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Tricyclic and Tetracyclic Graphs with Maximum and Minimum Eccentric Connectivity

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Abstract. Let G be a connected graph on n vertices. G is called tricyclic if it has $n+2$ edges, and tetracyclic if G has exactly $n+3$ edges. Suppose $\mathcal{C}_n$ and $\mathcal{D}_n$ denote the set of all tricyclic and tetracyclic $n-$vertex graphs, respectively. The aim of this paper is to calculate the minimum and maximum of eccentric connectivity index in $\mathcal{C}_n$ and $\mathcal{D}_n$.

Keywords: Tricyclic graph, Tetracyclic graph, Eccentric connectivity index.

2010 Mathematics subject classification: 05C12.

1. Introduction

In this section we recall some definitions that will be used in the paper. Let $G = (V, E)$ be a simple and finite undirected graph with v vertices and e edges and *Graph* denote the collection of all non-isomorphic finite graphs.

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Suppose G is a graph with vertex and edge sets of $V(G)$ and $E(G)$, respectively. If $x, y \in V(G)$ then the **distance** $d_G(x, y)$ (or d(x,y) for short) between x and y is defined as the length of a minimum path connecting x and y. The **Wiener index** is defined as the summation of distances between all pairs of vertices in the graph under consideration [20]. A topological index is called distance-based if it can be defined by distance function $d(-,-)$. It is worthy to mention here that Wiener did not consider the distance function $d(-,-)$ in his seminal paper, but Hosoya [10], was the first scientist presented a new simple formula for the Wiener index by using distance function. We encourage the readers to consult [4, 5, 12, 18] for more information on Wiener index.

The **cyclomatic number** of a connected graph G is defined as $c(G) = |E(G)| - |V(G)| + 1$. A graph G is called $k-$**cyclic**, if $c(G) = k$. In particular, if $c(G) = 1, 2, 3$ or 4 then G is called **unicyclic, bicyclic, tricyclic** or **tetracyclic graph**, respectively. Recently, some distance-based topological indices of unicyclic, bicyclic and tricyclic graphs are considered into account [9, 14, 16, 17, 22].

The eccentricity $\varepsilon_G(u)$ of a vertex u in a graph G is defined as the largest distance between u and other vertices of G. We will omit the subscript G when the graph is clear from the context. The eccentric connectivity index of G is defined as $\xi^c(G) = \sum_{u \in V(G)} deg(u)\varepsilon(u)$ [15]. We refer the interested readers to [1, 2] for some applications and [8, 11, 13, 21, 23] for the mathematical properties of this topological index.

Throughout this paper our notation is standard and taken mainly from the standard book of graph theory as [3, 19]. The complete, path, star and cycle graphs on n vertices are denoted by K_n, P_n, S_n and C_n, respectively. In this paper, we determine the maximum and minimum of the eccentric connectivity index in the classes of tricyclic and tetracyclic graphs in terms of its order.

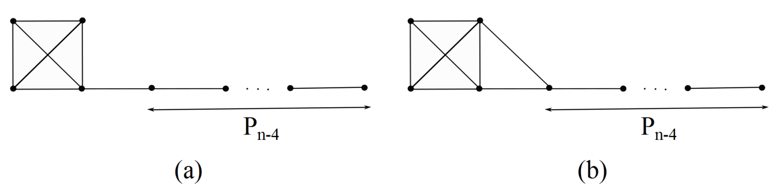

Figure 1. (a). A_n; (b). B_n.

2. Main Results

Let G and H be two simple and connected graphs with disjoint vertex sets. For given vertices $a \in V(G)$ and $b \in V(H)$, a splice of G and H is defined as the graph $(G \cdot H)_{(a,b)}$ obtained by identifying the vertices a and b. Similarly, a link of G and H is defined as the graph $(G \sim H)_{(a,b)}$ obtained by joining a

and b by an edge, see [7]. Suppose $\mathcal{C}_n$ and $\mathcal{D}_n$ denote the set of all tricyclic and tetracyclic $n-$vertex graphs, respectively. In this section the maximum and minimum of this topological index are computed in the classes of tricyclic and tetracyclic $n-$vertex graphs. Our work is a continuation of [22] that the authors computed the eccentric connectivity index of unicyclic graphs. In what follows, let A_n and B_n are graphs depicted in Figure 1. Furthermore, we will use $S_n + 3e$ and $S_n + 4e$ to denote the graphs obtained by inserting three and four arbitrary edges of $\bar{S_n}$ to S_n, respectively.

Theorem 2.1. *Suppose $G \in \mathcal{C}_n$, $n \geq 6$. Then $\xi^c(S_n + 3e) \leqslant \xi^c(G)$, with equality if and only if $G \cong S_n + 3e$.*

Proof. Let x is the number of vertices of degree $n-1$ in G. Then $\xi^c(G) \geqslant 4n + 8 - x(n-1)$, with equality if and only if each vertex of degree less than $n-1$ has eccentric connectivity 2. On the other hand, x is equal to 0 or 1. If $x = 0$, then $\xi^c(G) \geqslant 4n + 8$ and if $x = 1$, then $\xi^c(G) \geqslant 3n + 9$. So, each graph in which one vertex of degree $n-1$ and all other vertices of eccentric 2, has minimum eccentric connectivity. In other words, $S_n + 3e$ has minimum eccentric connectivity.

□

Using similar arguments as Theorem 2.1 one can prove the following result:

Theorem 2.2. *Let $G \in \mathcal{D}_n$, $n \geq 6$. Then $\xi^c(S_n + 4e) \leqslant \xi^c(G)$, with equality if and only if $G \cong S_n + 4e$.*

Lemma 2.3. *Let $G \in \mathcal{C}_n$ and u is a vertex of G. Suppose G' is obtained from G by adding a new vertex x, then joining x to u. Then $\xi^c(G') - \xi^c(G) \leqslant 2n + \varepsilon_G(u) + 6$ where $\varepsilon_G(u) \leqslant n-3$.*

Proof. It follows from the structure of G that, $\varepsilon_G(u) \leqslant n-3$. Now, assume that X is the sum of degrees over all vertices as v such that $\varepsilon_G(v) = \varepsilon_{G'}(v)$. Thus $X \geqslant 2[\frac{\varepsilon_G(u)+1}{2}] - 1$ and $\xi^c(G') - \xi^c(G) \leqslant 2n + 2\varepsilon_G(u) + 5 - X$. Therefore $\xi^c(G') - \xi^c(G) \leqslant 2n + 6 - 2[\frac{\varepsilon_G(u)+1}{2}] + 2\varepsilon_G(u) \leqslant 2n + \varepsilon_G(u) + 6$ that $\varepsilon_G(u) \leqslant n-3$. □

By Fig 2, it is not difficult to see that:

Lemma 2.4. *If $G \in \mathcal{C}_6$, then $\xi^c(G) \leqslant \xi^c(A_6)$.*

By definition of A_n, we calculate that:

Lemma 2.5. *For every $n \geqslant 6$,*

$$\xi^c(A_n) = \begin{cases} \frac{3}{2}n^2 + n - 18 & 2 \mid n \\ \frac{3}{2}n^2 + n - \frac{37}{2} & 2 \nmid n \end{cases}$$

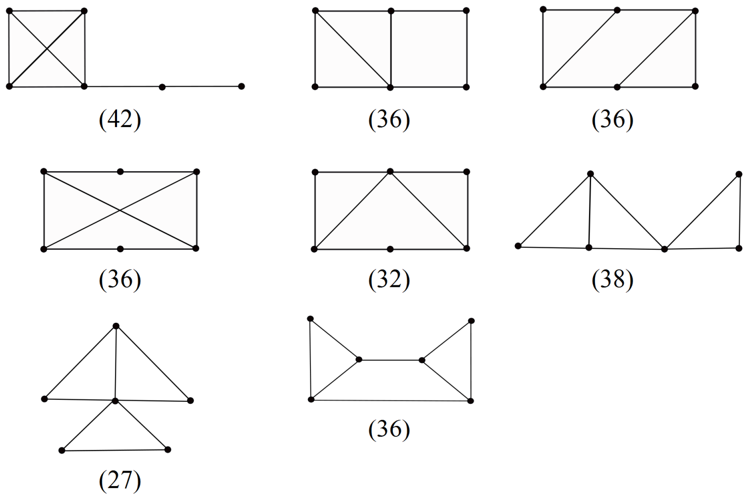

FIGURE 2. Some Tricyclic Graphs on Six Vertices with their Eccentric Connectivity Indices.

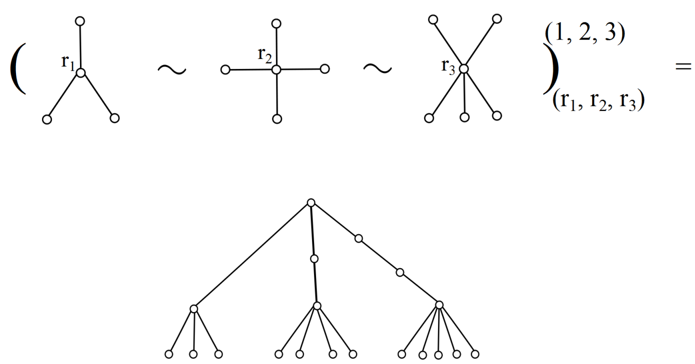

FIGURE 3. $(S_4 \sim S_5 \sim S_6)^{(1,2,3)}_{(r_1,r_2,r_3)}$.

A Hamiltonian cycle in G is a cycle such that visits each vertex exactly once. The graph G is called hamiltonian if it contains a Hamilton cycle. If G has vertices $v_1, v_2, ..., v_n$, the sequence $(deg(v_1), deg(v_2), ..., deg(v_n))$ is called a degree sequence of G. Suppose $G_1, ..., G_n$ are connected rooted graphs with root vertices $r_1, ..., r_n$, respectively. The quasilink $(G_1 \sim ... \sim G_n)^{(k_1,...,k_n)}_{(r_1,...,r_n)}$ is obtained by adding a new vertex x, then joining x to r_i by a path of length k_i, $i = 1, 2, ..., n$. As an example, Fig. 2 shows the quasilink of three stars S_4, S_5 and S_6. If $k_i = 0$, $i = 1, ..., n$, then the quasilink of $G_1, ..., G_n$ is isomorphic to the splice of $G_1, ..., G_n$. Furthermore, if $k_1 = 1$ and $k_2 = 0$ then $(G_1 \sim$

$G_2)^{(1,0)}_{(r_1,r_2)} \cong (G_1 \sim G_2)_{(r_1,r_2)}$. In what follows, we contract that at least one of k_i, is not equal to zero, $i = 1, 2, ..., n$.

Theorem 2.6. *Suppose $G \in \mathcal{C}_n$, $n \geq 6$. Then $\xi^c(G) \leqslant \xi^c(A_n)$.*

Proof. Induct on n. By Lemma 2.4, the result is valid for $n = 6$. Let $n \geqslant 7$ and assume the theorem holds for n. Suppose G is a tricyclic graph with $n+1$ vertices. If G has a pendent vertex as x, then by our assumption, $\xi^c(G - x) \leqslant \xi^c(A_n)$ and so, by Lemma 2.3, $\xi^c(G) \leqslant \xi^c(A_{n+1})$. If G does not have any pendent vertex, then it's degree sequence is equal to $(6, \underbrace{2, ..., 2}_{n-1})$, $(4, 4, \underbrace{2, ..., 2}_{n-2})$ or $(3, 3, 3, 3, \underbrace{2, ..., 2}_{n-4})$. Let G is Hamiltonian. Then for each vertex u of G, $\varepsilon(u) \leqslant \begin{cases} \frac{n}{2} & 2 \mid n \\ \frac{n-1}{2} & 2 \nmid n \end{cases}$. Thus $\xi^c(G) \leqslant \begin{cases} n^2 + 2n & 2 \mid n \\ n^2 + n - 2 & 2 \nmid n \end{cases}$ and so $\xi^c(G) \leqslant \xi^c(A_{n+1})$. Thus, if $(6, \underbrace{2, ..., 2}_{n-1})$ or $(4, 4, \underbrace{2, ..., 2}_{n-2})$ is a degree sequence of G, then $\xi^c(G) \leqslant \xi^c(A_{n+1})$. Assume that the degree sequence of G is $(3, 3, 3, 3, \underbrace{2, ..., 2}_{n-4})$. If G is not quasilink of some graphs, then by similar argument as above, $\xi^c(G) \leqslant \xi^c(A_{n+1})$. Otherwise, G is a quasilink of some graphs that one of them is a rooted cycle C and other one are connected rooted graphs isomorphic to G_1 with rooted vertex r_2. Let r_1 is the rooted vertex of C and, ur_1 and uv are their edges. If H is obtained from G by deleting edges uv and ur_1, then by adding edges ux and uy such that xy is an edge of G_1 with $d_{G_1}(x, r_2) = \varepsilon_{G_1}(r_2)$, we have $\xi^c(G) \leqslant \xi^c(H)$. On the other hand, H has a pendant vertex so, by the above argument $\xi^c(H) \leqslant \xi^c(A_{n+1})$, which completes the proof. □

Corollary 2.7. *Suppose $G \in \mathcal{C}_n$, $n \geq 6$. Then*

$$\xi^c(G) \leqslant \begin{cases} \frac{3}{2}n^2 + n - 18 & 2 \mid n \\ \frac{3}{2}n^2 + n - \frac{37}{2} & 2 \nmid n \end{cases}.$$

By Fig 4, we can write:

Lemma 2.8. *If $G \in \mathcal{D}_6$, then $\xi^c(G) \leqslant \xi^c(B_6)$.*

By a simple calculation, we can obtain:

Lemma 2.9. *For every $n \geqslant 7$,*

$$\xi^c(B_n) = \begin{cases} \frac{3}{2}n^2 + 3n - 30 & 2 \mid n \\ \frac{3}{2}n^2 + 3n - \frac{61}{2} & 2 \nmid n \end{cases}.$$

Now apply Lemma 2.8 and a similar technique as the proof of Theorem 2.6 to prove the following result:

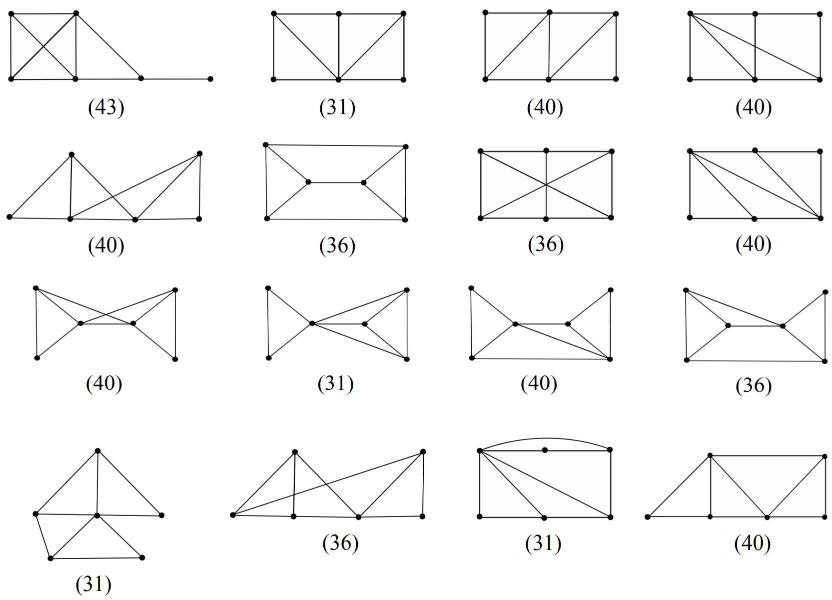

FIGURE 4. Some Tetracyclic Graphs on 6 Vertices with Their Eccentric connectivity Indices.

Theorem 2.10. *Suppose* $G \in \mathcal{D}_n$, $n \geq 6$. *Then* $\xi^c(G) \leqslant \xi^c(B_n)$.

Corollary 2.11. *Suppose* $G \in \mathcal{D}_n$, $n \geq 7$. *Then*

$$\xi^c(G) \leqslant \begin{cases} \frac{3}{2}n^2 + 3n - 30 & 2 \mid n \\ \frac{3}{2}n^2 + 3n - \frac{61}{2} & 2 \nmid n \end{cases}.$$

Proposition 2.12. *For every* $n \geqslant 6$, *we have* $\xi^c(A_n) \leqslant \xi^c(B_n)$.

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