in \tau(R)$ is clopen under $\tau(R)$-topology.

The annihilator of $f \in R$ in $\tau$ is defined to be the set $\{g \in R : fg = 0\}$ and is denoted by $Ann_R(f)$. A simple reasoning shows that if $X$ is equipped with the $Coz(R)$-topology, then $Ann_R(f) = \{g \in R : Coz(g) \subseteq \text{int}_X Z(f)\} = \{g \in R : cl_X (Coz(g)) \subseteq Z(f)\}$.

Proposition 1.4. The following statements are equivalent.

1. $\tau_{Z(R)} \subseteq \tau_{Coz(R)}$.
2. $Z(f)$ is clopen in $(X, \tau_{Coz(R)})$ for every $f \in R$.
3. For each $f \in R$, $Z(f) = \bigcup_{g \in Ann_R(f)} Coz(g)$.
4. For each $f \in R$, $(Ann_R(f), f)$ is a free ideal.

Proof. The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) are clear.

(c)$\Rightarrow$(d). This clear by the hypothesis and the fact that whenever $f \in R$ and $I$ is an ideal of $R$, then $\bigcap_{h \in f} Z(h) = \bigcap_{g \in f} (Z(f) \cap Z(g))$.

(d)$\Rightarrow$(a). Let $f \in R$ and $x \in Z(f)$. By (d), there exists $g \in Ann_R(f)$ such that $x \not\in Z(f) \cap Z(g)$. Hence, $x \not\in Z(g)$ and $x \in Coz(g) \subseteq Z(f)$ and so $Z(f) \in \tau_{Coz(R)}$. □

An immediate consequence of Propositions 1.3 and 1.4 is that $\tau_{Coz(R)} = \tau_{Z(R)}$ if and only if $Z(f)$ is clopen under both $Z(R)$-topology and $Coz(R)$-topology, for each $f \in R$.

2. Characterization of Maximal Fixed Ideals in Subrings

We remind that maximal fixed ideals of $C(X)$ coincide with its fixed maximal ideals and are of the form $M_x = \{f \in C(X) : f(x) = 0\}$, where $x \in X$. This fact is generalized for some special subalgebras of $C(X)$, such as intermediate subalgebras (subalgebras of $C(X)$ containing $C^+(X)$, see [7]), $C_c(X)$ (the subalgebra of $C(X)$ consisting of all functions with countable image, see [9]) and the subalgebras of the form $R + I$ where $I$ is an ideal of $C(X)$, see [13].
We will show that the same statement does not hold for arbitrary subrings of $\mathbb{R}^X$, in general.

**Remark 2.1.** (a) Every maximal fixed ideal and fixed maximal ideal of $R$ is of the form $M_x(R) = \{ f \in R : f(x) = 0 \}$ for some $x \in X$. However, parts (1) and (2) of Example 2.2 show that the ideals $M_x(R)$ are not necessarily maximal ideals or even maximal fixed ideals in $R$.

(b) Every maximal fixed ideal is both a maximal fixed ideal and a maximal $g$-ideal. But the converse is not necessarily true, in general, see part (1) of Example 2.2 and Example 2.3.

(c) A maximal fixed ideal need not be a maximal $g$-ideal, see Example 2.3.

(d) Every fixed maximal $g$-ideal is a maximal fixed ideal.

**Example 2.2.** (1) Let $X$ be a Tychonoff space, $x \in X$ and $R = \mathbb{Z} + M_x$. Then $M_x(R) = M_y$ is not a maximal ideal in $R$, since $2\mathbb{Z} + M_x$ is a proper ideal of $R$ and $M_x \subseteq 2\mathbb{Z} + M_x$. Therefore, $M_x(R)$ is a maximal fixed ideal and a maximal $g$-ideal which is not a maximal ideal.

(2) Let $X$ be a topological space with more than one point and $a \in X$. Also, let $t \in \mathbb{R}$ be a transcendental number and define $f : X \rightarrow \mathbb{R}$ by $f(a) = 0$ and $f(x) = t$, for every $x \neq a$. Set $R = \{ \sum_{i=0}^{n} m_i f^i : n \in \mathbb{N} \cup \{0\}, m_i \in \mathbb{Z} \}$. Evidently, $M_a(R) = (f)$ and $M_x(R) = \{0\}$, for every $x \neq a$. Therefore, $M_x(R)$ is not a maximal fixed ideal for any $x \neq a$.

In the next example we construct a subring $R$ such that, for some $x \in X$, $M_x(R)$ is a maximal fixed ideal which is not a maximal $g$-ideal.

**Example 2.3.** Let $X = \mathbb{R}$, $a \in \mathbb{R} \setminus \mathbb{Q}$, $b \in \mathbb{R} \setminus \{0\}$ and $t$ be a transcendental number. For every $\epsilon > 0$, define $f_\epsilon : X \rightarrow \mathbb{R}$ by $f_\epsilon(x) = 0$, if $|x - a| < \epsilon$ and $f_\epsilon(x) = b$, if $|x - a| \geq \epsilon$. Also, define $f : X \rightarrow \mathbb{R}$ by $f(x) = 0$, if $x \in \mathbb{Q}$ and $f(x) = t$, if $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $R$ be the algebra over $\mathbb{Q}$ generated by $\{f_\epsilon : \epsilon > 0\} \cup \{f, 1\}$. Evidently, $R$ is a subring of $\mathbb{R}^X$, and $M_a(R)$ equals to $(f_a)$ which is not a maximal ideal. It is easy to see that $M_x(R)$ is a maximal fixed ideal and $M_y(R) = I$, where $I$ is the ideal generated by $\{f_\epsilon : \epsilon > 0\}$. Clearly, $Z(f) \cap Z(g) \neq \emptyset$, for all $g \in I$. Hence $J = (I, f)$ is a $g$-ideal which strictly contains $I$. Therefore, $I$ is not a maximal $g$-ideal.

**Proposition 2.4.** The following statements hold for a subring $R$ of $\mathbb{R}^X$.

(a) $M_x(R)$ is a maximal $g$-ideal if and only if whenever $Z \in Z(R)$ and $x \notin Z$, then $x \notin cl_{Z(R)}Z$.

(b) For each $x \in X$, $M_x(R)$ is a maximal $g$-ideal if and only if every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Proof.** (a $\Rightarrow$). Let $f \in R$ and $x \notin Z(f)$, thus, the ideal $(M_x(R), f)$ contains an invertible element of $\mathbb{R}^X$. Hence, there are $g \in M_x(R)$ and $h \in R$ such that $Z(g + fh) = \emptyset$. Consequently, $x \notin Z(g)$ and $Z(f) \cap Z(g) = \emptyset$. 


(a $\iff$). Assume that $f \not\in M_x(R)$. Then there is some $g \in R$ such that $x \in Z(g)$ and $Z(f) \cap Z(g) = Z(f^2 + g^2) = \emptyset$. Hence, $(M_x(R), f)$ contains an invertible element of $\mathbb{R}^X$. Also, clearly, $M_x(R)$ is a $g$-ideal. Thus, $M_x(R)$ is a maximal $g$-ideal.

(b). An easy consequence of (a).

**Corollary 2.5.** If $M_x(R)$ is a maximal ideal for each $x \in X$, then every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Corollary 2.6.** Let $R$ be an invertible subring. Then every $Z \in Z(R)$ is clopen under $Z(R)$-topology if and only if $M_x(R)$ is a maximal ideal for each $x \in X$.

**Proof.** By our hypothesis and Proposition 2.4, this is clear. \(\Box\)

The following lemma is a restatement of the fact that the transcendental degree of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable, see [14].

**Lemma 2.7.** Let $S = \mathbb{Q}[y_1, \ldots, y_n]$ be the ring of $n$-variable polynomials with rational coefficients. Then there exists an uncountable set $X$ of transcendental numbers for which $F(a_1, \cdots, a_n) \neq 0$, for every distinct elements $a_1, \cdots, a_n$ of $X$ and every $F \in S$.

The following example shows that the converse of Corollary 2.5 does not hold, in general.

**Example 2.8.** Let $S$ be the polynomial ring $\mathbb{Q}[y_1, \ldots, y_n]$, where $n \in \mathbb{N}$ and $n > 1$. By Lemma 2.7, there exists an infinite set of transcendental numbers $X$ for which $F(a_1, \cdots, a_n) \neq 0$, for every $a_1, \cdots, a_n \in X$ and every $F \in S$. For each $a \in X$, define the function $f_a : X \to \mathbb{R}$ by $f_a(a) = 0$ and $f_a(x) = x$ for each $x \neq a$. Now, set

$$R = \{F(f_{a_1}, \ldots, f_{a_n}) : F \in S, \ n \in \mathbb{N}, \ a_1, \ldots, a_n \in X\}.$$ 

Hence, $M_x(R) = (f_a)$, for each $a \in X$, which is not a maximal ideal. However, every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Proposition 2.9.** If $R$ is a subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal $g$-ideal and a maximal fixed ideal for every $x \in X$.

**Proof.** It suffices to prove that every element of $Z(R)$ is closed under $Z(R)$-topology. To this aim, suppose that $a \in X$ and $a \not\in Z(f)$, for some $f \in R$. Put $g = f - f(a)$. Clearly, $Z(g) \in Z(R)$, $a \in Z(g)$ and $Z(g) \cap Z(f) = \emptyset$. \(\Box\)

**Corollary 2.10.** If $R$ is an invertible subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal ideal for each $x \in X$.

The converse of Corollary 2.10 does not hold, in general. For example, let $R$ denote the collection of all single variable polynomials over $\mathbb{R}$. Then, $M_x(R)$ is the maximal ideal $(x - r)$ for each $r \in \mathbb{R}$. However, $f = x^2 + 1$ is invertible in
which is not invertible in \( R \). Note that the subalgebras \( C_c(X) \) and \( R + I \), for each ideal \( I \) in \( C(X) \), satisfy Corollary 2.10 and so \( M_x(C_c(X)) \) and \( M_x(R + I) \) are maximal ideals of \( C_c(X) \) and \( R + I \), respectively, for each \( x \in X \). Remark that in parts (b) and (e) of the following proposition we assume that “\( = \)” is a partial order on \( X \).

**Proposition 2.11.** For a subring \( R \) of \( \mathbb{R}^X \), the following statements hold.

(a) The mapping \( x \mapsto M_x(R) \) is a one-one correspondence if and only if \((X, \tau_{Z(R)})\) is a \( T_0 \)-space.

(b) The mapping \( x \mapsto M_x(R) \) is an order isomorphism between \( X \) and the set of all maximal fixed ideals of \( R \) if and only if \((X, \tau_{Z(R)})\) is a \( T_1 \)-space.

(c) For every two distinct elements \( x, y \in X \), \( M_x(R) + M_y(R) \) is not a \( g \)-ideal if and only if \((X, \tau_{Z(R)})\) is a \( T_2 \)-space.

(d) The mapping \( x \mapsto M_x(R) \) is an order embedding between \( X \) and the set of all maximal \( g \)-ideals of \( R \) if and only if \((X, \tau_{Z(R)})\) is a \( T_0 \)-space and every element of \( Z(R) \) is clopen under \( Z(R) \)-topology.

**Proof.** (a). Let \( x, y \) be distinct points of \( X \), so \( M_x(R) \neq M_y(R) \), say \( M_x(R) \not\subseteq M_y(R) \). Hence, there exists \( f \in M_x(R) \setminus M_y(R) \). Thus \( x \in Z(f) \) and \( y \notin Z(f) \). It is clear that the above reasoning is reversible and hence we are done.

(b) \( \Rightarrow \). Suppose that \( x \) and \( y \) are two distinct points of \( X \). Since \( M_x(R) \subseteq M_y(R) \), there exists \( f \in M_x(R) \setminus M_y(R) \). Consequently, \( x \in Z(f) \) and \( y \notin Z(f) \).

(b) \( \Leftarrow \). Suppose that \( x \in X \) and \( I \) is a fixed ideal in \( R \) containing \( M_x(R) \). Take \( y \in \bigcap_{f \in I} Z(f) \). Clearly, \( M_x(R) \subseteq I \subseteq M_y(R) \). It suffices to show \( x = y \). Suppose that \( x \neq y \) and seek a contradiction. By our hypothesis, there exists \( f \in R \) such that \( x \in Z(f) \) and \( y \notin Z(f) \). Therefore, \( M_x(R) \not\subseteq M_y(R) \) and this is a contradiction. Now, by part (a), the proof is complete.

(c). For any two distinct points \( x, y \in X \), clearly, \( M_x(R) + M_y(R) \) is not a \( g \)-ideal if and only if there exist \( f \in M_x(R) \) and \( g \in M_y(R) \) such that \( Z(f) \cap Z(g) = \emptyset \).

(d) \( \Rightarrow \). By part (a), clearly, \((X, \tau_{Z(R)})\) is a \( T_0 \)-space. Now, Suppose that \( f \in R \) and \( x \notin Z(f) \). Since \( M_x(R) \) is a maximal \( g \)-ideal, it follows that \((M_x(R), f)\) has an invertible element of \( \mathbb{R}^X \) and so there exists \( g \in M_x(R) \), such that \( Z(g) \cap Z(f) = \emptyset \). Thus, \( Z(f) \) is closed and hence is clopen under \( Z(R) \)-topology.

(d) \( \Leftarrow \). Suppose that \( x \in X \), it suffices to show that \( M_x(R) \) is a maximal \( g \)-ideal. Assume that \( I \) is an ideal which properly contains \( M_x(R) \). Hence, there exists \( f \in I \) such that \( x \notin Z(f) \). By our hypothesis, there is \( g \in R \) such that \( x \in Z(g) \) and \( Z(g) \cap Z(f) = \emptyset \). Therefore, \( Z(f^2 + g^2) = \emptyset \) and \( f^2 + g^2 \in I \), hence, \( I \) is not a \( g \)-ideal.

It is easy to see that \( M_x(R) \), for each \( x \in X \), is a prime ideal of \( R \) and thus the hull-kernel topology may be defined on the family \( \{M_x(R) : x \in X\} \).
By considering this space, the next statement gives a relation between $Z(R)$-topology on $X$ and points of $X$.

**Proposition 2.12.** Let $R$ be a subring of $\mathbb{R}^X$ and $X$ equipped with the $\text{Coz}(R)$-topology. Then the mapping $\Phi : X \to \{M_x(R) : x \in X\}$ defined by $x \mapsto M_x(R)$ is a homeomorphism if and only if $(X, \tau_{Z(R)})$ is a $T_0$-space.

*Proof.* By part (a) of Theorem 2.12, $\Phi$ is a one-one correspondence if and only if $(X, \tau_{Z(R)})$ is a $T_0$-space. Also, if $f \in R$ and $x \in Z(f)$, then $f \in M_x(R)$ which means that basic closed sets of $X$ equipped with the $\text{Coz}(R)$-topology are mapped to the basic closed sets in $\{M_x(R) : x \in X\}$ equipped with the hull-kernel topology by the mapping $\Phi$ and therefore, it is a homeomorphism. \qed

3. $z_R$-Ideals and $z$-Ideals in Subrings

In this section we introduce $z_R$-ideals in a subring $R$ and via the $Z(R)$-topology and maximal $g$-ideals of $R$, various characterizations of these ideals are given.

**Definition 3.1.** A subset $F$ of $Z(R)$ is called $z_R$-filter on $X$, if

(a) $\emptyset \not\in F$. 
(b) If $Z_1, Z_2 \in F$, then $Z_1 \cap Z_2 \in F$. 
(c) If $Z_1 \in F$, $Z_2 \in Z(R)$ and $Z_1 \subseteq Z_2$, then $Z_2 \in F$. 

Moreover, $F$ is called a prime $z_R$-filter, if whenever $Z_1 \cup Z_2 \in F$, then $Z_1 \in F$ or $Z_2 \in F$ for each $Z_1, Z_2 \in Z(R)$. Also, $F$ is called a $z_R$-ultrafilter, if $F$ is maximal among $z_R$-filters on $X$.

The following proposition immediately follows from Definition 3.1.

**Proposition 3.2.** For any subring $R$, the following statements hold.

(a) $I \subseteq R$ is a $g$-ideal in $R$ if and only if $Z_R(I) = \{Z(f) : f \in I\}$ is a $z_R$-filter on $X$. 
(b) $F$ is a $z_R$-filter on $X$ if and only if $Z_R^{-1}(F) = \{f \in R : Z(f) \in F\}$ is a $g$-ideal. 
(c) $F$ is a prime $z_R$-filter on $X$ if and only if $Z_R^{-1}(F)$ is a prime $g$-ideal. 
(d) $A$ is a $z_R$-ultrafilter on $X$ if and only if $Z_R^{-1}(A)$ is a maximal $g$-ideal. 
(e) If $M$ is a maximal $g$-ideal in $R$, then $Z_R(M)$ is a $z_R$-ultrafilter on $X$.

It is easy to see that for an ideal $I$ of $R$ we always have $I \subseteq Z_R^{-1}(Z_R(I))$ and the inclusion may be proper. We call an ideal $I$ in $R$ a $z_R$-ideal, if $I = Z_R^{-1}(Z_R(I))$. It follows that every $z_R$-ideal is semiprime and arbitrary intersections of $z_R$-ideals is a $z_R$-ideal. Also, the zero ideal, the ideals of the form $M_x(R)$, maximal $g$-ideals and $Z^{-1}(F)$, for each $z_R$-filter $F$, are all $z_R$-ideals of $R$. For each $f \in R$, the intersection of all the maximal ideals, maximal $g$-ideals and maximal fixed ideals of $R$ containing $f$ are denoted by $M_f(R)$, $MG_f(R)$ and $MF_f(R)$, respectively. It is easy to observe that $MG_f(R)$ is a $z_R$-ideal for each $f \in R$. 


Obviously, \( MG_f \cap MG_g = MG_{fg} , \quad MF_f \cap MF_g = MF_{fg} , \quad MG_f^2 + g^2 = MG_{(f,g)} \) and \( MF_f^2 + g^2 = MF_{(f,g)} \) for all \( f, g \in R \).

**Proposition 3.3.** Let \((X, \tau_{Z(R)})\) be a \( T_1 \)-space. Then the following statements hold.

(a) The following statements are equivalent.
   1. \( g \in MF_f(R) \).
   2. \( MF_g(R) \subseteq MF_f(R) \).
   3. \( Z(f) \subseteq Z(g) \).
   (b) \( MF_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \} \).
   (c) An ideal \( I \) of \( R \) is a \( z_R \)-ideal if and only if \( MF_f(R) \subseteq I \) for every \( f \in I \).

**Proof.**

(a: \( 1 \Rightarrow 2 \)). Evident.

(a: \( 2 \Rightarrow 3 \)). Let \( x \in Z(f) \). Then \( f \in M_x(R) \) and thus \( MF_g(R) \subseteq MF_f(R) \subseteq M_x(R) \). This implies \( g \in M_x(R) \) and hence \( x \in Z(g) \).

(a: \( 3 \Rightarrow 1 \)). If \( g \notin MF_f(R) \), then there exists \( x \in X \) such that \( f \in M_x(R) \) and \( g \notin M_x(R) \). Therefore, \( x \in Z(f) \setminus Z(g) \) and so \( Z(f) \subseteq Z(g) \).

(b) and (c) obviously follow from part (a). \( \square \)

**Lemma 3.4.** Assume that every \( Z \in Z(R) \) is clopen under \( Z(R) \)-topology. Then \( MG_f(R) = MF_f(R) \), for every \( f \in R \).

**Proof.** Suppose that \( f \in R \). By part (b) of Proposition 2.4, \( M_x(R) \) is a maximal \( g \)-ideal for each \( x \in X \). Consequently, \( MG_f(R) \subseteq MF_f(R) \). Now, assume that \( g \notin MG_f(R) \). Hence, there exists a maximal \( g \)-ideal \( M \) in \( R \) such that \( f \in M \) and \( g \notin M \). Thus, there exists \( h \in M \) such that \( Z(g) \cap Z(h) = \emptyset \). Since \( f^2 + h^2 \in M \) and \( M \) is a \( g \)-ideal, there is a point \( x \in Z(f^2 + h^2) = Z(f) \cap Z(h) \). Clearly, \( g \notin M_x(R) \) and \( f \in M_x(R) \). Therefore, \( g \notin MF_f(R) \). \( \square \)

Proposition 3.3 and Lemma 3.4 imply the next statement.

**Proposition 3.5.** Let \((X, \tau_{Z(R)})\) be a \( T_1 \)-space and every \( Z \in Z(R) \) be a clopen set under \( Z(R) \)-topology. Then the following statements hold.

(a) The following statements are equivalent.
   1. \( g \in MG_f(R) \).
   2. \( MG_g(R) \subseteq MG_f(R) \).
   3. \( Z(f) \subseteq Z(g) \).
   (b) \( MG_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \} \).
   (c) An ideal \( I \) of \( R \) is a \( z_R \)-ideal if and only if \( MG_f(R) \subseteq I \) for every \( f \in I \).

The following corollary follows from Corollary 2.6 and Proposition 3.5.

**Corollary 3.6.** Let \( R \) be an invertible subalgebra of \( \mathbb{R}^X \). Then the following statements hold.
(a) The following conditions are equivalent:
(1) \( g \in M_f(R) \).
(2) \( M_g(R) \subseteq M_f(R) \).
(3) \( Z(f) \subseteq Z(g) \).

(b) \( M_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \} \).

(c) An ideal \( I \) of \( R \) is \( z_R \)-ideal if and only if \( M_f(R) \subseteq I \) for every \( f \in I \).

It follows from Corollary 3.6 that for an invertible subalgebra \( R \), the notion of \( z_R \)-ideal coincides with the notion of \( z \)-ideal. The next statement extends this fact and shows that this coincidence is equivalent to invertibility of \( R \).

**Theorem 3.7.** Let \( R \) be a subring of \( \mathbb{R}^X \). The following statements are equivalent.

(a) Every maximal ideal in \( R \) is a \( g \)-ideal.

(b) Every maximal \( g \)-ideal of \( R \) is a maximal ideal and if \( J \) is a maximal ideal of \( R \), then every maximal element in the set of \( g \)-ideals contained in \( J \) is a prime ideal.

(c) Every maximal ideal in \( R \) is a \( g \)-ideal.

(d) \( R \) is an invertible subring.

(e) Every \( z \)-ideal of \( R \) is a \( z_R \)-ideal.

Moreover, if \( R \) is a subalgebra and one of (a)-(c) holds, then every \( z_R \)-ideal is a \( z \)-ideal.

**Proof.** (a) \( \Rightarrow \) (b). This is clear.

(b) \( \Rightarrow \) (c). Suppose that \( M \) is a maximal ideal and \( P \) is a maximal element of \( G_M \), where \( G_M \) is the set of all \( g \)-ideals contained in \( M \). Assume that \( J \) is a maximal ideal of \( R \) containing \( P \). Then \( M \cap J = P \). As \( M \cap J \) is prime and both \( M \) and \( J \) are maximal ideals, we have \( M = J \). Hence, \( M \) is a maximal \( g \)-ideal.

(c) \( \Rightarrow \) (d). Suppose that \( Z(f) = \emptyset \) for \( f \in R \) and, on the contrary, \( f \) is a non-unit element of \( R \). Clearly, there exists a maximal ideal \( M \) of \( R \) containing \( f \). By our hypothesis, \( M \) is a \( g \)-ideal which contradicts with \( Z(f) = \emptyset \).

(d) \( \Rightarrow \) (e). Suppose that \( I \) is a \( z \)-ideal and \( Z(f) \subseteq Z(g) \) where \( f \in I \) and \( g \in R \). Since \( I \) is a \( z \)-ideal, it follows that \( M_f \subseteq I \). It suffices to prove that \( g \in M_f \). To see this, suppose that \( M \) is a maximal ideal containing \( f \). As \( R \) is invertible, \( M \) is a \( g \)-ideal and so it is a maximal \( g \)-ideal. Obviously, \( M \) is a \( z_R \)-ideal and so \( g \in M \).

(e) \( \Rightarrow \) (a). Suppose that \( M \) is a maximal ideal and, on the contrary, \( M \) is not a \( g \)-ideal. Thus, there exists \( f \in M \) such that \( Z(f) = \emptyset \). By (e), \( M \) is a \( z_R \)-ideal and since \( f \in M \), it follows that \( M = R \), which is a contradiction.

Now, suppose that one of (a)-(c) holds, \( R \) is a subalgebra and \( I \) is a \( z_R \)-ideal of \( R \). By our hypothesis, \( M F_f(R) = M_f(R) \) for every \( f \in R \), and thus we are done. \( \square \)
It is well-known that every minimal prime ideals over a $z$-ideal is also a $z$-ideal, see [10, Theorem 14.7]. The same statement holds for $z_R$-ideals as the following proposition shows.

**Proposition 3.8.** Let $I$ be a $z_R$-ideal of $R$ and $P$ a prime ideal in $R$ minimal over $I$. Then $P$ is a $z_R$-ideal.

**Proof.** Assume that $Z(f) = Z(g)$ and $f \in P$. Thus, there exists $h \notin P$, such that $fh \in I$. Since $Z(fh) = Z(gh)$ and $I$ is a $z_R$-ideal, it follows that $gh \in I \subseteq P$. As $h \notin P$, clearly, this implies that $g \in P$. $\square$

An immediate consequence of Proposition 3.8 is that every minimal prime ideal in a subring $R$ is a $z_R$-ideal. By the following statement, we extend some fundamental statements about $z$-ideals in the literature of $C(X)$ to the subrings of $\mathbb{R}^X$, namely, [10, 2.9, 5.3 and 5.5]. The proofs are left to the reader.

**Proposition 3.9.** Let $R$ be a lattice-ordered subring of $\mathbb{R}^X$ and $I$ be a $z_R$-ideal in $R$. Then the following statements hold.

(a) The following statements are equivalent

1. $I$ is a prime ideal;
2. $I$ contains a prime ideal;
3. if $fg = 0$, then $f \in I$ or $g \in I$;
4. for each $f \in R$, there is a $Z \in Z_R(I)$ on which $f$ does not change sign.

(b) Every prime $g$-ideal of $R$ is contained in a unique maximal $g$-ideal.

(c) If $P$ is a prime ideal of $R$, then $Z_R(P)$ is a prime $z_R$-filter on $X$.

(d) If $P$ is a prime $z_R$-filter on $X$, then $Z_R^{-1}(P)$ is a prime ideal in $R$.

(e) Every $z_R$-ideal of $R$ is absolutely convex.

Thus, if $I$ is an absolutely convex ideal of $R$, then $R/I$ is a lattice ring.

(f) $I(f) \geq 0$ if and only if $f \geq 0$ on some $Z \in Z_R(I)$.

(g) Suppose that there exists $Z \in Z_R(I)$ such that $f(x) > 0$, for every $x \in Z$, then $I(f) > 0$. The converse is true whenever $I$ is a maximal $g$-ideal.

4. $z_R^o$-Ideals and $z^o$-Ideals in Subrings

In this section we generalize the concept of $z^o$-ideals of $C(X)$ to the subrings of $\mathbb{R}^X$ and introduce $z_R^o$-ideal. Coincidence of $z_R^o$-ideals with $z^o$-ideals of $R$ is discussed. Note that, for each element $f$ of a commutative rings $S$, we use $P_f(S)$ to denote the intersection of all the minimal prime ideals in $S$ containing $f$.

**Definition 4.1.** An ideal $I$ of a subring $R$ of $\mathbb{R}^X$ is called a $z_R^o$-ideal, if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$, where $f \in I$ and $g \in R$, implies $g \in I$.

The following statement investigates some characterizations of $z_R^o$-ideals in subrings.

**Theorem 4.2.** Let $R$ be a subring of $\mathbb{R}^X$ and $I$ be an ideal in $R$. The following statements are equivalent.
(a) $I$ is a $z^*_R$-ideal.

(b) Whenever $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$, then $g \in I$. 

(c) $R \cap P_f(C) \subseteq I$ for each $f \in I$. 

(d) Whenever $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$, then $g \in I$. 

Proof. (a$\Rightarrow$b). First note that by [3, Lemma 2.1] we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ if and only if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ for each $f, g \in C(X)$. Now, let $I$ be a $z^*_R$-ideal in $R$ and $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$. Thus, by our hypothesis, we have $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ which implies that $g \in I$.

(b$\Rightarrow$c). By [3, Proposition 2.3], we have $P_f(C) = \{g \in C(X) : \text{Ann}_C(f) \subseteq \text{Ann}_C(g)\}$. Thus the proof is evident.

(c$\Rightarrow$d). Let $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$. As $f \in I$, by our hypothesis, $P_f(C) \cap R \subseteq I$ and thus $P_g(C) \cap R \subseteq I$ which implies that $g \in I$.

(d$\Rightarrow$a). Let $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ where $f \in I$ and $g \in R$. Therefore, by [3, Lemma 2.1], we have $P_f(C) \subseteq P_g(C)$ and hence $P_f(C) \cap R \subseteq P_g(C) \cap R$. Thus we are done by our hypothesis. 

\[\square\]

Lemma 4.3. Let $R$ be a subring of $\mathbb{R}^X$, then for each $f \in R$ we have $P_f(C) \subseteq P_f(R)$. 

Proof. Let $g \in P_f(C)$. By [3, Proposition 2.3], we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$. Therefore, $\text{Ann}_R(f) = \text{Ann}_C(f) \cap R \subseteq \text{Ann}_C(g) \cap R = \text{Ann}_R(g)$. Thus, by [2, Proposition 1.5] we are done. 

\[\square\]

Theorem 4.4. Let $R$ be a subring of $\mathbb{R}^X$. Then every $z^*_R$-ideal in $R$ is a $z^0$-ideal if and only if $P_f(R) = P_f(C)$ for each $f \in R$. 

Proof. ($\Rightarrow$). Assume on the contrary that there exists some $f \in R$ such that $P_f(R) \neq P_f(C)$. Thus, using Theorem 4.2 we have $P_f(C) \subseteq P_f(R)$. Again by Theorem 4.2, $P_f(C) \cap R$ is a $z^*_R$-ideal in $R$. Also, it is clear that this ideal is not a $z^0$-ideal, since $P_f(R) \subseteq P_f(C) \cap R$.

($\Leftarrow$). Let $I$ be a $z^*_R$-ideal in $R$ and $f \in I$. By Theorem 4.2, $P_f(C) \cap R \subseteq I$. Thus, by our hypothesis, $P_f(R) \subseteq I$ which means that $I$ is a $z^0$-ideal in $R$. 

\[\square\]

From Theorem 4.2 it follows that every $z^0$-ideal in a subring $R$ is a $z^*_R$-ideal. However, the converse of this fact does not hold, in general. The following example gives an example of a subring $R$ which has a $z^*_R$-ideal that is not a $z^0$-ideal.

Example 4.5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$. It is clear that $f \in C(\mathbb{R})$. Now, let $R = \{\sum_{i=0}^{n} r_i f^i : r_i \in \mathbb{R}, n = 0, 1, ...\}$. It is easy to see that $P_f(R) = R$, however, $P_f(C) \cap R \neq R$. Also, by Theorem 4.2, $P_f(C) \cap R$ is $z^*_R$-ideal and it is clear that this ideal is not a $z^0$-ideal.
The next theorem gives a sufficient conditions on $X$ in order that $z^n_R$-ideals in a subring $R$ coincide with $z^\circ$-ideals of $R$.

**Theorem 4.6.** Let $R$ be a subring of $\mathbb{R}^X$ and $X$ be equipped with the $\text{Coz}(R)$-topology. Then an ideal $I$ in $R$ is a $z^\circ$-ideal if and only if it is a $z^n_R$-ideal.

**Proof.** Let $I$ be a $z^n_R$-ideal in $R$ and $f \in I$. As $X$ is equipped with the $\text{Coz}(R)$-topology, we have $g \in \text{Ann}_R(f)$ if and only if $\text{Coz}(g) \subseteq \text{int}_X Z(f)$ for each $f, g \in R$. Therefore, $P_f(R) = \text{Ann}_R \text{Ann}_R(f) = \{g \in R : \text{Coz}(g) \cap \text{int}_X Z(f) = \emptyset\} = \{g \in R : \text{Ann}_R(f) \subseteq \text{Ann}_R(g)\}$. Hence, $P_f(R) \subseteq I$ which means that $I$ is a $z^\circ$-ideal in $R$. This completes the proof, since, as former stated, every $z^\circ$-ideal in $R$ is a $z^n_R$-ideal. \qed

Note that the condition that $X$ is equipped with the $\text{Coz}(R)$-topology is a sufficient condition for coincidence of $z^n_R$-ideals with $z^\circ$-ideals in a given subring $R$. The next example shows that this condition is not necessary.

**Example 4.7.** Let $X = \mathbb{R} \setminus \{0\}$ with the topology inherits from the usual topology on $\mathbb{R}$. Also, let $f : X \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$. It is clear that $f \in C(X)$ and $f^2 = f$. Now, set $R = \{r + sf : r, s \in \mathbb{R}\}$. It is clear that $R$ is a subring of $C(X)$. Also, by a routine reasoning, one can proves that the only ideals of $R$ are the ideals $(0)$, $(f)$, $(1 - f)$ and $R$. Moreover, the minimal prime ideals of $R$ are only the ideals $(f)$ and $(1 - f)$. These imply that every $z^n_R$-ideal is a $z^\circ$-ideal in $R$. However, clearly, $X$ is not equipped with the $\text{Coz}(R)$-topology.

It follows from Theorem 4.6 that for an intermediate subalgebra $A(X)$ of $C(X)$, $z^n_A$-ideals coincide with $z^\circ$-ideals of $A(X)$. However, the same statement does not true for $z_A$-ideals and $z$-ideals in $A(X)$, in general, see [6, Theorem 2.2]. Moreover, Theorem 3.7 together with Theorem 4.6 imply that in the subalgebras of $C(X)$ which are of the form $\mathbb{R} + I$, where $I$ is a free ideal in $C(X)$, $zR + I$-ideals coincide with $z$-ideals of $\mathbb{R} + I$ and $z^n_R + I$-ideals coincide with $z^\circ$-ideals, too. Note that whenever $I$ is a free ideal in $C(X)$, then $\mathbb{R} + I$ determines the topology of $X$.

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**References**


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