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# Tricyclic and Tetracyclic Graphs with Maximum and Minimum Eccentric Connectivity

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Abstract. Let $G$ be a connected graph on $n$ vertices. $G$ is called tricyclic if it has $n+2$ edges, and tetracyclic if $G$ has exactly $n+3$ edges. Suppose $\mathcal{C}_n$ and $\mathcal{D}_n$ denote the set of all tricyclic and tetracyclic $n-$vertex graphs, respectively. The aim of this paper is to calculate the minimum and maximum of eccentric connectivity index in $\mathcal{C}_n$ and $\mathcal{D}_n$.

**Keywords:** Tricyclic graph, Tetracyclic graph, Eccentric connectivity index.

**2010 Mathematics subject classification:** 05C12.

## 1. Introduction

In this section we recall some definitions that will be used in the paper. Let $G = (V, E)$ be a simple and finite undirected graph with $v$ vertices and $e$ edges and *Graph* denote the collection of all non-isomorphic finite graphs.

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Suppose $G$ is a graph with vertex and edge sets of $V(G)$ and $E(G)$, respectively. If $x, y \in V(G)$ then the **distance** $d_G(x, y)$ (or d(x,y) for short) between $x$ and $y$ is defined as the length of a minimum path connecting $x$ and $y$. The **Wiener index** is defined as the summation of distances between all pairs of vertices in the graph under consideration [20]. A topological index is called distance-based if it can be defined by distance function $d(-,-)$. It is worthy to mention here that Wiener did not consider the distance function $d(-,-)$ in his seminal paper, but Hosoya [10], was the first scientist presented a new simple formula for the Wiener index by using distance function. We encourage the readers to consult [4, 5, 12, 18] for more information on Wiener index.

The **cyclomatic number** of a connected graph $G$ is defined as $c(G) = |E(G)| - |V(G)| + 1$. A graph $G$ is called $k-$**cyclic**, if $c(G) = k$. In particular, if $c(G) = 1, 2, 3$ or 4 then $G$ is called **unicyclic, bicyclic, tricyclic** or **tetracyclic graph**, respectively. Recently, some distance-based topological indices of unicyclic, bicyclic and tricyclic graphs are considered into account [9, 14, 16, 17, 22].

The eccentricity $\varepsilon_G(u)$ of a vertex $u$ in a graph $G$ is defined as the largest distance between $u$ and other vertices of $G$. We will omit the subscript $G$ when the graph is clear from the context. The eccentric connectivity index of $G$ is defined as $\xi^c(G) = \sum_{u \in V(G)} deg(u)\varepsilon(u)$ [15]. We refer the interested readers to [1, 2] for some applications and [8, 11, 13, 21, 23] for the mathematical properties of this topological index.

Throughout this paper our notation is standard and taken mainly from the standard book of graph theory as [3, 19]. The complete, path, star and cycle graphs on $n$ vertices are denoted by $K_n, P_n$, $S_n$ and $C_n$, respectively. In this paper, we determine the maximum and minimum of the eccentric connectivity index in the classes of tricyclic and tetracyclic graphs in terms of its order.

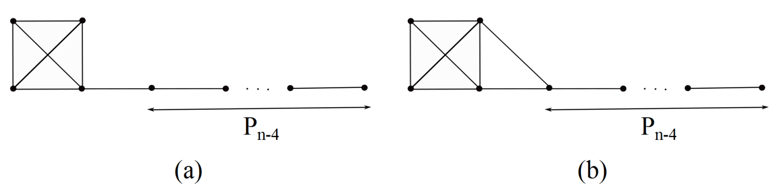


FIGURE 1. (a). $A_n$; (b). $B_n$.

## 2. Main Results

Let $G$ and $H$ be two simple and connected graphs with disjoint vertex sets. For given vertices $a \in V(G)$ and $b \in V(H)$, a splice of $G$ and $H$ is defined as the graph $(G \cdot H)_{(a,b)}$ obtained by identifying the vertices $a$ and $b$. Similarly, a link of $G$ and $H$ is defined as the graph $(G \sim H)_{(a,b)}$ obtained by joining $a$

and $b$ by an edge, see [7]. Suppose $\mathcal{C}_n$ and $\mathcal{D}_n$ denote the set of all tricyclic and tetracyclic $n-$vertex graphs, respectively. In this section the maximum and minimum of this topological index are computed in the classes of tricyclic and tetracyclic $n-$vertex graphs. Our work is a continuation of [22] that the authors computed the eccentric connectivity index of unicyclic graphs. In what follows, let $A_n$ and $B_n$ are graphs depicted in Figure 1. Furthermore, we will use $S_n + 3e$ and $S_n + 4e$ to denote the graphs obtained by inserting three and four arbitrary edges of $\bar{S_n}$ to $S_n$, respectively.

**Theorem 2.1.** *Suppose $G \in \mathcal{C}_n$, $n \geq 6$. Then $\xi^c(S_n + 3e) \leqslant \xi^c(G)$, with equality if and only if $G \cong S_n + 3e$.*

*Proof.* Let $x$ is the number of vertices of degree $n-1$ in $G$. Then $\xi^c(G) \geqslant 4n + 8 - x(n-1)$, with equality if and only if each vertex of degree less than $n-1$ has eccentric connectivity 2. On the other hand, $x$ is equal to 0 or 1. If $x = 0$, then $\xi^c(G) \geqslant 4n + 8$ and if $x = 1$, then $\xi^c(G) \geqslant 3n + 9$. So, each graph in which one vertex of degree $n-1$ and all other vertices of eccentric 2, has minimum eccentric connectivity. In other words, $S_n + 3e$ has minimum eccentric connectivity.

□

Using similar arguments as Theorem 2.1 one can prove the following result:

**Theorem 2.2.** *Let $G \in \mathcal{D}_n$, $n \geq 6$. Then $\xi^c(S_n + 4e) \leqslant \xi^c(G)$, with equality if and only if $G \cong S_n + 4e$.*

**Lemma 2.3.** *Let $G \in \mathcal{C}_n$ and $u$ is a vertex of $G$. Suppose $G'$ is obtained from $G$ by adding a new vertex $x$, then joining $x$ to $u$. Then $\xi^c(G') - \xi^c(G) \leqslant 2n + \varepsilon_G(u) + 6$ where $\varepsilon_G(u) \leqslant n-3$.*

*Proof.* It follows from the structure of $G$ that, $\varepsilon_G(u) \leqslant n-3$. Now, assume that $X$ is the sum of degrees over all vertices as $v$ such that $\varepsilon_G(v) = \varepsilon_{G'}(v)$. Thus $X \geqslant 2[\frac{\varepsilon_G(u)+1}{2}] - 1$ and $\xi^c(G') - \xi^c(G) \leqslant 2n + 2\varepsilon_G(u) + 5 - X$. Therefore $\xi^c(G') - \xi^c(G) \leqslant 2n + 6 - 2[\frac{\varepsilon_G(u)+1}{2}] + 2\varepsilon_G(u) \leqslant 2n + \varepsilon_G(u) + 6$ that $\varepsilon_G(u) \leqslant n-3$. □

By Fig 2, it is not difficult to see that:

**Lemma 2.4.** *If $G \in \mathcal{C}_6$, then $\xi^c(G) \leqslant \xi^c(A_6)$.*

By definition of $A_n$, we calculate that:

**Lemma 2.5.** *For every $n \geqslant 6$,*

$$\xi^c(A_n) = \begin{cases} \frac{3}{2}n^2 + n - 18 & 2 \mid n \\ \frac{3}{2}n^2 + n - \frac{37}{2} & 2 \nmid n \end{cases}$$

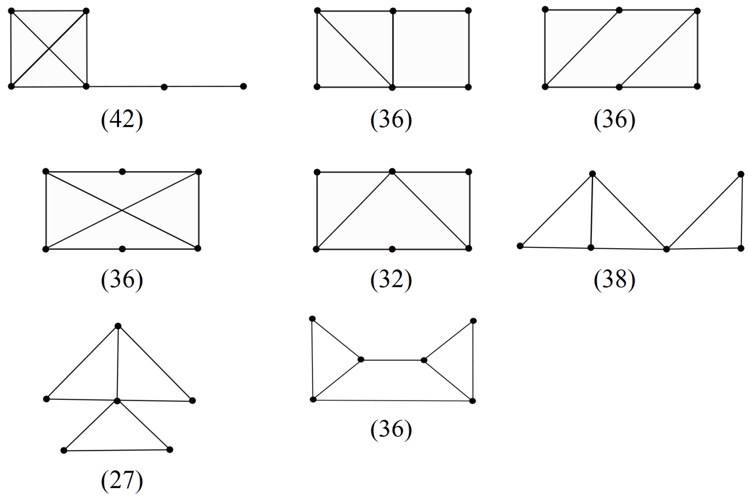


FIGURE 2. Some Tricyclic Graphs on Six Vertices with their Eccentric Connectivity Indices.

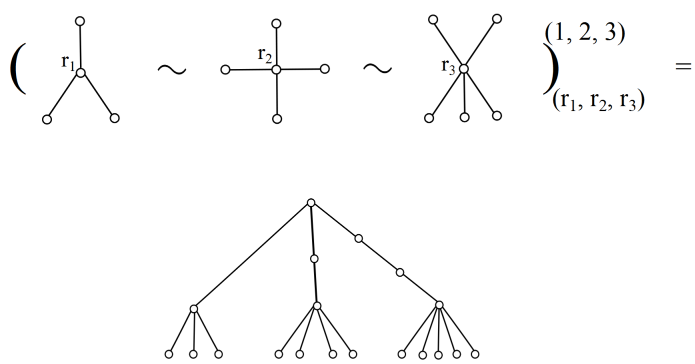


FIGURE 3. $(S_4 \sim S_5 \sim S_6)^{(1,2,3)}_{(r_1,r_2,r_3)}$.

A Hamiltonian cycle in $G$ is a cycle such that visits each vertex exactly once. The graph $G$ is called hamiltonian if it contains a Hamilton cycle. If $G$ has vertices $v_1, v_2, ..., v_n$, the sequence $(deg(v_1), deg(v_2), ..., deg(v_n))$ is called a degree sequence of $G$. Suppose $G_1, ..., G_n$ are connected rooted graphs with root vertices $r_1, ..., r_n$, respectively. The quasilink $(G_1 \sim ... \sim G_n)^{(k_1,...,k_n)}_{(r_1,...,r_n)}$ is obtained by adding a new vertex $x$, then joining $x$ to $r_i$ by a path of length $k_i$, $i = 1, 2, ..., n$. As an example, Fig. 2 shows the quasilink of three stars $S_4, S_5$ and $S_6$. If $k_i = 0$, $i = 1, ..., n$, then the quasilink of $G_1, ..., G_n$ is isomorphic to the splice of $G_1, ..., G_n$. Furthermore, if $k_1 = 1$ and $k_2 = 0$ then $(G_1 \sim$

$G_2)^{(1,0)}_{(r_1,r_2)} \cong (G_1 \sim G_2)_{(r_1,r_2)}$. In what follows, we contract that at least one of $k_i$, is not equal to zero, $i = 1, 2, ..., n$.

**Theorem 2.6.** *Suppose $G \in \mathcal{C}_n$, $n \geq 6$. Then $\xi^c(G) \leqslant \xi^c(A_n)$.*

*Proof.* Induct on $n$. By Lemma 2.4, the result is valid for $n = 6$. Let $n \geqslant 7$ and assume the theorem holds for $n$. Suppose $G$ is a tricyclic graph with $n+1$ vertices. If $G$ has a pendent vertex as $x$, then by our assumption, $\xi^c(G-x) \leqslant \xi^c(A_n)$ and so, by Lemma 2.3, $\xi^c(G) \leqslant \xi^c(A_{n+1})$. If $G$ does not have any pendent vertex, then it's degree sequence is equal to $(6, \underbrace{2, ..., 2}_{n-1})$, $(4, 4, \underbrace{2, ..., 2}_{n-2})$ or $(3, 3, 3, 3, \underbrace{2, ..., 2}_{n-4})$. Let $G$ is Hamiltonian. Then for each vertex $u$ of $G$, $\varepsilon(u) \leqslant \begin{cases} \frac{n}{2} & 2 \mid n \\ \frac{n-1}{2} & 2 \nmid n \end{cases}$. Thus $\xi^c(G) \leqslant \begin{cases} n^2+2n & 2 \mid n \\ n^2+n-2 & 2 \nmid n \end{cases}$ and so $\xi^c(G) \leqslant \xi^c(A_{n+1})$. Thus, if $(6, \underbrace{2, ..., 2}_{n-1})$ or $(4, 4, \underbrace{2, ..., 2}_{n-2})$ is a degree sequence of $G$, then $\xi^c(G) \leqslant \xi^c(A_{n+1})$. Assume that the degree sequence of $G$ is $(3, 3, 3, 3, \underbrace{2, ..., 2}_{n-4})$. If $G$ is not quasilink of some graphs, then by similar argument as above, $\xi^c(G) \leqslant \xi^c(A_{n+1})$. Otherwise, $G$ is a quasilink of some graphs that one of them is a rooted cycle $C$ and other one are connected rooted graphs isomorphic to $G_1$ with rooted vertex $r_2$. Let $r_1$ is the rooted vertex of $C$ and, $ur_1$ and $uv$ are their edges. If $H$ is obtained from $G$ by deleting edges $uv$ and $ur_1$, then by adding edges $ux$ and $uy$ such that $xy$ is an edge of $G_1$ with $d_{G_1}(x, r_2) = \varepsilon_{G_1}(r_2)$, we have $\xi^c(G) \leqslant \xi^c(H)$. On the other hand, $H$ has a pendant vertex so, by the above argument $\xi^c(H) \leqslant \xi^c(A_{n+1})$, which completes the proof. □

**Corollary 2.7.** *Suppose $G \in \mathcal{C}_n$, $n \geq 6$. Then*

$$\xi^c(G) \leqslant \begin{cases} \frac{3}{2}n^2 + n - 18 & 2 \mid n \\ \frac{3}{2}n^2 + n - \frac{37}{2} & 2 \nmid n \end{cases}.$$

By Fig 4, we can write:

**Lemma 2.8.** *If $G \in \mathcal{D}_6$, then $\xi^c(G) \leqslant \xi^c(B_6)$.*

By a simple calculation, we can obtain:

**Lemma 2.9.** *For every $n \geqslant 7$,*

$$\xi^c(B_n) = \begin{cases} \frac{3}{2}n^2 + 3n - 30 & 2 \mid n \\ \frac{3}{2}n^2 + 3n - \frac{61}{2} & 2 \nmid n \end{cases}.$$

Now apply Lemma 2.8 and a similar technique as the proof of Theorem 2.6 to prove the following result:

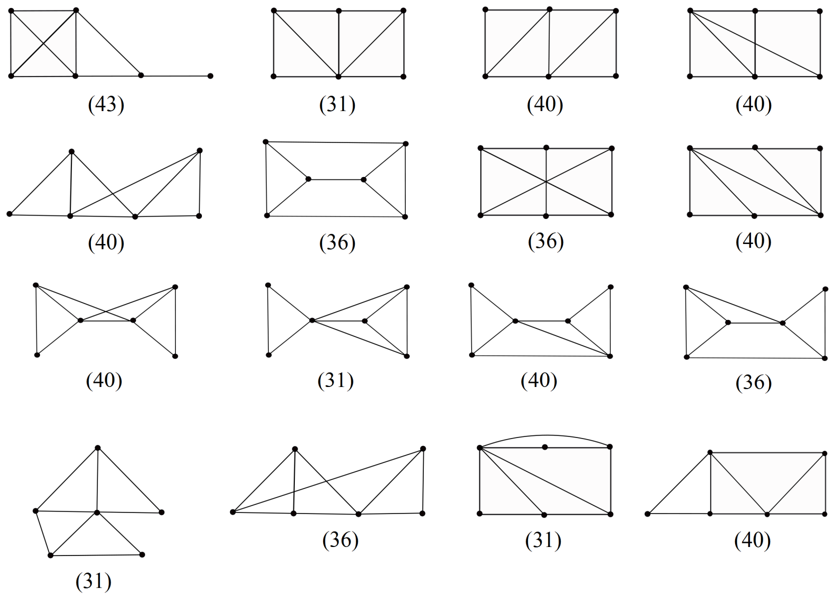


Figure 4. Some Tetracyclic Graphs on 6 Vertices with Their Eccentric connectivity Indices.

**Theorem 2.10.** *Suppose $G \in \mathcal{D}_n$, $n \geq 6$. Then $\xi^c(G) \leqslant \xi^c(B_n)$.*

**Corollary 2.11.** *Suppose $G \in \mathcal{D}_n$, $n \geq 7$. Then*

$$\xi^c(G) \leqslant \begin{cases} \frac{3}{2}n^2 + 3n - 30 & 2 \mid n \\ \frac{3}{2}n^2 + 3n - \frac{61}{2} & 2 \nmid n \end{cases}.$$

**Proposition 2.12.** *For every $n \geqslant 6$, we have $\xi^c(A_n) \leqslant \xi^c(B_n)$.*

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