Abstract. Let $X$ be a topological space and $R$ be a subring of $\mathbb{R}^X$. By determining some special topologies on $X$ associated with the subring $R$, characterizations of maximal fixed and maximal $g$-ideals in $R$ of the form $M_r(R)$ are given. Moreover, the classes of $z_R$-ideals and $z_R^\circ$-ideals are introduced in $R$ which are topological generalizations of $z$-ideals and $z^\circ$-ideals of $C(X)$, respectively. Various characterizations of these ideals are established. Also, coincidence of $z_R$-ideals with $z$-ideals and $z_R^\circ$-ideals with $z^\circ$-ideals in $R$ are investigated. It turns out that some fundamental statements in the context of $C(X)$ are extended to the subrings of $\mathbb{R}^X$.

Keywords: $Z(R)$-topology, $Coz(R)$-topology, $g$-ideal, $z_R$-ideal, $z_R^\circ$-ideal, invertible subring.


1. Introduction

For a topological space $X$, $\mathbb{R}^X$ denotes the algebra of all real-valued functions and $C(X)$ (resp., $C^*(X)$) denotes the subalgebra of $\mathbb{R}^X$ consisting of all continuous functions (resp., bounded continuous functions). Moreover, we use $R$ to denote a unital subring of $\mathbb{R}^X$. Note that topological spaces which are considered in this paper are not necessarily Tychonoff. For each $f \in \mathbb{R}^X$,
\( Z(f) = \{ x \in X : f(x) = 0 \} \) denotes the zero-set of \( f \) and \( Coz(f) \) denotes the complement of \( Z(f) \) with respect to \( X \). We denote by \( Z(R) \) the collection of all the zero-sets of elements of \( R \), we use \( Z(X) \) instead of \( Z(C(X)) \). We denote by \( M_x(R) \) the set \( \{ f \in R : x \in Z(f) \} \), \( M_x(C(X)) \) is denoted by \( M_x \). The subring \( R \) is called invertible, if \( f \in R \) and \( Z(f) = \emptyset \) implies that \( f \) is invertible in \( R \). Moreover, \( R \) is called a lattice-ordered subring if it is a sublattice of \( \mathbb{R}^X \) (i.e., \( f \wedge g \) and \( f \vee g \) are in \( R \) for each \( f, g \in R \)). It is clear that \( C(X) \) is an invertible lattice-ordered subring of \( \mathbb{R}^X \). However, the same statement does not hold for \( C^*(X) \). A proper ideal \( I \) of \( R \) is called a growing ideal, briefly, a \( g \)-ideal, if contains no invertible element of \( \mathbb{R}^X \), i.e., \( Z(f) \neq \emptyset \) for each \( f \in I \). It is evident that a subring \( R \) is invertible if and only if every ideal every ideal of \( R \) is a \( g \)-ideal. Clearly, \( M^{*p} \), for each \( p \in \beta \mathbb{X} \setminus \nu \mathbb{X} \), is not a \( g \)-ideal of \( C^*(X) \). An ideal \( I \) of \( R \) is called fixed if \( \bigcap_{f \in I} Z(f) \neq \emptyset \), otherwise, it is called free. By a maximal fixed ideal of \( R \), we mean a fixed ideal which is maximal in the set of all fixed ideals of \( R \). An ideal \( I \) in a commutative ring \( S \) is called a \( z \)-ideal (resp., \( z^* \)-ideal) if \( M_a(S) \subseteq I \) (resp., \( P_a(S) \subseteq I \)), for each \( a \in I \), where \( M_a(S) \) (resp., \( P_a(S) \)) denotes the intersection of all the maximal (resp., minimal prime) ideals of \( S \) containing \( a \). It is well-known that in \( C(X) \) an ideal \( I \) is a \( z \)-ideal (resp., \( z^* \)-ideal) if and only if whenever \( Z(f) \subseteq Z(g) \) (resp., \( \text{int}_X Z(f) \subseteq \text{int}_X Z(g) \)), \( f \in I \) and \( g \in C(X) \), then \( g \in I \).

This paper consists of 4 sections. Section 1, as we have already noticed, is the introduction, in which we determine two special topologies on \( X \) which the subring \( R \) generate, namely, \( Z(R) \)-topology and \( Coz(R) \)-topology. Comparison and coincidence of these topologies are studied. Section 2 deals with maximal ideals in \( R \), specially, maximal fixed and maximal \( g \)-ideals. Using the \( Z(R) \)-topology, characterizations of maximal fixed ideals of \( R \), which are of the form \( M_x(R) \), are given. Moreover, relations between mapping \( x \rightarrow M_x(R) \) and the separation properties of the topological space \( (X, \tau_{Z(R)}) \) will be found. In section 3, we introduce the notion of \( z_R \)-ideal in a subring \( R \) as a natural topological generalization of the notion of \( z \)-ideal in \( C(X) \). Various characterizations of these ideals via \( Z(R) \)-topology are given and relations between \( z_R \)-ideals and \( z \)-ideals in \( R \) (by their algebraic descriptions) are discussed. Section 4 deals with \( z^*_R \)-ideals of \( R \) which are natural topological generalizations of \( z^* \)-ideals of \( C(X) \). Using \( Coz(R) \)-topology, coincidence of \( z^*_R \)-ideals with \( z^* \)-ideals of \( R \) (by their algebraic descriptions) are established.

**Definition 1.1.** For each subring \( R \) of \( \mathbb{R}^X \), clearly, \( Z(R) \) and \( Coz(R) \) constitute bases for some topologies on \( X \). The induced topologies are called \( Z(R) \)-topology and \( Coz(R) \)-topology, respectively, and are denoted by \( \tau_{Z(R)} \) and \( \tau_{Coz(R)} \), respectively.

In the next three statements we compare these topologies. Note that two subsets \( S_1, S_2 \) of \( \mathbb{R}^X \) are called zero-set equivalent, if \( Z(S_1) = Z(S_2) \).
Proposition 1.2. Let $R$ be a subring of $\mathbb{R}^X$, if $S$ and $C(R)$ are zero-set equivalent subsets of $\mathbb{R}^R$ and $gof \in R$ for each $f \in R$ and each $g \in S$, then $\tau_{Coz(R)} \subseteq \tau_Z(R)$ and the equality does not hold, in general.

Proof. We are to show that $Coz(R) \subseteq \tau_Z(R)$. If $x \notin Z(f)$ where $f \in R$, then there is a $g$ in $S$ such that $f(x) \in Z(g)$ and $f^{-1}(Z(g)) \cap Z(f) = \emptyset$. Therefore, $gof \in R$, $x \in Z(gof)$ and $Z(gof) \cap Z(f) = \emptyset$ which proves the inclusion. Now, we show that the inclusion may be proper. Let $(X, \tau_X)$ be a Tychonoff space which has at least one non-open zero-set $Z$. Set $R = C(X)$, then $\tau_{Coz(R)} = \tau_X$, whereas $Z \notin \tau_X$ and hence, $\tau_{Coz(R)} \subsetneq \tau_Z(R)$. $\Box$

Proof of the following proposition is standard.

Proposition 1.3. The following statements are equivalent.

(a) $\tau_{Coz(R)} \subseteq \tau_Z(R)$.

(b) Every $Z \in \mathcal{Z}(R)$ is clopen under $\mathcal{Z}(R)$-topology.

(c) For each $f \in R$, $Z(f) = \bigcup_{g \in \text{Ann}_R(f)} \text{Coz}(g)$.

(d) For each $f \in R$, $(\text{Ann}_R(f), f)$ is a free ideal.

Proof. The implications (a)$\Rightarrow$(b)$\Rightarrow$(c) are clear.

(c)$\Rightarrow$(d). This clear by the hypothesis and the fact that whenever $f \in R$ and $I$ is an ideal of $R$, then $\bigcap_{h \in (I,f)} Z(h) = \bigcap_{g \in I} (Z(f) \cap Z(g))$.

(d)$\Rightarrow$(a). Let $f \in R$ and $x \in Z(f)$. By (d), there exists $g \in \text{Ann}_R(f)$ such that $x \notin Z(f) \cap Z(g)$. Hence, $x \notin Z(g)$ and $x \in \text{Coz}(g) \subseteq Z(f)$ and so $Z(f) \in \tau_{Coz(R)}$. $\Box$

An immediate consequence of Propositions 1.3 and 1.4 is that $\tau_{Coz(R)} = \tau_Z(R)$ if and only if $Z(f)$ is clopen under both $Z(R)$-topology and $Coz(R)$-topology, for each $f \in R$.

2. Characterization of Maximal Fixed Ideals in Subrings

We remind that maximal fixed ideals of $C(X)$ coincide with its fixed maximal ideals and are of the form $M_x = \{f \in C(X) : f(x) = 0\}$, where $x \in X$. This fact is generalized for some special subalgebras of $C(X)$, such as intermediate subalgebras (subalgebras of $C(X)$ containing $C^+(X)$, see [7]), $C_c(X)$ (the subalgebra of $C(X)$ consisting of all functions with countable image, see [9]) and the subalgebras of the form $R + I$ where $I$ is an ideal of $C(X)$, see [13].
We will show that the same statement does not hold for arbitrary subrings of $\mathbb{R}^X$, in general.

Remark 2.1. (a) Every maximal fixed ideal and fixed maximal ideal of $R$ is of the form $M_x(R) = \{f \in R : f(x) = 0\}$ for some $x \in X$. However, parts (1) and (2) of Example 2.2 show that the ideals $M_x(R)$ are not necessarily maximal ideals or even maximal fixed ideals in $R$.

(b) Every fixed maximal ideal is both a maximal fixed ideal and a maximal $g$-ideal. But the converse is not necessarily true, in general, see part (1) of Example 2.2 and Example 2.3.

(c) A maximal fixed ideal need not be a maximal $g$-ideal, see Example 2.3.

(d) Every fixed maximal $g$-ideal is a maximal fixed ideal.

Example 2.2. (1) Let $X$ be a Tychonoff space, $x \in X$ and $R = \mathbb{Z} + M_x$. Then $M_x(R) = M_x$ is not a maximal ideal in $R$, since $2\mathbb{Z} + M_x$ is a proper ideal of $R$ and $M_x \subseteq 2\mathbb{Z} + M_x$. Therefore, $M_x(R)$ is a maximal fixed ideal and a maximal $g$-ideal which is not a maximal ideal.

(2) Let $X$ be a topological space with more than one point and $a \in X$. Also, let $t \in \mathbb{R}$ be a transcendental number and define $f : X \rightarrow \mathbb{R}$ by $f(a) = 0$ and $f(x) = t$, for every $x \neq a$. Set $R = \{\sum_{i=0}^{n} m_i f^i : n \in \mathbb{N} \cup \{0\}, m_i \in \mathbb{Z}\}$. Evidently, $M_a(R) = \{f\}$ and $M_x(R) = \{0\}$, for every $x \neq a$. Therefore, $M_x(R)$ is not a maximal fixed ideal for any $x \neq a$.

In the next example we construct a subring $R$ such that, for some $x \in X$, $M_x(R)$ is a maximal fixed ideal which is not a maximal $g$-ideal.

Example 2.3. Let $X = \mathbb{R}$, $a \in \mathbb{R} \setminus \mathbb{Q}$, $b \in \mathbb{R} \setminus \{0\}$ and $t$ be a transcendental number. For every $\epsilon > 0$, define $f_\epsilon : X \rightarrow \mathbb{R}$ by $f_\epsilon(x) = 0$, if $|x - a| < \epsilon$ and $f_\epsilon(x) = b$, if $|x - a| \geq \epsilon$. Also, define $f : X \rightarrow \mathbb{R}$ by $f(x) = 0$, if $x \in \mathbb{Q}$ and $f(x) = t$, if $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $R$ be the algebra over $\mathbb{Q}$ generated by $\{f_\epsilon : \epsilon > 0\} \cup \{f, 1\}$. Evidently, $R$ is a subring of $\mathbb{R}^X$, and $M_a(R)$ equals to $(f_a)$ which is not a maximal ideal. It is easy to see that $M_x(R)$ is a maximal fixed ideal and $M_a(R) = I$, where $I$ is the ideal generated by $\{f_\epsilon : \epsilon > 0\}$. Clearly, $Z(f) \cap Z(g) \neq \emptyset$, for all $g \in I$. Hence $J = (I, f)$ is a $g$-ideal which strictly contains $I$. Therefore, $I$ is not a maximal $g$-ideal.

Proposition 2.4. The following statements hold for a subring $R$ of $\mathbb{R}^X$.

(a) $M_x(R)$ is a maximal $g$-ideal if and only if whenever $Z \in Z(R)$ and $x \notin Z$, then $x \notin \text{cl}_{Z(R)} Z$.

(b) For each $x \in X$, $M_x(R)$ is a maximal $g$-ideal if and only if every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

Proof. (a $\Rightarrow$). Let $f \in R$ and $x \notin Z(f)$, thus, the ideal $(M_x(R), f)$ contains an invertible element of $\mathbb{R}^X$. Hence, there are $g \in M_x(R)$ and $h \in R$ such that $Z(g + fh) = \emptyset$. Consequently, $x \in Z(g)$ and $Z(f) \cap Z(g) = \emptyset$. 
(a $\iff$). Assume that $f \not\in M_x(R)$. Then there is some $g \in R$ such that $x \in Z(g)$ and $Z(f) \cap Z(g) = Z(f^2 + g^2) = \emptyset$. Hence, $(M_x(R), f)$ contains an invertible element of $\mathbb{R}^X$. Also, clearly, $M_x(R)$ is a $g$-ideal. Thus, $M_x(R)$ is a maximal $g$-ideal.

(b). An easy consequence of (a). □

**Corollary 2.5.** If $M_x(R)$ is a maximal ideal for each $x \in X$, then every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Corollary 2.6.** Let $R$ be an invertible subring. Then every $Z \in Z(R)$ is clopen under $Z(R)$-topology if and only if $M_x(R)$ is a maximal ideal for each $x \in X$.

**Proof.** By our hypothesis and Proposition 2.4, this is clear. □

The following lemma is a restatement of the fact that the transcendental degree of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable, see [14].

**Lemma 2.7.** Let $S = \mathbb{Q}[y_1, \ldots, y_n]$ be the ring of $n$-variable polynomials with rational coefficients. Then there exists an uncountable set $X$ of transcendental numbers for which $F(a_1, \ldots, a_n) \neq 0$, for every distinct elements $a_1, \ldots, a_n$ of $X$ and every $F \in S$.

The following example shows that the converse of Corollary 2.5 does not hold, in general.

**Example 2.8.** Let $S$ be the polynomial ring $\mathbb{Q}[y_1, \ldots, y_n]$, where $n \in \mathbb{N}$ and $n > 1$. By Lemma 2.7, there exists an infinite set of transcendental numbers $X$ for which $F(a_1, \ldots, a_n) \neq 0$, for every $a_1, \ldots, a_n \in X$ and every $F \in S$. For each $a \in X$, define the function $f_a : X \rightarrow \mathbb{R}$ by $f_a(a) = 0$ and $f_a(x) = x$ for each $x \neq a$. Now, set

$$
R = \{F(f_{a_1}, \ldots, f_{a_n}) : F \in S, \ a_n \in \mathbb{N}, \ a_1, \ldots, a_n \in X\}.
$$

Hence, $M_x(R) = (f_a)$, for each $a \in X$, which is not a maximal ideal. However, every $Z \in Z(R)$ is clopen under $Z(R)$-topology.

**Proposition 2.9.** If $R$ is a subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal $g$-ideal and a maximal fixed ideal for every $x \in X$.

**Proof.** It suffices to prove that every element of $Z(R)$ is closed under $Z(R)$-topology. To this aim, suppose that $a \in X$ and $a \notin Z(f)$, for some $f \in R$. Put $g = f - f(a)$. Clearly, $Z(g) \in Z(R)$, $a \in Z(g)$ and $Z(g) \cap Z(f) = \emptyset$. □

**Corollary 2.10.** If $R$ is an invertible subalgebra of $\mathbb{R}^X$, then $M_x(R)$ is a maximal ideal for each $x \in X$.

The converse of Corollary 2.10 does not hold, in general. For example, let $R$ denote the collection of all single variable polynomials over $\mathbb{R}$. Then, $M_x(R)$ is the maximal ideal $(x - r)$ for each $r \in \mathbb{R}$. However, $f = x^2 + 1$ is invertible in...
\(\mathbb{R}^R\) which is not invertible in \(R\). Note that the subalgebras \(C_c(X)\) and \(R + I\), for each ideal \(I\) in \(C(X)\), satisfy Corollary 2.10 and so \(M_x(C_c(X))\) and \(M_x(R + I)\) are maximal ideals of \(C_c(X)\) and \(R + I\), respectively, for each \(x \in X\). Remark that in parts (b) and (c) of the following proposition we assume that \(\sim\) is a partial order on \(X\).

**Proposition 2.11.** For a subring \(R\) of \(\mathbb{R}^X\), the following statements hold.

(a) The mapping \(x \mapsto M_x(R)\) is a one-one correspondence if and only if \((X, \tau_{Z(R)})\) is a \(T_0\)-space.

(b) The mapping \(x \mapsto M_x(R)\) is an order isomorphism between \(X\) and the set of all maximal fixed ideals of \(R\) if and only if \((X, \tau_{Z(R)})\) is a \(T_1\)-space.

(c) For every two distinct elements \(x, y \in X\), \(M_x(R) + M_y(R)\) is not a \(g\)-ideal if and only if \((X, \tau_{Z(R)})\) is a \(T_2\)-space.

(d) The mapping \(x \mapsto M_x(R)\) is an order embedding between \(X\) and the set of all maximal \(g\)-ideals of \(R\) if and only if \((X, \tau_{Z(R)})\) is a \(T_0\)-space and every element of \(Z(R)\) is clopen under \(Z(R)\)-topology.

**Proof.** (a). Let \(x, y\) be distinct points of \(X\), so \(M_x(R) \neq M_y(R)\), say \(M_x(R) \not\subseteq M_y(R)\). Hence, there exists \(f \in M_x(R) \setminus M_y(R)\). Thus \(x \in Z(f)\) and \(y \notin Z(f)\). It is clear that the above reasoning is reversible and hence we are done.

(b \(\Rightarrow\)). Suppose that \(x\) and \(y\) are two distinct points of \(X\). Since \(M_x(R) \not\subseteq M_y(R)\), there exists \(f \in M_x(R) \setminus M_y(R)\). Consequently, \(x \in Z(f)\) and \(y \notin Z(f)\).

(b \(\Leftarrow\)). Suppose that \(x \in X\) and \(I\) is a fixed ideal in \(R\) containing \(M_x(R)\). Take \(y \in \cap_{f \in I} Z(f)\). Clearly, \(M_x(R) \subseteq I \subseteq M_y(R)\). It suffices to show \(x = y\). Suppose that \(x \neq y\) and seek a contradiction. By our hypothesis, there exists \(f \in R\) such that \(x \in Z(f)\) and \(y \notin Z(f)\). Therefore, \(M_x(R) \not\subseteq M_y(R)\) and this is a contradiction. Now, by part (a), the proof is complete.

(c). For any two distinct points \(x, y \in X\), clearly, \(M_x(R) + M_y(R)\) is not a \(g\)-ideal if and only if there exist \(f \in M_x(R)\) and \(g \in M_y(R)\) such that \(Z(f) \cap Z(g) = \emptyset\).

(d \(\Rightarrow\)). By part (a), clearly, \((X, \tau_{Z(R)})\) is a \(T_0\)-space. Now, Suppose that \(f \in R\) and \(x \notin Z(f)\). Since \(M_x(R)\) is a maximal \(g\)-ideal, it follows that \((M_x(R), f)\) has an invertible element of \(\mathbb{R}^X\) and so there exists \(g \in M_x(R)\), such that \(Z(g) \cap Z(f) = \emptyset\). Thus, \(Z(f)\) is closed and hence is clopen under \(Z(R)\)-topology.

(d \(\Leftarrow\)). Suppose that \(x \in X\), it suffices to show that \(M_x(R)\) is a maximal \(g\)-ideal. Assume that \(I\) is an ideal which properly contains \(M_x(R)\). Hence, there exists \(f \in I\) such that \(x \notin Z(f)\). By our hypothesis, there is \(g \in R\) such that \(x \in Z(g)\) and \(Z(g) \cap Z(f) = \emptyset\). Therefore, \(Z(f^2 + g^2) = \emptyset\) and \(f^2 + g^2 \in I\), hence, \(I\) is not a \(g\)-ideal. \(\square\)

It is easy to see that \(M_x(R)\), for each \(x \in X\), is a prime ideal of \(R\) and thus the hull-kernel topology may be defined on the family \(\{M_x(R) : x \in X\}\).
By considering this space, the next statement gives a relation between \( Z(R) \)-topology on \( X \) and points of \( X \).

**Proposition 2.12.** Let \( R \) be a subring of \( \mathbb{R}^X \) and \( X \) equipped with the \( Coz(R) \)-topology. Then the mapping \( \Phi : X \rightarrow \{ M_a(R) : x \in X \} \) defined by \( x \mapsto M_a(R) \) is a homeomorphism if and only if \( (X, \tau_{Z(R)}) \) is a \( T_0 \)-space.

**Proof.** By part (a) of Theorem 2.12, \( \Phi \) is a one-one correspondence if and only if \( (X, \tau_{Z(R)}) \) is a \( T_0 \)-space. Also, if \( f \in R \) and \( x \in Z(f) \), then \( f \in M_a(R) \) which means that basic closed sets of \( X \) equipped with the \( Coz(R) \)-topology are mapped to the basic closed sets in \( \{ M_a(R) : x \in X \} \) equipped with the hull-kernel topology by the mapping \( \Phi \) and therefore, it is a homeomorphism. \( \square \)

3. \( z_R \)-Ideals and \( z \)-Ideals in Subrings

In this section we introduce \( z_R \)-ideals in a subring \( R \) and via the \( Z(R) \)-topology and maximal \( g \)-ideals of \( R \), various characterizations of these ideals are given.

**Definition 3.1.** A subset \( F \) of \( Z(R) \) is called a \( z_R \)-filter on \( X \), if

(a) \( \emptyset \notin F \).

(b) If \( Z_1, Z_2 \in F \), then \( Z_1 \cap Z_2 \in F \).

(c) If \( Z_1 \in F \), \( Z_2 \in Z(R) \) and \( Z_1 \subseteq Z_2 \), then \( Z_2 \in F \).

Moreover, \( F \) is called a prime \( z_R \)-filter, if whenever \( Z_1 \cup Z_2 \in F \), then \( Z_1 \in F \) or \( Z_2 \in F \) for each \( Z_1, Z_2 \in Z(R) \). Also, \( F \) is called a \( z_R \)-ultrafilter, if \( F \) is maximal among \( z_R \)-filters on \( X \).

The following proposition immediately follows from Definition 3.1.

**Proposition 3.2.** For any subring \( R \), the following statements hold.

(a) \( I \subseteq R \) is a \( g \)-ideal in \( R \) if and only if \( Z_R(I) = \{ Z(f) : f \in I \} \) is a \( z_R \)-filter on \( X \).

(b) \( F \) is a \( z_R \)-filter on \( X \) if and only if \( Z_R^{-1}(F) = \{ f \in R : Z(f) \in F \} \) is a \( g \)-ideal.

(c) \( F \) is a prime \( z_R \)-filter on \( X \) if and only if \( Z_R^{-1}(F) \) is a prime \( g \)-ideal.

(d) \( A \) is a \( z_R \)-ultrafilter on \( X \) if and only if \( Z_R^{-1}(A) \) is a maximal \( g \)-ideal.

(e) If \( M \) is a maximal \( g \)-ideal in \( R \), then \( Z_R(M) \) is a \( z_R \)-ultrafilter on \( X \).

It is easy to see that for an ideal \( I \) of \( R \) we always have \( I \subseteq Z_R^{-1}(Z_R(I)) \) and the inclusion may be proper. We call an ideal \( I \) in \( R \) a \( z_R \)-ideal, if \( I = Z_R^{-1}(Z_R(I)) \). It follows that every \( z_R \)-ideal is semiprime and arbitrary intersections of \( z_R \)-ideals is a \( z_R \)-ideal. Also, the zero ideal, the ideals of the form \( M_a(R) \), maximal \( g \)-ideals and \( Z^{-1}(F) \), for each \( z_R \)-filter \( F \), are all \( z_R \)-ideals of \( R \). For each \( f \in R \), the intersection of all the maximal ideals, maximal \( g \)-ideals and maximal fixed ideals of \( R \) containing \( f \) are denoted by \( M_f(R) \), \( MG_f(R) \) and \( MF_f(R) \), respectively. It is easy to observe that \( MG_f(R) \) is a \( z_R \)-ideal for each \( f \in R \).
Obviously, \( MG_f \cap MG_g = MG_{fg} \), \( MF_f \cap MF_g = MF_{fg} \), \( MG^{2+g^2} = MG_{(f,g)} \) and \( MF^{2+g^2} = MF_{(f,g)} \) for all \( f, g \in R \).

**Proposition 3.3.** Let \((X, \tau_Z(R))\) be a \( T_1 \)-space. Then the following statements hold.

(a) The following statements are equivalent.

1. \( g \in MF_f(R) \).
2. \( MF_g(R) \subseteq MF_f(R) \).
3. \( Z(f) \subseteq Z(g) \).

(b) \( MF_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \} \).

(c) An ideal \( I \) of \( R \) is a \( z_R \)-ideal if and only if \( MF_f(R) \subseteq I \) for every \( f \in I \).

**Proof.** (a: \( 1 \Rightarrow 2 \)). Evident.

(a: \( 2 \Rightarrow 3 \)). Let \( x \in Z(f) \). Then \( f \in M_x(R) \) and thus \( MF_g(R) \subseteq MF_f(R) \subseteq M_x(R) \). This implies \( g \in M_x(R) \) and hence \( x \in Z(g) \).

(a: \( 3 \Rightarrow 1 \)). If \( g \notin MF_f(R) \), then there exists \( x \in X \) such that \( f \in M_x(R) \) and \( g \notin M_x(R) \). Therefore, \( x \in Z(f) \setminus Z(g) \) and so \( Z(f) \not\subseteq Z(g) \).

(b) and (c) obviously follow from part (a).

**Lemma 3.4.** Assume that every \( Z \in Z(R) \) is clopen under \( Z(R) \)-topology. Then \( MG_f(R) = MF_f(R) \), for every \( f \in R \).

**Proof.** Suppose that \( f \in R \). By part (b) of Proposition 2.4, \( M_g(R) \) is a maximal \( g \)-ideal for each \( x \in X \). Consequently, \( MG_f(R) \subseteq MF_f(R) \). Now, assume that \( g \notin MG_f(R) \). Hence, there exists a maximal \( g \)-ideal \( M \) in \( R \) such that \( f \in M \) and \( g \notin M \). Thus, there exists \( h \in M \) such that \( Z(g) \cap Z(h) = \emptyset \). Since \( f^2 + h^2 \in M \) and \( M \) is a \( g \)-ideal, there is a point \( x \in Z(f^2 + h^2) = Z(f) \cap Z(h) \). Clearly, \( g \notin M_x(R) \) and \( f \in M_x(R) \). Therefore, \( g \notin MF_f(R) \).

Proposition 3.3 and Lemma 3.4 imply the next statement.

**Proposition 3.5.** Let \((X, \tau_Z(R))\) be a \( T_1 \)-space and every \( Z \in Z(R) \) be a clopen set under \( Z(R) \)-topology. Then the following statements hold.

(a) The following statements are equivalent.

1. \( g \in MG_f(R) \).
2. \( MG_g(R) \subseteq MG_f(R) \).
3. \( Z(f) \subseteq Z(g) \).

(b) \( MG_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \} \).

(c) An ideal \( I \) of \( R \) is a \( z_R \)-ideal if and only if \( MG_f(R) \subseteq I \) for every \( f \in I \).

The following corollary follows from Corollary 2.6 and Proposition 3.5.

**Corollary 3.6.** Let \( R \) be an invertible subalgebra of \( \mathbb{R}^X \). Then the following statements hold.
(a) The following conditions are equivalent;
(1) \( g \in M_f(R) \).
(2) \( M_g(R) \subseteq M_f(R) \).
(3) \( Z(f) \subseteq Z(g) \).
(b) \( M_f(R) = \{ g \in R : Z(f) \subseteq Z(g) \} \).
(c) An ideal \( I \) of \( R \) is \( z_R \)-ideal if and only if \( M_f(R) \subseteq I \) for every \( f \in I \).

It follows from Corollary 3.6 that for an invertible subalgebra \( R \), the notion of \( z_R \)-ideal coincides with the notion of \( z \)-ideal. The next statement extends this fact and shows that this coincidence is equivalent to invertibility of \( R \).

**Theorem 3.7.** Let \( R \) be a subring of \( \mathbb{R}^X \). The following statements are equivalent.

(a) Every maximal ideal in \( R \) is a \( g \)-ideal.
(b) Every maximal \( g \)-ideal of \( R \) is a maximal ideal and if \( J \) is a maximal ideal of \( R \), then every maximal element in the set of \( g \)-ideals contained in \( J \) is a prime ideal.
(c) Every maximal ideal in \( R \) is a \( g \)-ideal.
(d) \( R \) is an invertible subring.
(e) Every \( z \)-ideal of \( R \) is a \( z_R \)-ideal.

Moreover, if \( R \) is a subalgebra and one of (a)-(c) holds, then every \( z_R \)-ideal is a \( z \)-ideal.

**Proof.** (a) \( \Rightarrow \) (b). This is clear.

(b) \( \Rightarrow \) (c). Suppose that \( M \) is a maximal ideal and \( P \) is a maximal element of \( G_M \), where \( G_M \) is the set of all \( g \)-ideals contained in \( M \). Assume that \( J \) is a maximal ideal of \( R \) containing \( P \). Then \( M \cap J = P \). As \( M \cap J \) is prime and both \( M \) and \( J \) are maximal ideal, we have \( M = J \). Hence, \( M \) is a maximal \( g \)-ideal.

(c) \( \Rightarrow \) (d). Suppose that \( Z(f) = \emptyset \) for \( f \in R \) and, on the contrary, \( f \) is a non-unit element of \( R \). Clearly, there exists a maximal ideal \( M \) of \( R \) containing \( f \).

By our hypothesis, \( M \) is a \( g \)-ideal which contradicts with \( Z(f) = \emptyset \).

(d) \( \Rightarrow \) (e). Suppose that \( I \) is a \( z \)-ideal and \( Z(f) \subseteq Z(g) \) where \( f \in I \) and \( g \in R \). Since \( I \) is a \( z \)-ideal, it follows that \( M_f \subseteq I \). It suffices to prove that \( g \in M_f \). To see this, suppose that \( M \) is a maximal ideal containing \( f \). As \( R \) is invertible, \( M \) is a \( g \)-ideal and so it is a maximal \( g \)-ideal. Obviously, \( M \) is a \( z_R \)-ideal and so \( g \in M \).

(e) \( \Rightarrow \) (a). Suppose that \( M \) is a maximal ideal and, on the contrary, \( M \) is not a \( g \)-ideal. Thus, there exists \( f \in M \) such that \( Z(f) = \emptyset \). By (e), \( M \) is a \( z_R \)-ideal and since \( f \in M \), it follows that \( M = R \), which is a contradiction.

Now, suppose that one of (a)-(e) holds, \( R \) is a subalgebra and \( I \) is a \( z_R \)-ideal of \( R \). By our hypothesis, \( M_f(R) = M_f(R) \) for every \( f \in R \), and thus we are done.
It is well-known that every minimal prime ideals over a $z$-ideal is also a $z$-ideal, see [10, Theorem 14.7]. The same statement holds for $z_R$-ideals as the following proposition shows.

**Proposition 3.8.** Let $I$ be a $z_R$-ideal of $R$ and $P$ a prime ideal in $R$ minimal over $I$. Then $P$ is a $z_R$-ideal.

**Proof.** Assume that $Z(f) = Z(g)$ and $f \in P$. Thus, there exists $h \notin P$, such that $fh \in I$. Since $Z(fh) = Z(gh)$ and $I$ is a $z_R$-ideal, it follows that $gh \in I \subseteq P$. As $h \notin P$, clearly, this implies that $g \in P$. \hfill $\Box$

An immediate consequence of Proposition 3.8 is that every minimal prime ideal in a subring $R$ is a $z_R$-ideal. By the following statement, we extend some fundamental statements about $z$-ideals in the literature of $C(X)$ to the subrings of $\mathbb{R}^X$, namely, [10, 2.9, 5.3 and 5.5]. The proofs are left to the reader.

**Proposition 3.9.** Let $R$ be a lattice-ordered subring of $\mathbb{R}^X$ and $I$ be a $z_R$-ideal in $R$. Then the following statements hold.

(a) The following statements are equivalent

1. $I$ is a prime ideal;
2. $I$ contains a prime ideal;
3. if $fg = 0$, then $f \in I$ or $g \in I$;
4. for each $f \in R$, there is a $Z \in Z_R(I)$ on which $f$ does not change sign.

(b) Every prime $g$-ideal of $R$ is contained in a unique maximal $g$-ideal.

(c) If $P$ is a prime ideal of $R$, then $Z_R(P)$ is a prime $z_R$-filter on $X$.

(d) If $\mathcal{P}$ is a prime $z_R$-filter on $X$, then $Z_R^{-1}(\mathcal{P})$ is a prime ideal in $R$.

(e) Every $z_R$-ideal of $R$ is absolutely convex.

Thus, if $I$ is an absolutely convex ideal of $R$, then $R/I$ is a lattice ring.

(f) $I(f) \geq 0$ if and only if $f \geq 0$ on some $Z \in Z_R(I)$.

(g) Suppose that there exists $Z \in Z_R(I)$ such that $f(x) > 0$, for every $x \in Z$, then $I(f) > 0$. The converse is true whenever $I$ is a maximal $g$-ideal.

4. $z_R^\circ$-Ideals and $z^\circ$-Ideals in Subrings

In this section we generalize the concept of $z^\circ$-ideals of $C(X)$ to the subrings of $\mathbb{R}^X$ and introduce $z_R^\circ$-ideal. Coincidence of $z_R^\circ$-ideals with $z^\circ$-ideals of $R$ is discussed. Note that, for each element $f$ of a commutative rings $S$, we use $P_f(S)$ to denote the intersection of all the minimal prime ideals in $S$ containing $f$.

**Definition 4.1.** An ideal $I$ of a subring $R$ of $\mathbb{R}^X$ is called a $z_R^\circ$-ideal, if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$, where $f \in I$ and $g \in R$, implies $g \in I$.

The following statement investigates some characterizations of $z_R^\circ$-ideals in subrings.

**Theorem 4.2.** Let $R$ be a subring of $\mathbb{R}^X$ and $I$ be an ideal in $R$. The following statements are equivalent.
(a) $I$ is a $z_R^o$-ideal.
(b) Whenever $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$, then $g \in I$.
(c) $R \cap P_f(C) \subseteq I$ for each $f \in I$.
(d) Whenever $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$, then $g \in I$.

Proof. (a$\Rightarrow$b). First note that by [3, Lemma 2.1] we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ if and only if $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ for each $f, g \in C(X)$. Now, let $I$ be a $z_R^o$-ideal in $R$ and $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$ where $f \in I$ and $g \in R$. Thus, by our hypothesis, we have $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ which implies that $g \in I$.

(b$\Rightarrow$c). By [3, Proposition 2.3], we have $P_f(C) = \{g \in C(X) : \text{Ann}_C(f) \subseteq \text{Ann}_C(g)\}$. Thus the proof is evident.

(c$\Rightarrow$d). Let $P_g(C) \cap R \subseteq P_f(C) \cap R$, where $f \in I$ and $g \in R$. As $f \in I$, by our hypothesis, $P_f(C) \cap R \subseteq I$ and thus $P_g(C) \cap R \subseteq I$ which implies that $g \in I$.

(d$\Rightarrow$a). Let $\text{int}_X Z(f) \subseteq \text{int}_X Z(g)$ where $f \in I$ and $g \in R$. Therefore, by [3, Lemma 2.1], we have $P_f(C) \subseteq P_g(C)$ and hence $P_f(C) \cap R \subseteq P_g(C) \cap R$. Thus we are done by our hypothesis. \qed

Lemma 4.3. Let $R$ be a subring of $\mathbb{R}^X$, then for each $f \in R$ we have $P_f(C) \subseteq P_f(R)$. \qed

Proof. Let $g \in P_f(C)$. By [3, Proposition 2.3], we have $\text{Ann}_C(f) \subseteq \text{Ann}_C(g)$. Therefore, $\text{Ann}_R(f) = \text{Ann}_C(f) \cap R \subseteq \text{Ann}_C(g) \cap R = \text{Ann}_R(g)$. Thus, by [2, Proposition 1.5] we are done.

Theorem 4.4. Let $R$ be a subring of $\mathbb{R}^X$. Then every $z_R^o$-ideal in $R$ is a $z^o$-ideal if and only if $P_f(R) = P_f(C)$ for each $f \in R$.

Proof. (\Rightarrow). Assume on the contrary that there exists some $f \in R$ such that $P_f(R) \neq P_f(C)$. Thus, using Theorem 4.2 we have $P_f(C) \subseteq P_f(R)$. Again by Theorem 4.2, $P_f(C) \cap R$ is a $z_R^o$-ideal in $R$. Also, it is clear that this ideal is not a $z^o$-ideal, since, $P_f(R) \not\subseteq P_f(C) \cap R$.

(\Leftarrow). Let $I$ be a $z_R^o$-ideal in $R$ and $f \in I$. By Theorem 4.2, $P_f(C) \cap R \subseteq I$. Thus, by our hypothesis, $P_f(R) \subseteq I$ which means that $I$ is a $z^o$-ideal in $R$. \qed

From Theorem 4.2 it follows that every $z^o$-ideal in a subring $R$ is a $z_R^o$-ideal. However, the converse of this fact does not hold, in general. The following example gives an example of a subring $R$ which has a $z_R^o$-ideal that is not a $z^o$-ideal.

Example 4.5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$. It is clear that $f \in C(\mathbb{R})$. Now, let $R = \{\sum_{i=0}^{n} r_i f^i : r_i \in \mathbb{R}, n = 0, 1, \ldots\}$. It is easy to see that $P_f(R) = R$, however, $P_f(C) \cap R \neq R$. Also, by Theorem 4.2, $P_f(C) \cap R$ is $z_R^o$-ideal and it is clear that this ideal is not a $z^o$-ideal.
The next theorem gives a sufficient conditions on $X$ in order that $z^0_R$-ideals in a subring $R$ coincide with $z^0$-ideals of $R$.

**Theorem 4.6.** Let $R$ be a subring of $\mathbb{R}^X$ and $X$ be equipped with the $\text{Coz}(R)$-topology. Then an ideal $I$ in $R$ is a $z^0$-ideal if and only if it is a $z^0_R$-ideal.

**Proof.** Let $I$ be a $z^0_R$-ideal in $R$ and $f \in I$. As $X$ is equipped with the $\text{Coz}(R)$-topology, we have $g \in \text{Ann}_R(f)$ if and only if $\text{Coz}(g) \subseteq \text{int}_X Z(f)$ for each $f, g \in R$. Therefore, $P_f(R) = \text{Ann}_R \text{Ann}_R(f) = \{g \in R : \text{Coz}(g) \cap \text{int}_X Z(f) = \emptyset\} = \{g \in R : \text{Ann}_R(f) \subseteq \text{Ann}_R(g)\}$. Hence, $P_f(R) \subseteq I$ which means that $I$ is a $z^0$-ideal in $R$. This completes the proof, since, as former stated, every $z^0$-ideal in $R$ is a $z^0_R$-ideal.

Note that the condition that $X$ is equipped with the $\text{Coz}(R)$-topology is a sufficient condition for coincidence of $z^0_R$-ideals with $z^0$-ideals in a given subring $R$. The next example shows that this condition is not necessary.

**Example 4.7.** Let $X = \mathbb{R} \setminus \{0\}$ with the topology inherits from the usual topology on $\mathbb{R}$. Also, let $f : X \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$. It is clear that $f \in C(X)$ and $f^2 = f$. Now, set $R = \{r + sf : r, s \in \mathbb{R}\}$. It is clear that $R$ is a subring of $C(X)$. Also, by a routine reasoning, one can proves that the only ideals of $R$ are the ideals $(0), (f), (1 - f)$ and $R$. Moreover, the minimal prime ideals of $R$ are only the ideals ($f$) and $(1 - f)$. These imply that every $z^0_R$-ideal is a $z^0$-ideal in $R$. However, clearly, $X$ is not equipped with the $\text{Coz}(R)$-topology.

It follows from Theorem 4.6 that for an intermediate subalgebra $A(X)$ of $C(X)$, $z^0_A$-ideals coincide with $z^0$-ideals of $A(X)$. However, the same statement does not true for $z_A$-ideals and $z$-ideals in $A(X)$, in general, see [6, Theorem 2.2]. Moreover, Theorem 3.7 together with Theorem 4.6 imply that in the subalgebras of $C(X)$ which are of the form $\mathbb{R} + I$, where $I$ is a free ideal in $C(X)$, $z_{\mathbb{R} + I}$-ideals coincide with $z$-ideals of $\mathbb{R} + I$ and $z^0_{\mathbb{R} + I}$-ideals coincide with $z^0$-ideals, too. Note that whenever $I$ is a free ideal in $C(X)$, then $\mathbb{R} + I$ determines the topology of $X$.

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