z\textsubscript{R}-Ideals and z\textsubscript{R}\textsuperscript{\circ}-Ideals in Subrings of \(\mathbb{R}^X\)

Ali Rezaei Aliabad and Mehdi Parsinia*

Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran.

E-mail: aliabady_r@scu.ac.ir
E-mail: parsiniamehdi@gmail.com

Abstract. Let \(X\) be a topological space and \(R\) be a subring of \(\mathbb{R}^X\). By determining some special topologies on \(X\) associated with the subring \(R\), characterizations of maximal fixed and maximal \(g\)-ideals in \(R\) of the form \(M_x(R)\) are given. Moreover, the classes of \(z_R\)-ideals and \(z_R\textsuperscript{\circ}\)-ideals are introduced in \(R\) which are topological generalizations of \(z\)-ideals and \(z\textsuperscript{\circ}\)-ideals of \(C(X)\), respectively. Various characterizations of these ideals are established. Also, coincidence of \(z_R\)-ideals with \(z\)-ideals and \(z_R\textsuperscript{\circ}\)-ideals with \(z\textsuperscript{\circ}\)-ideals in \(R\) are investigated. It turns out that some fundamental statements in the context of \(C(X)\) are extended to the subrings of \(\mathbb{R}^X\).

Keywords: \(Z(R)\)-topology, \(Coz(R)\)-topology, \(g\)-ideal, \(z_R\)-ideal, \(z_R\textsuperscript{\circ}\)-ideal, invertible subring.


1. Introduction

For a topological space \(X\), \(\mathbb{R}^X\) denotes the algebra of all real-valued functions and \(C(X)\) (resp., \(C^*(X)\)) denotes the subalgebra of \(\mathbb{R}^X\) consisting of all continuous functions (resp., bounded continuous functions). Moreover, we use \(R\) to denote a unital subring of \(\mathbb{R}^X\). Note that topological spaces which are considered in this paper are not necessarily Tychonoff. For each \(f \in \mathbb{R}^X\),

*Corresponding Author

Received 12 May 2016; Accepted 14 January 2017
©2019 Academic Center for Education, Culture and Research TMU

Downloaded from ijmsi.ir at 3:37 +0430 on Saturday July 4th 2020
\( Z(f) = \{ x \in X : f(x) = 0 \} \) denotes the zero-set of \( f \) and \( \text{Coz}(f) \) denotes the complement of \( Z(f) \) with respect to \( X \). We denote by \( Z(R) \) the collection of all the zero-sets of elements of \( R \), we use \( Z(X) \) instead of \( Z(C(X)) \). We denote by \( M_s(R) \) the set \( \{ f \in R : x \in Z(f) \} \), \( M_s(C(X)) \) is denoted by \( M_s \). The subring \( R \) is called invertible, if \( f \in R \) and \( Z(f) = \emptyset \) implies that \( f \) is invertible in \( R \). Moreover, \( R \) is called a lattice-ordered subring if it is a sublattice of \( \mathbb{R}^X \) (i.e., \( f \land g \) and \( f \lor g \) are in \( R \) for each \( f, g \in R \)). It is clear that \( C(X) \) is an invertible lattice-ordered subring of \( \mathbb{R}^X \). However, the same statement does not hold for \( C^*(X) \). A proper ideal \( I \) of \( R \) is called a growing ideal, briefly, a \( g \)-ideal, if contains no invertible element of \( \mathbb{R}^X \), i.e., \( Z(f) \neq \emptyset \) for each \( f \in I \). It is evident that a subring \( R \) is invertible if and only if every ideal every ideal of \( R \) is a \( g \)-ideal. Clearly, \( M^{*p} \), for each \( p \in \beta X \setminus \nu X \), is not a \( g \)-ideal of \( C^*(X) \). An ideal \( I \) of \( R \) is called fixed if \( \bigcap_{I \subseteq I} Z(f) \neq \emptyset \), otherwise, it is called free. By a maximal fixed ideal of \( R \), we mean a fixed ideal which is maximal in the set of all fixed ideals of \( R \). An ideal \( I \) in a commutative ring \( S \) is called a \( z \)-ideal (resp., \( z^a \)-ideal) if \( M_a(S) \subseteq I \) (resp., \( P_a(S) \subseteq I \)), for each \( a \in I \), where \( M_a(S) \) (resp., \( P_a(S) \)) denotes the intersection of all the maximal (resp., minimal prime) ideals of \( S \) containing \( a \). It is well-known that in \( C(X) \) an ideal \( I \) is a \( z \)-ideal (resp., \( z^a \)-ideal) if and only if whenever \( Z(f) \subseteq Z(g) \) (resp., \( \text{int}_X Z(f) \subseteq \text{int}_X Z(g) \)), \( f \in I \) and \( g \in C(X) \), then \( g \in I \).

This paper consists of 4 sections. Section 1, as we have already noticed, is the introduction, in which we determine two special topologies on \( X \) which the subring \( R \) generate, namely, \( Z(R) \)-topology and \( \text{Coz}(R) \)-topology. Comparison and coincidence of these topologies are studied. Section 2 deals with maximal ideals in \( R \), specially, maximal fixed and maximal \( g \)-ideals. Using the \( Z(R) \)-topology, characterizations of maximal fixed ideals of \( R \), which are of the form \( M_s(R) \), are given. Moreover, relations between mapping “\( x \longrightarrow M_s(R) \)” and the separation properties of the topological space \( (X, \tau_{Z(R)}) \) will be found. In section 3, we introduce the notion of \( z_R \)-ideal in a subring \( R \) as a natural topological generalization of the notion of \( z \)-ideal in \( C(X) \). Various characterizations of these ideals via \( Z(R) \)-topology are given and relations between \( z_R \)-ideals and \( z \)-ideals in \( R \) (by their algebraic descriptions) are discussed. Section 4 deals with \( z^a_R \)-ideals of \( R \) which are natural topological generalizations of \( z^a \)-ideals of \( C(X) \). Using \( \text{Coz}(R) \)-topology, coincidence of \( z^a_R \)-ideals with \( z^a \)-ideals of \( R \) (by their algebraic descriptions) are established.

**Definition 1.1.** For each subring \( R \) of \( \mathbb{R}^X \), clearly, \( Z(R) \) and \( \text{Coz}(R) \) constitute bases for some topologies on \( X \). The induced topologies are called \( Z(R) \)-topology and \( \text{Coz}(R) \)-topology, respectively, and are denoted by \( \tau_{Z(R)} \) and \( \tau_{\text{Coz}(R)} \), respectively.

In the next three statements we compare these topologies. Note that two subsets \( S_1, S_2 \) of \( \mathbb{R}^X \) are called zero-set equivalent, if \( Z(S_1) = Z(S_2) \).
Proposition 1.2. Let $R$ be a subring of $\mathbb{R}^X$, if $S$ and $C(\mathbb{R})$ are zero-set equivalent subsets of $\mathbb{R}^R$ and $gof \in R$ for each $f \in R$ and each $g \in S$, then $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$ and the equality does not hold, in general.

Proof. We are to show that $Coz(R) \subseteq \tau_{Z(R)}$. If $x \notin Z(f)$ where $f \in R$, then there is a $g$ in $S$ such that $f(x) \in Z(g)$ and $f^{-1}(Z(g)) \cap Z(f) = \emptyset$. Therefore, $gof \in R$, $x \in Z(gof)$ and $Z(gof) \cap Z(f) = \emptyset$ which proves the inclusion. Now, we show that the inclusion may be proper. Let $(X, \tau_X)$ be a Tychonoff space which has at least one non-open zero-set $Z$. Set $R = C(X)$, then $\tau_{Coz(R)} = \tau_X$, whereas $Z \notin \tau_X$ and hence, $\tau_{Coz(R)} \subsetneq \tau_{Z(R)}$.

Proof of the following proposition is standard.

Proposition 1.3. The following statements are equivalent.

(a) $\tau_{Coz(R)} \subseteq \tau_{Z(R)}$.
(b) Every $Z \in \mathbb{Z}(R)$ is clopen under $Z(R)$-topology.

The annihilator of $f \in R$ in $R$ is defined to be the set $\{g \in R : fg = 0\}$ and is denoted by $Ann_R(f)$. A simple reasoning shows that if $X$ is equipped with the $Coz(R)$-topology, then $Ann_R(f) = \{g \in R : Coz(g) \subseteq \text{int}_X Z(f)\} = \{g \in R : \text{cl}_X(Coz(g)) \subseteq Z(f)\}$.

Proposition 1.4. The following statements are equivalent.

(a) $\tau_{Z(R)} \subseteq \tau_{Coz(R)}$.
(b) $Z(f)$ is clopen in $(X, \tau_{Coz(R)})$ for every $f \in R$.
(c) For each $f \in R$, $Z(f) = \bigcup_{g \in Ann_R(f)} Coz(g)$.
(d) For each $f \in R$, $(Ann_R(f), f)$ is a free ideal.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear.

$(c) \Rightarrow (d)$. This clear by the hypothesis and the fact that whenever $f \in R$ and $I$ is an ideal of $R$, then $\bigcap_{h \in I} Z(h) = \bigcap_{g \in I} (Z(f) \cap Z(g))$.

$(d) \Rightarrow (a)$. Let $f \in R$ and $x \in Z(f)$. By (d), there exists $g \in Ann_R(f)$ such that $x \notin Z(f) \cap Z(g)$. Hence, $x \notin Z(g)$ and $x \in Coz(g) \subseteq Z(f)$ and so $Z(f) \in \tau_{Coz(R)}$.

An immediate consequence of Propositions 1.3 and 1.4 is that $\tau_{Coz(R)} = \tau_{Z(R)}$ if and only if $Z(f)$ is clopen under both $Z(R)$-topology and $Coz(R)$-topology, for each $f \in R$.

2. Characterization of Maximal Fixed Ideals in Subrings

We remind that maximal fixed ideals of $C(X)$ coincide with its fixed maximal ideals and are of the form $M_x = \{f \in C(X) : f(x) = 0\}$, where $x \in X$. This fact is generalized for some special subalgebras of $C(X)$, such as intermediate subalgebras (subalgebras of $C(X)$ containing $C^*(X)$, see [7]), $C_c(X)$ (the subalgebra of $C(X)$ consisting of all functions with countable image, see [9]) and the subalgebras of the form $R + I$ where $I$ is an ideal of $C(X)$, see [13].
We will show that the same statement does not hold for arbitrary subrings of \( \mathbb{R}^X \), in general.

**Remark 2.1.** (a) Every maximal fixed ideal and fixed maximal ideal of \( R \) is of the form \( M_x(R) = \{ f \in R : f(x) = 0 \} \) for some \( x \in X \). However, parts (1) and (2) of Example 2.2 show that the ideals \( M_x(R) \) are not necessarily maximal ideals or even maximal fixed ideals in \( R \).

(b) Every maximal fixed ideal is both a maximal fixed ideal and a maximal \( g \)-ideal. But the converse is not necessarily true, in general, see part (1) of Example 2.2 and Example 2.3.

(c) A maximal fixed ideal need not be a maximal \( g \)-ideal, see Example 2.3.

(d) Every fixed maximal \( g \)-ideal is a maximal fixed ideal.

**Example 2.2.** (1) Let \( X \) be a Tychonoff space, \( x \in X \) and \( R = \mathbb{Z} + M_x \). Then \( M_x(R) = M_x \) is not a maximal ideal in \( R \), since \( 2\mathbb{Z} + M_x \) is a proper ideal of \( R \) and \( M_x \subseteq 2\mathbb{Z} + M_x \). Therefore, \( M_x(R) \) is a maximal fixed ideal and a maximal \( g \)-ideal which is not a maximal ideal.

(2) Let \( X \) be a topological space with more than one point and \( a \in X \). Also, let \( t \in \mathbb{R} \) be a transcendental number and define \( f : X \rightarrow \mathbb{R} \) by \( f(a) = 0 \) and \( f(x) = t \), for every \( x \neq a \). Set \( R = \{ \sum_{i=0}^{n} m_i f^i : n \in \mathbb{N} \cup \{0\}, m_i \in \mathbb{Z} \} \). Evidently, \( M_a(R) = (f) \) and \( M_x(R) = \{0\} \), for every \( x \neq a \). Therefore, \( M_x(R) \) is not a maximal fixed ideal for any \( x \neq a \).

In the next example we construct a subring \( R \) such that, for some \( x \in X \), \( M_x(R) \) is a maximal fixed ideal which is not a maximal \( g \)-ideal.

**Example 2.3.** Let \( X = \mathbb{R} \), \( a \in \mathbb{R} \setminus \mathbb{Q} \), \( b \in \mathbb{R} \setminus \{0\} \) and \( t \) be a transcendental number. For every \( \epsilon > 0 \), define \( f_\epsilon : X \rightarrow \mathbb{R} \) by \( f_\epsilon(x) = 0 \), if \( |x - a| < \epsilon \) and \( f_\epsilon(x) = b \), if \( |x - a| \geq \epsilon \). Also, define \( f : X \rightarrow \mathbb{R} \) by \( f(x) = 0 \), if \( x \in \mathbb{Q} \) and \( f(x) = t \), if \( x \in \mathbb{R} \setminus \mathbb{Q} \). Let \( R \) be the algebra over \( \mathbb{Q} \) generated by \( \{ f_\epsilon : \epsilon > 0 \} \cup \{ f, 1 \} \). Evidently, \( R \) is a subring of \( \mathbb{R}^X \), and \( M_a(R) \) equals to \( (f_a) \) which is not a maximal ideal. It is easy to see that \( M_a(R) \) is a maximal fixed ideal and \( M_b(R) = I \), where \( I \) is the ideal generated by \( \{ f_\epsilon : \epsilon > 0 \} \). Clearly, \( Z(f) \cap Z(g) \neq \emptyset \), for all \( g \in I \). Hence \( J = (I, f) \) is a \( g \)-ideal which strictly contains \( I \). Therefore, \( I \) is not a maximal \( g \)-ideal.

**Proposition 2.4.** The following statements hold for a subring \( R \) of \( \mathbb{R}^X \).

(a) \( M_x(R) \) is a maximal \( g \)-ideal if and only if whenever \( Z \in Z(R) \) and \( x \notin Z \), then \( x \notin \text{cl}_{Z(R)} Z \).

(b) For each \( x \in X \), \( M_x(R) \) is a maximal \( g \)-ideal if and only if every \( Z \in Z(R) \) is clopen under \( Z(R) \)-topology.

**Proof.** (a \( \Rightarrow \)). Let \( f \in R \) and \( x \notin Z(f) \), thus, the ideal \( (M_x(R), f) \) contains an invertible element of \( \mathbb{R}^X \). Hence, there are \( g \in M_x(R) \) and \( h \in R \) such that \( Z(g + fh) = \emptyset \). Consequently, \( x \in Z(g) \) and \( Z(f) \cap Z(g) = \emptyset \).
(a ⇐). Assume that \( f \not\in M_x(R) \). Then there is some \( g \in R \) such that \( x \in Z(g) \) and \( Z(f) \cap Z(g) = Z(f^2 + g^2) = \emptyset \). Hence, \((M_x(R), f)\) contains an invertible element of \( \mathbb{R}^X \). Also, clearly, \( M_x(R) \) is a \( g \)-ideal. Thus, \( M_x(R) \) is a maximal \( g \)-ideal.

(b). An easy consequence of (a). \( \square \)

**Corollary 2.5.** If \( M_x(R) \) is a maximal ideal for each \( x \in X \), then every \( Z \in Z(R) \) is clopen under \( Z(R) \)-topology.

**Corollary 2.6.** Let \( R \) be an invertible subring. Then every \( Z \in Z(R) \) is clopen under \( Z(R) \)-topology if and only if \( M_x(R) \) is a maximal ideal for each \( x \in X \).

**Proof.** By our hypothesis and Proposition 2.4, this is clear. \( \square \)

The following lemma is a restatement of the fact that the transcendental degree of \( \mathbb{R} \) over \( \mathbb{Q} \) is uncountable, see [14].

**Lemma 2.7.** Let \( S = \mathbb{Q}[y_1, \ldots, y_n] \) be the ring of \( n \)-variable polynomials with rational coefficients. Then there exists an uncountable set \( X \) of transcendental numbers for which \( F(a_1, \cdots, a_n) \neq 0 \), for every distinct elements \( a_1, \cdots, a_n \) of \( X \) and every \( F \in S \).

The following example shows that the converse of Corollary 2.5 does not hold, in general.

**Example 2.8.** Let \( S \) be the polynomial ring \( \mathbb{Q}[y_1, \ldots, y_n] \), where \( n \in \mathbb{N} \) and \( n > 1 \). By Lemma 2.7, there exists an infinite set of transcendental numbers \( X \) for which \( F(a_1, \cdots, a_n) \neq 0 \), for every \( a_1, \cdots, a_n \in X \) and every \( F \in S \). For each \( a \in X \), define the function \( f_a : X \rightarrow \mathbb{R} \) by \( f_a(a) = 0 \) and \( f_a(x) = x \) for each \( x \neq a \). Now, set

\[
R = \{ F(f_{a_1}, \ldots, f_{a_n}) : F \in S, n \in \mathbb{N}, a_1, \ldots, a_n \in X \}.
\]

Hence, \( M_x(R) = (f_a) \), for each \( a \in X \), which is not a maximal ideal. However, every \( Z \in Z(R) \) is clopen under \( Z(R) \)-topology.

**Proposition 2.9.** If \( R \) is a subalgebra of \( \mathbb{R}^X \), then \( M_x(R) \) is a maximal \( g \)-ideal and a maximal fixed ideal for every \( x \in X \).

**Proof.** It suffices to prove that every element of \( Z(R) \) is closed under \( Z(R) \)-topology. To this aim, suppose that \( a \in X \) and \( a \notin Z(f) \), for some \( f \in R \). Put \( g = f - f(a) \). Clearly, \( Z(g) \in Z(R), a \in Z(g) \) and \( Z(g) \cap Z(f) = \emptyset \). \( \square \)

**Corollary 2.10.** If \( R \) is an invertible subalgebra of \( \mathbb{R}^X \), then \( M_x(R) \) is a maximal ideal for each \( x \in X \).

The converse of Corollary 2.10 does not hold, in general. For example, let \( R \) denote the collection of all single variable polynomials over \( \mathbb{R} \). Then, \( M_x(R) \) is the maximal ideal \( (x - r) \) for each \( r \in \mathbb{R} \). However, \( f = x^2 + 1 \) is invertible in
\[ R^2 \] which is not invertible in \( R \). Note that the subalgebras \( C_c(X) \) and \( R + I \), for each ideal \( I \) in \( C(X) \), satisfy Corollary 2.10 and so \( M_x(C_c(X)) \) and \( M_x(R + I) \) are maximal ideals of \( C_c(X) \) and \( R + I \), respectively, for each \( x \in X \). Remark that in parts (b) and (e) of the following proposition we assume that "\(^{-}\)" is a partial order on \( X \).

**Proposition 2.11.** For a subring \( R \) of \( \mathbb{R}^X \), the following statements hold.

(a) The mapping \( x \mapsto M_x(R) \) is a one-one correspondence if and only if \( (X,\tau_{Z(R)}) \) is a \( T_0 \)-space.

(b) The mapping \( x \mapsto M_x(R) \) is an order isomorphism between \( X \) and the set of all maximal fixed ideals of \( R \) if and only if \( (X,\tau_{Z(R)}) \) is a \( T_1 \)-space.

(c) For every two distinct elements \( x, y \in X \), \( M_x(R) + M_y(R) \) is not a \( g \)-ideal if and only if \( (X,\tau_{Z(R)}) \) is a \( T_2 \)-space.

(d) The mapping \( x \mapsto M_x(R) \) is an order embedding between \( X \) and the set of all maximal \( g \)-ideals of \( R \) if and only if \( (X,\tau_{Z(R)}) \) is a \( T_0 \)-space and every element of \( Z(R) \) is clopen under \( Z(R) \)-topology.

**Proof.** (a). Let \( x, y \) be distinct points of \( X \), so \( M_x(R) \neq M_y(R) \), say \( M_x(R) \not\subseteq M_y(R) \). Hence, there exists \( f \in M_x(R) \setminus M_y(R) \). Thus \( x \in Z(f) \) and \( y \notin Z(f) \). It is clear that the above reasoning is reversible and hence we are done.

(b \( \Rightarrow \)). Suppose that \( x \) and \( y \) are two distinct points of \( X \). Since \( M_x(R) \not\subseteq M_y(R) \), there exists \( f \in M_x(R) \setminus M_y(R) \). Consequently, \( x \in Z(f) \) and \( y \notin Z(f) \).

(b \( \Leftarrow \)). Suppose that \( x \in X \) and \( I \) is a fixed ideal in \( R \) containing \( M_x(R) \). Take \( y \in \bigcap_{f \in I} Z(f) \). Clearly, \( M_x(R) \subseteq I \subseteq M_y(R) \). It suffices to show \( x = y \). Suppose that \( x \neq y \) and seek a contradiction. By our hypothesis, there exists \( f \in R \) such that \( x \in Z(f) \) and \( y \notin Z(f) \). Therefore, \( M_x(R) \not\subseteq M_y(R) \) and this is a contradiction. Now, by part (a), the proof is complete.

(c). For any two distinct points \( x, y \in X \), clearly, \( M_x(R) + M_y(R) \) is not a \( g \)-ideal if and only if there exist \( f \in M_x(R) \) and \( g \in M_y(R) \) such that \( Z(f) \cap Z(g) = \emptyset \).

(d \( \Rightarrow \)). By part (a), clearly, \( (X,\tau_{Z(R)}) \) is a \( T_0 \)-space. Now, suppose that \( f \in R \) and \( x \notin Z(f) \). Since \( M_x(R) \) is a maximal \( g \)-ideal, it follows that \( (M_x(R), f) \) has an invertible element of \( \mathbb{R}^X \) and so there exists \( g \in M_x(R) \), such that \( Z(g) \cap Z(f) = \emptyset \). Thus, \( Z(f) \) is closed and hence is clopen under \( Z(R) \)-topology.

(d \( \Leftarrow \)). Suppose that \( x \in X \), it suffices to show that \( M_x(R) \) is a maximal \( g \)-ideal. Assume that \( I \) is an ideal which properly contains \( M_x(R) \). Hence, there exists \( f \in I \) such that \( x \notin Z(f) \). By our hypothesis, there is \( g \in R \) such that \( x \in Z(g) \) and \( Z(g) \cap Z(f) = \emptyset \). Therefore, \( Z(f^2 + g^2) = \emptyset \) and \( f^2 + g^2 \in I \), hence, \( I \) is not a \( g \)-ideal.

\( \square \)

It is easy to see that \( M_x(R) \), for each \( x \in X \), is a prime ideal of \( R \) and thus the hull-kernel topology may be defined on the family \( \{M_x(R) : x \in X \} \).
By considering this space, the next statement gives a relation between Z(R)-topology on X and points of X.

**Proposition 2.12.** Let R be a subring of \( \mathbb{R}^X \) and X equipped with the Coz(R)-topology. Then the mapping \( \Phi : X \to \{M_x(R) : x \in X\} \) defined by \( x \mapsto M_x(R) \) is a homeomorphism if and only if \( (X, \tau_{Z(R)}) \) is a T_0-space.

**Proof.** By part (a) of Theorem 2.12, \( \Phi \) is a one-one correspondence if and only if \( (X, \tau_{Z(R)}) \) is a T_0-space. Also, if \( f \in R \) and \( x \in Z(f) \), then \( f \in M_x(R) \) which means that basic closed sets of X equipped with the Coz(R)-topology are mapped to the basic closed sets in \( \{M_x(R) : x \in X\} \) equipped with the hull-kernel topology by the mapping \( \Phi \) and therefore, it is a homeomorphism. \( \square \)

3. \( z_R \)-Ideals and \( z \)-Ideals in Subrings

In this section we introduce \( z_R \)-ideals in a subring \( R \) and via the Z(R)-topology and maximal g-ideals of \( R \), various characterizations of these ideals are given.

**Definition 3.1.** A subset \( F \) of \( Z(R) \) is called \( z_R \)-filter on \( X \), if

(a) \( \emptyset \notin F \).

(b) If \( Z_1, Z_2 \in F \), then \( Z_1 \cap Z_2 \in F \).

(c) If \( Z_1 \in F \), \( Z_2 \in Z(R) \) and \( Z_1 \subseteq Z_2 \), then \( Z_2 \in F \).

Moreover, \( F \) is called a prime \( z_R \)-filter, if whenever \( Z_1 \cup Z_2 \in F \), then \( Z_1 \in F \) or \( Z_2 \in F \) for each \( Z_1, Z_2 \in Z(R) \). Also, \( F \) is called a \( z_R \)-ultrafilter, if \( F \) is maximal among \( z_R \)-filters on \( X \).

The following proposition immediately follows from Definition 3.1.

**Proposition 3.2.** For any subring \( R \), the following statements hold.

(a) \( I \subseteq R \) is a g-ideal in \( R \) if and only if \( Z_R(I) = \{Z(f) : f \in I\} \) is a \( z_R \)-filter on \( X \).

(b) \( F \) is a \( z_R \)-filter on \( X \) if and only if \( Z_\mathcal{R}^{-1}(F) = \{f \in R : Z(f) \in F\} \) is a g-ideal.

(c) \( F \) is a prime \( z_R \)-filter on \( X \) if and only if \( Z_\mathcal{R}^{-1}(F) \) is a prime g-ideal.

(d) \( A \) is a \( z_R \)-ultrafilter on \( X \) if and only if \( Z_\mathcal{R}^{-1}(A) \) is a maximal g-ideal.

(e) If \( M \) is a maximal g-ideal in \( R \), then \( Z_R(M) \) is a \( z_R \)-ultrafilter on \( X \).

It is easy to see that for an ideal \( I \) of \( R \) we always have \( I \subseteq Z_\mathcal{R}^{-1}Z_R(I) \) and the inclusion may be proper. We call an ideal \( I \) in \( R \) a \( z_R \)-ideal, if \( I = Z_\mathcal{R}^{-1}Z_R(I) \). It follows that every \( z_R \)-ideal is semiprime and arbitrary intersections of \( z_R \)-ideals is a \( z_R \)-ideal. Also, the zero ideal, the ideals of the form \( M_x(R) \), maximal g-ideals and \( Z^{-1}(F) \), for each \( z_R \)-filter \( F \), are all \( z_R \)-ideals of \( R \). For each \( f \in R \), the intersection of all the maximal ideals, maximal g-ideals and maximal fixed ideals of \( R \) containing \( f \) are denoted by \( M_f(R), MG_f(R) \) and \( MF_f(R) \), respectively. It is easy to observe that \( MG_f(R) \) is a \( z_R \)-ideal for each \( f \in R \).
Obviously, $MG_f \cap MG_g = MG_{fg}$, $MF_f \cap MF_g = MF_{fg}$, $MG_{f^2 + g^2} = MG_{(f,g)}$ and $MF_{f^2 + g^2} = MF_{(f,g)}$ for all $f, g \in R$.

**Proposition 3.3.** Let $(X, \tau_{Z(R)})$ be a $T_1$-space. Then the following statements hold.

(a) The following statements are equivalent.

1. $g \in MF_f(R)$.
2. $MF_g(R) \subseteq MF_f(R)$.
3. $Z(f) \subseteq Z(g)$.

(b) $MF_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$.

(c) An ideal $I$ of $R$ is a $z_R$-ideal if and only if $MF_f(R) \subseteq I$ for every $f \in I$.

**Proof.**

(a: $1 \Rightarrow 2$). Evident.

(a: $2 \Rightarrow 3$). Let $x \in Z(f)$. Then $f \in M_z(R)$ and thus $MF_g(R) \subseteq MF_f(R) \subseteq M_z(R)$. This implies $g \in M_z(R)$ and hence $x \in Z(g)$.

(a: $3 \Rightarrow 1$). If $g \notin MF_f(R)$, then there exists $x \in X$ such that $f \in M_z(R)$ and $g \notin M_z(R)$. Therefore, $x \in Z(f) \setminus Z(g)$ and so $Z(f) \subseteq Z(g)$.

(b) and (c) obviously follow from part (a).

**Lemma 3.4.** Assume that every $Z \in Z(R)$ is clopen under $Z(R)$-topology. Then $MG_f(R) = MF_f(R)$, for every $f \in R$.

**Proof.** Suppose that $f \in R$. By part (b) of Proposition 2.4, $M_z(R)$ is a maximal $g$-ideal for each $x \in X$. Consequently, $MG_f(R) \subseteq MF_f(R)$. Now, assume that $g \notin MG_f(R)$. Hence, there exists a maximal $g$-ideal $M$ in $R$ such that $f \in M$ and $g \notin M$. Thus, there exists $h \in M$ such that $Z(g) \cap Z(h) = \emptyset$. Since $f^2 + h^2 \in M$ and $M$ is a $g$-ideal, there is a point $x \in Z(f^2 + h^2) = Z(f) \cap Z(h)$. Clearly, $g \notin M_z(R)$ and $f \in M_z(R)$. Therefore, $g \notin MF_f(R)$. □

Proposition 3.3 and Lemma 3.4 imply the next statement.

**Proposition 3.5.** Let $(X, \tau_{Z(R)})$ be a $T_1$-space and every $Z \in Z(R)$ be a clopen set under $Z(R)$-topology. Then the following statements hold.

(a) The following statements are equivalent.

1. $g \in MG_f(R)$.
2. $MG_g(R) \subseteq MG_f(R)$.
3. $Z(f) \subseteq Z(g)$.

(b) $MG_f(R) = \{g \in R : Z(f) \subseteq Z(g)\}$.

(c) An ideal $I$ of $R$ is a $z_R$-ideal if and only if $MG_f(R) \subseteq I$ for every $f \in I$.

The following corollary follows from Corollary 2.6 and Proposition 3.5.

**Corollary 3.6.** Let $R$ be an invertible subalgebra of $\mathbb{R}^X$. Then the following statements hold.
(a) The following conditions are equivalent:
(1) $g \in M_f(R)$.
(2) $M_g(R) \subseteq M_f(R)$.
(3) $Z(f) \subseteq Z(g)$.

(b) $M_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \}$.

(c) An ideal $I$ of $R$ is $z_R$-ideal if and only if $M_f(R) \subseteq I$ for every $f \in I$.

It follows from Corollary 3.6 that for an invertible subalgebra $R$, the notion of $z_R$-ideal coincides with the notion of $z$-ideal. The next statement extend this fact and shows that this coincidence is equivalent to invertibility of $R$.

**Theorem 3.7.** Let $R$ be a subring of $\mathbb{R}^X$. The following statements are equivalent.

(a) Every maximal ideal in $R$ is a $g$-ideal.
(b) Every maximal $g$-ideal of $R$ is a maximal ideal and if $J$ is a maximal ideal of $R$, then every maximal element in the set of $g$-ideals contained in $J$ is a prime ideal.
(c) Every maximal ideal in $R$ is a $g$-ideal.
(d) $R$ is an invertible subring.
(e) Every $z$-ideal of $R$ is a $z_R$-ideal.

Moreover, if $R$ is a subalgebra and one of (a)-(c) holds, then every $z_R$-ideal is a $z$-ideal.

**Proof.** (a) $\Rightarrow$ (b). This is clear.
(b)$\Rightarrow$(c). Suppose that $M$ is a maximal ideal and $P$ is a maximal element of $G_M$, where $G_M$ is the set of all $g$-ideals contained in $M$. Assume that $J$ is a maximal ideal of $R$ containing $P$. Then $M \cap J = P$. As $M \cap J$ is prime and both $M$ and $J$ are maximal ideal, we have $M = J$. Hence, $M$ is a maximal $g$-ideal.
(c)$\Rightarrow$(d). Suppose that $Z(f) = \emptyset$ for $f \in R$ and, on the contrary, $f$ is a non-unit element of $R$. Clearly, there exists a maximal ideal $M$ of $R$ containing $f$. By our hypothesis, $M$ is a $g$-ideal which contradicts with $Z(f) = \emptyset$.
(d)$\Rightarrow$(e). Suppose that $I$ is a $z$-ideal and $Z(f) \subseteq Z(g)$ where $f \in I$ and $g \in R$. Since $I$ is a $z$-ideal, it follows that $M_f \subseteq I$. It suffices to prove that $g \in M_f$. To see this, suppose that $M$ is a maximal ideal containing $f$. As $R$ is invertible, $M$ is a $g$-ideal and so it is a maximal $g$-ideal. Obviously, $M$ is a $z_R$-ideal and so $g \in M$.
(e)$\Rightarrow$(a). Suppose that $M$ is a maximal ideal and, on the contrary, $M$ is not a $g$-ideal. Thus, there exists $f \in M$ such that $Z(f) = \emptyset$. By (e), $M$ is a $z_R$-ideal and since $f \in M$, it follows that $M = R$, which is a contradiction.

Now, suppose that one of (a)-(c) holds, $R$ is a subalgebra and $I$ is a $z_R$-ideal of $R$. By our hypothesis, $M_f(R) = M_f(R)$ for every $f \in R$, and thus we are done. $\square$
It is well-known that every minimal prime ideals over a \( z \)-ideal is also a \( z \)-ideal, see \cite[Theorem 14.7]{10}. The same statement holds for \( z_R \)-ideals as the following proposition shows.

**Proposition 3.8.** Let \( I \) be a \( z_R \)-ideal of \( R \) and \( P \) a prime ideal in \( R \) minimal over \( I \). Then \( P \) is a \( z_R \)-ideal.

*Proof.* Assume that \( Z(f) = Z(g) \) and \( f \in P \). Thus, there exists \( h \notin P \), such that \( fh \in I \). Since \( Z(fh) = Z(gh) \) and \( I \) is a \( z_R \)-ideal, it follows that \( gh \in I \subseteq P \). As \( h \notin P \), clearly, this implies that \( g \notin P \). \( \square \)

An immediate consequence of Proposition 3.8 is that every minimal prime ideal in a subring \( R \) is a \( z_R \)-ideal. By the following statement, we extend some fundamental statements about \( z \)-ideals in the literature of \( C(X) \) to the subrings of \( \mathbb{R}^X \), namely, \cite[2.9, 5.3 and 5.5]{10}. The proofs are left to the reader.

**Proposition 3.9.** Let \( R \) be a lattice-ordered subring of \( \mathbb{R}^X \) and \( I \) be a \( z_R \)-ideal in \( R \). Then the following statements hold.

(a) The following statements are equivalent

1. \( I \) is a prime ideal;
2. \( I \) contains a prime ideal;
3. if \( fg = 0 \), then \( f \in I \) or \( g \in I \);
4. for each \( f \in R \), there is a \( Z \in Z_R(I) \) on which \( f \) does not change sign.

(b) Every prime \( g \)-ideal of \( R \) is contained in a unique maximal \( g \)-ideal.

(c) If \( P \) is a prime ideal of \( R \), then \( Z_R(P) \) is a prime \( z_R \)-filter on \( X \).

(d) If \( P \) is a prime \( z_R \)-filter on \( X \), then \( Z_R^{-1}(P) \) is a prime ideal in \( R \).

(e) Every \( z_R \)-ideal of \( R \) is absolutely convex.

Thus, if \( I \) is an absolutely convex ideal of \( R \), then \( R/I \) is a lattice ring.

(f) \( I(f) \geq 0 \) if and only if \( f \geq 0 \) on some \( Z \in Z_R(I) \).

(g) Suppose that there exists \( Z \in Z_R(I) \) such that \( f(x) > 0 \), for every \( x \in Z \), then \( I(f) > 0 \). The converse is true whenever \( I \) is a maximal \( g \)-ideal.

4. \( z_R^0 \)-Ideals and \( z^2 \)-Ideals in Subrings

In this section we generalize the concept of \( z^2 \)-ideals of \( C(X) \) to the subrings of \( \mathbb{R}^X \) and introduce \( z_R^0 \)-ideal. Coincidence of \( z_R^0 \)-ideals with \( z^2 \)-ideals of \( R \) is discussed. Note that, for each element \( f \) of a commutative rings \( S \), we use \( P_f(S) \) to denote the intersection of all the minimal prime ideals in \( S \) containing \( f \).

**Definition 4.1.** An ideal \( I \) of a subring \( R \) of \( \mathbb{R}^X \) is called a \( z_R^0 \)-ideal, if \( \text{int}_X Z(f) \subseteq \text{int}_X Z(g) \), where \( f \in I \) and \( g \in R \), implies \( g \in I \).

The following statement investigates some characterizations of \( z_R^0 \)-ideals in subrings.

**Theorem 4.2.** Let \( R \) be a subring of \( \mathbb{R}^X \) and \( I \) be an ideal in \( R \). The following statements are equivalent.
(a) I is a $z_{\mathbb{R}}^\circ$-ideal.
(b) Whenever $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$, then $g \in I$.
(c) $R \cap P_f(C) \subseteq I$ for each $f \in I$.
(d) Whenever $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$, then $g \in I$.

Proof. (a$\Rightarrow$b). First note that by [3, Lemma 2.1] we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ if and only if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ for each $f, g \in C(X)$. Now, let $I$ be a $z_{\mathbb{R}}^\circ$-ideal in $R$ and $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$. Thus, by our hypothesis, we have $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ which implies that $g \in I$.

(b$\Rightarrow$c). By [3, Proposition 2.3], we have $P_f(C) = \{g \in C(X) : \text{Ann}_C(f) \subseteq \text{Ann}_C(g)\}$. Thus the proof is evident.

(c$\Rightarrow$d). Let $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$. As $f \in I$, by our hypothesis, $P_f(C) \cap R \subseteq I$ and thus $P_g(C) \cap R \subseteq I$ which implies that $g \in I$.

(d$\Rightarrow$a). Let $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ where $f \in I$ and $g \in R$. Therefore, by [3, Lemma 2.1], we have $P_f(C) \subseteq P_g(C)$ and hence $P_f(C) \cap R \subseteq P_g(C) \cap R$. Thus we are done by our hypothesis. □

**Lemma 4.3.** Let $R$ be a subring of $\mathbb{R}^X$, then for each $f \in R$ we have $P_f(C) \subseteq P_f(R)$.

Proof. Let $g \in P_f(C)$. By [3, Proposition 2.3], we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$. Therefore, $\text{Ann}_R(f) = \text{Ann}_C(f) \cap R \subseteq \text{Ann}_C(g) \cap R = R$. Thus, by [2, Proposition 1.5] we are done. □

**Theorem 4.4.** Let $R$ be a subring of $\mathbb{R}^X$. Then every $z_{\mathbb{R}}^\circ$-ideal in $R$ is a $z^\circ$-ideal if and only if $P_f(R) = P_f(C)$ for each $f \in R$.

Proof. ($\Rightarrow$). Assume on the contrary that there exists some $f \in R$ such that $P_f(R) \neq P_f(C)$. Thus, using Theorem 4.2 we have $P_f(C) \subseteq P_f(R)$. Again by Theorem 4.2, $P_f(C) \cap R$ is a $z_{\mathbb{R}}^\circ$-ideal in $R$. Also, it is clear that this ideal is not a $z^\circ$-ideal, since, $P_f(R) \nsubseteq P_f(C) \cap R$.

($\Leftarrow$). Let $I$ be a $z_{\mathbb{R}}^\circ$-ideal in $R$ and $f \in I$. By Theorem 4.2, $P_f(C) \cap R \subseteq I$. Thus, by our hypothesis, $P_f(R) \subseteq I$ which means that $I$ is a $z^\circ$-ideal in $R$. □

From Theorem 4.2 it follows that every $z^\circ$-ideal in a subring $R$ is a $z_{\mathbb{R}}^\circ$-ideal. However, the converse of this fact does not hold, in general. The following example gives an example of a subring $R$ which has a $z_{\mathbb{R}}^\circ$-ideal that is not a $z^\circ$-ideal.

**Example 4.5.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$. It is clear that $f \in C(\mathbb{R})$. Now, let $R = \{\sum_{i=0}^n r_i f^i : r_i \in \mathbb{R}, n = 0, 1, \ldots\}$. It is easy to see that $P_f(R) = R$, however, $P_f(C) \cap R \neq R$. Also, by Theorem 4.2, $P_f(C) \cap R$ is $z_{\mathbb{R}}^\circ$-ideal and it is clear that this ideal is not a $z^\circ$-ideal.
The next theorem gives a sufficient conditions on $X$ in order that $z_R^0$-ideals in a subring $R$ coincide with $z^0$-ideals of $R$.

**Theorem 4.6.** Let $R$ be a subring of $\mathbb{R}^X$ and $X$ be equipped with the $\text{Coz}(R)$-topology. Then an ideal $I$ in $R$ is a $z^0$-ideal if and only if it is a $z_R^0$-ideal.

**Proof.** Let $I$ be a $z_R^0$-ideal in $R$ and $f \in I$. As $X$ is equipped with the $\text{Coz}(R)$-topology, we have $g \in \text{Ann}_R(f)$ if and only if $\text{Coz}(g) \subseteq \text{int}_X Z(f)$ for each $f, g \in R$. Therefore, $P_I(R) = \text{Ann}_R \text{Ann}_R(f) = \{g \in R : \text{Coz}(g) \cap \text{int}_X Z(f) = \emptyset\} = \{g \in R : \text{Ann}_R(f) \subseteq \text{Ann}_R(g)\}$. Hence, $P_I(R) \subseteq I$ which means that $I$ is a $z^0$-ideal in $R$. This completes the proof, since, as former stated, every $z^0$-ideal in $R$ is a $z_R^0$-ideal.

□

Note that the condition that $X$ is equipped with the $\text{Coz}(R)$-topology is a sufficient condition for coincidence of $z_R^0$-ideals with $z^0$-ideals in a given subring $R$. The next example shows that this condition is not necessary.

**Example 4.7.** Let $X = \mathbb{R} \setminus \{0\}$ with the topology inherits from the usual topology on $\mathbb{R}$. Also, let $f : X \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$. It is clear that $f \in C(X)$ and $f^2 = f$. Now, set $R = \{r + sf : r, s \in \mathbb{R}\}$. It is clear that $R$ is a subring of $C(X)$. Also, by a routine reasoning, one can proves that the only ideals of $R$ are the ideals $(0)$, $(f)$, $(1-f)$ and $R$. Moreover, the minimal prime ideals of $R$ are only the ideals $(f)$ and $(1-f)$. These imply that every $z_R^0$-ideal is a $z^0$-ideal in $R$. However, clearly, $X$ is not equipped with the $\text{Coz}(R)$-topology.

It follows from Theorem 4.6 that for an intermediate subalgebra $A(X)$ of $C(X)$, $z_A^0$-ideals coincide with $z^0$-ideals of $A(X)$. However, the same statement does not true for $z_A$-ideals and $z$-ideals in $A(X)$, in general, see [6, Theorem 2.2]. Moreover, Theorem 3.7 together with Theorem 4.6 imply that in the subalgebras of $C(X)$ which are of the form $\mathbb{R} + I$, where $I$ is a free ideal in $C(X)$, $z_{\mathbb{R} + I}$-ideals coincide with $z$-ideals of $\mathbb{R} + I$ and $z_{\mathbb{R} + I}^0$-ideals coincide with $z^0$-ideals, too. Note that whenever $I$ is a free ideal in $C(X)$, then $\mathbb{R} + I$ determines the topology of $X$.

**Acknowledgments**

The authors would like to thank to the referee for careful reading the paper.

**References**

4. F. Azarpanah, M. Karavan, On Nonregular Ideals and \( z^\circ \)-Ideals in \( C(X) \), Cech. Math., 55, (130), (2005), 397-407.
5. F. Azarpanah, R. Mohamadian, \( \sqrt{z} \)-Ideals and \( \sqrt{z^\circ} \)-Ideals in \( C(X) \), Acta. Math. Sinica, English Series, 23 (2007), 989-1006.
8. J.M. Dominguez, J. Gomez Perez, M.A. Mulero, Intermediate Algebras between \( C^*(X) \) and \( C(X) \) as Rings of Fractions of \( C^*(X) \), Topology Appl., 77, (1997), 115-130.
12. D.Plank, On a Class of Subalgebras of \( C(X) \) with Application to \( \beta X \setminus X \), Fund. Math., 64, (1969), 41-54.