

## Domination and Signed Domination Number of Cayley Graphs

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**ABSTRACT.** In this paper, we investigate domination number as well as signed domination numbers of  $Cay(G : S)$  for all cyclic group  $G$  of order  $n$ , where  $n \in \{p^m, pq\}$  and  $S = \{k < n : gcd(k, n) = 1\}$ . We also introduce some families of connected regular graphs  $\Gamma$  such that  $\gamma_S(\Gamma) \in \{2, 3, 4, 5\}$ .

**Keywords:** Cayley graph, Cyclic group, Domination number, Signed domination number.

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### 1. INTRODUCTION

By a graph  $\Gamma$  we mean a simple graph with vertex set  $V(\Gamma)$  and edge set  $E(\Gamma)$ . A graph is said to be *connected* if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining  $v_i$  and  $v_j$  is called the *distance* between  $v_i$  and  $v_j$  and denoted by  $d(v_i, v_j)$ . A graph  $\Gamma$  is said to be *regular* of degree  $k$  or, *k-regular* if every vertex has degree  $k$ . A subset  $P$  of vertices of  $\Gamma$  is a *k-packing* if  $d(x, y) > k$  for all pairs of distinct vertices  $x$  and  $y$  of  $P$  [9].

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Let  $G$  be a non-trivial group,  $S$  be an inverse closed subset of  $G$  which does not contain the identity element of  $G$ , i.e.  $S = S^{-1} = \{s^{-1} : s \in S\}$ . The Cayley graph of  $G$  denoted by  $Cay(G : S)$ , is a graph with vertex set  $G$  and two vertices  $a$  and  $b$  are adjacent if and only if  $ab^{-1} \in S$ . The Cayley graph  $Cay(G : S)$  is connected if and only if  $S$  generates  $G$ .

A set  $D \subseteq V$  of vertices in a graph  $\Gamma$  is a dominating set if every vertex  $v \in V$  is an element of  $D$  or adjacent to an element of  $D$ . The domination number  $\gamma(\Gamma)$  of a graph  $\Gamma$  is the minimum cardinality of a dominating set of  $\Gamma$ .

For a vertex  $v \in V(\Gamma)$ , the closed neighborhood  $N[v]$  of  $v$  is the set consisting  $v$  and all of its neighbors. For a function  $f : V(\Gamma) \rightarrow \{-1, 1\}$  and a subset  $W$  of  $V$  we define  $f(W) = \sum_{u \in W} f(u)$ . A signed dominating function of  $\Gamma$  is a function  $f : V(\Gamma) \rightarrow \{-1, 1\}$  such that  $f(N[v]) > 0$  for all  $v \in V(\Gamma)$ . The weight of a function  $f$  is  $\omega(f) = \sum_{v \in V} f(v)$ . The signed domination number  $\gamma_s(\Gamma)$  is the minimum weight of a signed dominating function of  $\Gamma$ . A signed dominating function of weight  $\gamma_s(\Gamma)$  is called a  $\gamma_s(\Gamma)$ -function. We denote  $f(N[v])$  by  $f[v]$ . Also for  $A \subseteq V(\Gamma)$  and signed dominating function  $f$ , set  $\{v \in A : f(v) = -1\}$  is denoted by  $A_f^-$ .

Finding some kinds of domination numbers of graphs is certainly one of the most important properties in any graph. (See for instance [2, 3, 5, 6, 11, 13])

These motivated us to consider on domination and signed domination number of Cayley graphs of cyclic group of orders  $p^n, pq$ , where  $p$  and  $q$  are prime numbers.

## 2. CAYLEY GRAPHS OF ORDER $p^n$

In this section  $p$  is a prime number and  $B(1, n) = \{k < n : \gcd(k, n) = 1\}$ .

**Lemma 2.1.** *Let  $G$  be a group and  $H$  be a proper subgroup of  $G$  such that  $[G : H] = t$ . If  $S = G \setminus H$ , then  $Cay(G : S)$  is a complete  $t$ -partite graph.*

*Proof.* One can see  $G = \langle S \rangle$  and  $e \notin S = S^{-1}$ . Let  $a \in G$ . If  $x, y \in Ha$ , then  $x = h_1a, y = h_2a$ . Since  $xy^{-1} \in H$ ,  $xy \notin E(Cay(G : S))$ . So induced subgraph on every coset of  $H$  is empty. Let  $Ha$  and  $Hb$  two disjoint cosets of  $H$  and  $x \in Ha, y \in Hb$ . Hence,  $xy^{-1} \in S$ . So  $xy \in E(Cay(G : S))$ . Therefore,  $Cay(G : S) = K_{|H|, |H|, \dots, |H|}$ .  $\square$

**Lemma 2.2.** *Let  $G$  be a group of order  $n$  and  $G = \langle S \rangle$ , where  $S = S^{-1}$  and  $0 \notin S$ . Then  $\gamma(Cay(G : S)) = 1$  if and only if  $S = G \setminus \{0\}$ .*

*Proof.* The proof is straightforward.  $\square$

**Theorem 2.3.** [13] Let  $K_{a,b}$  be a complete bipartite graph with  $b \leq a$ . Then

$$\gamma_S(K_{a,b}) = \begin{cases} a + 1 & \text{if } b = 1, \\ b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even,} \\ b + 1 & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is odd,} \\ 4 & \text{if } b \geq 4 \text{ and } a, b \text{ are both even,} \\ 6 & \text{if } b \geq 4 \text{ and } a, b \text{ are both odd,} \\ 5 & \text{if } b \geq 4 \text{ and } a, b \text{ have different parity.} \end{cases}$$

**Theorem 2.4.** Let  $\mathbb{Z}_{2^n} = \langle S \rangle$  and  $S = B(1, 2^n)$ . Then

- i.  $\text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2^{n-1}, 2^{n-1}}$
- ii.  $\gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2$ .
- iii.

$$\gamma_S(\text{Cay}(\mathbb{Z}_{2^n} : S)) = \begin{cases} 2 & \text{if } n = 1, 2, \\ 4 & \text{if } n \geq 3. \end{cases}$$

*Proof.* i. Let  $H = \mathbb{Z}_{2^n} \setminus S$ . Then  $H = \{i : 2 \mid i\}$ . It is not hard to see that  $H$  is a subgroup of  $\mathbb{Z}_{2^n}$  and  $[\mathbb{Z}_{2^n} : H] = 2$ . Hence, by Lemma 2.1,  $\text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2^{n-1}, 2^{n-1}}$ .

ii. By part i.  $\text{Cay}(\mathbb{Z}_{2^n} : S)$  is a complete bipartite graph. So

$$\gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2.$$

iii. The proof is straightforward by Theorem 2.3. □

**Corollary 2.5.** For any integer  $n > 2$ , there is a  $2^{n-1}$ -regular graph  $\Gamma$  with  $2^n$  vertices such that  $\gamma_S(\Gamma) = 4$ .

**Theorem 2.6.** Let  $\mathbb{Z}_{p^n} = \langle S \rangle$  ( $p$  odd prime) and  $S = B(1, p^n)$ . Then following statements hold:

- i.  $\text{Cay}(\mathbb{Z}_{p^n} : S)$  is a complete  $p$ -partite graph.
- ii.  $\gamma(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 2$ .
- iii.  $\gamma_S(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 3$ .

*Proof.* i. Let  $H = \mathbb{Z}_{p^n} \setminus S$ . Then  $H = \{i : p \mid i\}$ .  $H$  is a subgroup of  $\mathbb{Z}_{p^n}$  and  $|H| = p^n - \Phi(p^n) = p^{n-1}$ . So  $[\mathbb{Z}_{p^n} : H] = p$ . Hence, by Lemma 2.1,  $\text{Cay}(\mathbb{Z}_{p^n} : S)$  is a complete  $p$ -partite graph of size  $p^{n-1}$ .

ii. Since  $\text{Cay}(\mathbb{Z}_{p^n} : S)$  is a complete  $p$ -partite graph,  $D = \{a, b\}$  is a minimal dominating set where  $a, b$  are not in the same partition.

iii. Let  $\Gamma = \text{Cay}(\mathbb{Z}_{p^n} : S)$ . Let  $V(\Gamma) = \bigcup_{i=1}^p A_i$  where  $A_i = \{v_{ij} : 1 \leq j \leq p^{n-1}\}$ . Define  $f : V(\Gamma) \rightarrow \{-1, 1\}$

$$f(v_{ij}) = \begin{cases} -1 & \text{if } 1 \leq i \leq \lfloor \frac{p}{2} \rfloor - 1 \text{ and } 1 \leq j \leq \lceil \frac{p^{n-1}}{2} \rceil, \\ -1 & \text{if } \lfloor \frac{p}{2} \rfloor \leq i \leq p \text{ and } 1 \leq j \leq \lfloor \frac{p^{n-1}}{2} \rfloor, \\ 1 & \text{otherwise.} \end{cases}$$

Let  $v \in \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor - 1} A_i$ . So  $|N(v) \cap V_f^-| = \frac{1}{2}(p^n - p^{n-1} - 4)$ . So  $f[v] = f(v) + 4 \geq 3$ . If  $v \in \bigcup_{i=\lfloor \frac{p}{2} \rfloor}^p A_i$ , then  $|N(v) \cap V_f^-| = \frac{1}{2}(p^n - p^{n-1} - 2)$ . So  $f[v] = f(v) + 2 \geq 1$ . Hence,  $f$  is a signed dominating function. Since  $|V_f^-| = \frac{1}{2}(p^n - 3)$ ,  $\omega(f) = 3$ . So  $\gamma_S(\Gamma) \leq 3$ . On the contrary, suppose  $\gamma_S(\Gamma) < 3$ . So there is a  $\gamma_S$ -function  $g$  such that  $\omega(g) < 3$ . So  $|V_g^-| > \frac{1}{2}(p^n - 3)$ . Let  $|V_g^-| = \frac{1}{2}(p^n - 1)$ . If  $A_i \cap V_g^- = \emptyset$  for some  $1 \leq i \leq p$ , then  $g[v] = 1 - p^{n-1}$  for every  $v \in A_i$ . Hence,  $A_i \cap V_g^- \neq \emptyset$  for every  $1 \leq i \leq p$ . If  $|A_i \cap V_g^-| \geq \lceil \frac{p^{n-1}}{2} \rceil$  for every  $1 \leq i \leq p$ , then  $|V_g^-| \geq \frac{1}{2}(p^n + p)$ . This is impossible. So there is  $j \in \{1, 2, \dots, p\}$  such that  $|A_j \cap V_g^-| \leq \lfloor \frac{p^{n-1}}{2} \rfloor$ . Let  $u \in A_j \cap V_g^-$ . So  $g[u] = \deg(u) + 1 - 2|N(u) \cap V_g^-| < 0$ . This is contradiction. Therefore  $\gamma_S(\Gamma) = 3$ .  $\square$

**Corollary 2.7.** *For every integer  $n$ , there is a  $(p^n - p^{n-1})$ -regular graph  $\Gamma$  with  $p^n$  vertices such that  $\gamma_S(\Gamma) = 3$ .*

### 3. CAYLEY GRAPHS OF ORDER $pq$

In this section  $p$  and  $q$  are distinct prime numbers where  $p < q$ . Let  $B(1, pq)$  be a generator of  $\mathbb{Z}_{pq}$ . For  $1 \leq i \leq p$  and  $1 \leq j \leq q$ , set

$$A_i = \{i + kp : 0 \leq k \leq q - 1\}$$

and

$$B_j = \{j + k'q : 0 \leq k' \leq p - 1\}.$$

With these notations in mind we will prove the following results.

**Lemma 3.1.** *Let  $\mathbb{Z}_{pq} = \langle S \rangle$  and  $S = B(1, pq)$ . Then following statements hold.*

- i.  $V(\text{Cay}(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^p A_i$  and  $\text{Cay}(\mathbb{Z}_{pq} : S)$  is a  $p$ -partite graph.
- ii.  $V(\text{Cay}(\mathbb{Z}_{pq} : S)) = \bigcup_{j=1}^q B_j$  and  $\text{Cay}(\mathbb{Z}_{pq} : S)$  is a  $q$ -partite graph.
- iii. Let  $1 \leq i \leq p$ . For any  $x \in A_i$  there is some  $1 \leq j \leq q$  such that  $x \in B_j$ .
- iv.  $|A_i \cap B_j| = 1$  for every  $i, j$ .

*Proof.* i. Let  $s \in V(\text{Cay}(\mathbb{Z}_{pq} : S))$ . If  $p \mid s$ , then  $s \in A_p$ . Otherwise,  $s \in A_i$  where  $s = kp + i$  for some  $1 \leq k \leq (p - 1)$ . Thus  $V(\text{Cay}(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^p A_i$ . Since  $1 \leq i \neq j \leq p$ ,  $A_i \cap A_j = \emptyset$ . We show that the

induced subgraph on  $A_i$  is empty. Let  $l + t \in E(\text{Cay}(\mathbb{Z}_{pq} : S))$ . If  $l, t \in A_s$  for some  $1 \leq s \leq p$ , then  $l = s + kp, t = s + k'p$ . So  $p \mid (l - t)$ . This is impossible.

- ii. The proof is likewise part i.
- iii. Let  $1 \leq i \leq p$  and let  $x \in A_i$ . If  $x \leq q$ , then  $x \in B_x$ . If not,  $x = i + kp > q$  such that  $1 \leq k \leq q - 1$ . Hence,  $x \equiv t \pmod{q}$  where  $1 \leq t \leq q$ , and so  $x \in B_t$ .
- iv. By Case iii and since  $|A_i| = q$  and also for every  $j \neq j', B_j \cap B_{j'} = \emptyset$ , the result reaches.

□

**Theorem 3.2.** [6] For any graph  $\Gamma$ ,  $\left\lceil \frac{n}{1+\Delta(\Gamma)} \right\rceil \leq \gamma(\Gamma) \leq n - \Delta(\Gamma)$  where  $\Delta(\Gamma)$  is the maximum degree of  $\Gamma$ .

**Theorem 3.3.** Let  $\mathbb{Z}_{pq} = \langle S \rangle$  and  $S = B(1, pq)$ . Then the following is hold.

$$\gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) = \begin{cases} 2 & p = 2; \\ 3 & p > 2. \end{cases}$$

*Proof.* Let  $p = 2$ . By Lemma 3.1,  $D = \{i, i + q\}$  is a dominating set. Since  $\text{Cay}(\mathbb{Z}_{pq} : S)$  is a  $(q - 1)$ -regular graph, by Theorem 3.2,  $\gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) \geq 2$ . Thus  $\gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) = 2$ .

Let  $p > 2$ . We define  $D = \{1, 2, s\}$  where  $s \in A_1 \setminus N(2)$ . Since 1, 2 are adjacent,  $N(1) \cup N(2) = V(\text{Cay}(\mathbb{Z}_{pq} : S)) \setminus D$ . Thus  $D$  is a dominating set. As a consequence,  $\gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) \leq 2$ . It is enough to show that  $\gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) \neq 2$ . Let  $D' = \{x, y\}$ . We show that  $D'$  is not a dominating set. If  $x, y \in A_i$  for some  $1 \leq i \leq p$ , then for every  $z \in A_i \setminus D'$ ,  $z \notin N(D')$ . If not,  $x \in A_i$  and  $y \in A_j$  for some  $1 \leq i \neq j \leq p$ . If  $x, y$  are adjacent, then there is  $x' \in A_i \setminus \{x\}$  such that  $x' \notin N(y)$ . Thus  $D'$  is not dominating set. If  $x$  and  $y$  are not adjacent, then there is  $z \in A_l, l \neq i, j$ , such that the induced subgraph on  $\{x, y, z\}$  is empty. Hence,  $D'$  is not a dominating set and the proof is completed.

□

**Theorem 3.4.** Let  $\mathbb{Z}_{pq} = \langle S \rangle$  where  $p \in \{2, 3, 5\}$  and  $S = B(1, pq)$ . Then

$$\gamma_s(\text{Cay}(\mathbb{Z}_{pq} : S)) = p.$$

*Proof.* Let  $A = \{1, 1 + p, \dots, 1 + (\lfloor \frac{q}{2} \rfloor - 1)p\}$  and  $B = \{i + tq : i \in A \text{ and } 1 \leq t \leq p - 1\}$ . We define  $f : V(\text{Cay}(\mathbb{Z}_{pq} : S)) \rightarrow \{-1, 1\}$  such that

$$f(x) = \begin{cases} -1 & x \in A \cup B, \\ 1 & \text{otherwise.} \end{cases}$$

Let  $v \in V(\text{Cay}(\mathbb{Z}_{pq} : S))$ . If  $f(v) = -1$ , then

$$f[v] = -1 + (p - 1)(q - 1) - 2 \left( \left( \lfloor \frac{q}{2} \rfloor - 1 \right) (p - 1) \right) = 2p - 3.$$

Otherwise,

$$f[v] = 1 + (p-1)(q-1) - 2 \left\lfloor \frac{q}{2} \right\rfloor (p-1) = 1.$$

Hence,  $f$  is a dominating function. Also

$$\omega(f) = pq - 2(|A| + |B|) = pq - 2 \left( \left\lfloor \frac{q}{2} \right\rfloor + (p-1) \left\lfloor \frac{q}{2} \right\rfloor \right) = p.$$

It is enough to show that  $f$  has the minimal wait. Let, to the contrary,  $g$  be a dominating function and  $\omega(g) < \omega(f)$ . So  $|V_g^-| > |V_f^-|$ . Without loss of generality, suppose that  $|V_g^-| = p \lfloor \frac{q}{2} \rfloor + 1$ . Let  $A_i^- = A_i \cap V_g^-$ ,  $A_i^+ = A_i \setminus A_i^-$  and  $B_j^- = B_j \cap V_g^-$ . We will reach the contradiction by three steps.

Step 1. For every  $1 \leq i \leq p$ ,  $A_i^- \neq \emptyset$ .

On the contrary, let  $A_s^- = \emptyset$  for some  $1 \leq s \leq p$ . Let  $u \in A_s$ . Then by Lemma 3.1,  $u \in A_s \cap B_t$  for some  $1 \leq t \leq q$ . So

$$g[u] = (p-1)(q-1) + 1 - 2(|V_g^-| - |B_t^-|) \geq 1.$$

Thus  $|B_t^-| \geq \lceil \frac{q}{2} \rceil$ . Hence,  $|V_g^-| \geq |A_s| \lceil \frac{q}{2} \rceil$ . This implies  $q + (q-p) \lfloor \frac{q}{2} \rfloor < 1$ . This is a contradiction. Hence,  $A_s^- \neq \emptyset$ .

Similar argument applies for  $B_j$ . Therefore,  $B_j^- \neq \emptyset$  for every  $1 \leq j \leq q$ .

Step 2. For every  $1 \leq i \leq p$ ,  $|A_i^-| \geq \lfloor \frac{q}{2} \rfloor$ .

On the contrary, Let  $|A_l^-| < \lfloor \frac{q}{2} \rfloor$  for some  $1 \leq l \leq p$ . Without loss of generality suppose that  $|A_l^-| = \lfloor \frac{q}{2} \rfloor - 1$ . Let  $v \in A_l$ . By Lemma 3.1,  $v \in A_l \cap B_k$  for some  $1 \leq k \leq q$ . If  $g(v) = -1$ , then  $g[v] = (p-1)(q-1) - 1 - 2(|V_g^-| - |A_l^-| - |B_k^-| + 2) \geq 1$ . Then  $|B_k^- \setminus \{v\}| \geq 4$ . If  $g(v) = 1$ , then  $|B_k^- \setminus \{v\}| \geq 2$ . Hence,  $|V_g^-| \geq 4|A_l^-| + |A_l^-| + 2|A_l^+|$ . As a consequence  $p > 8$ . This is impossible.

Therefore, for every  $1 \leq i \leq p$ ,  $|A_i^-| \geq \lfloor \frac{q}{2} \rfloor$  and since  $|V_g^-| = p \lfloor \frac{q}{2} \rfloor + 1$ , we may suppose that  $|A_1^-| = \lceil \frac{q}{2} \rceil$  and  $|A_i^-| = \lfloor \frac{q}{2} \rfloor$  for  $2 \leq i \leq p$ .

Step 3. For every  $1 \leq j \leq q$ ,  $|B_j^-| \geq \lceil \frac{p}{2} \rceil$ .

On the contrary, let  $|B_h^-| < \lceil \frac{p}{2} \rceil$  for some  $1 \leq h \leq q$ . Suppose that  $|B_h^-| = \lfloor \frac{p}{2} \rfloor$ . By Lemma 3.1,  $B_h \cap A_i \neq \emptyset$  for any  $1 \leq i \leq p$ . Let  $z \in B_h^- \cap A_i$ . Thus

$$\begin{aligned} g[z] &= -1 + (p-1)(q-1) - 2(|V_g^-| - |A_i^-| - |B_h^-| + 2) \\ &\leq -1 + (p-1)(q-1) - 2 \left( p \left\lfloor \frac{q}{2} \right\rfloor + 1 - \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{p}{2} \right\rfloor + 2 \right) \\ &\leq p - 6 \end{aligned}$$

Since  $p \in \{2, 3, 5\}$ ,  $g[z] \leq -1$ . This is a contradiction.

By Step 3,  $|V_g^-| \geq q \lceil \frac{p}{2} \rceil$ . Hence,  $p \lfloor \frac{q}{2} \rfloor + 1 \geq q \lceil \frac{p}{2} \rceil$ . So  $p + q \leq 2$ . This is impossible. Therefore  $\gamma_S(\text{Cay}(G : S)) = \omega(f) = p$ .  $\square$

**Theorem 3.5.** Let  $\mathbb{Z}_{pq} = \langle S \rangle$  where  $p \geq 7$  and  $S = B(1, pq)$ . Then

$$\gamma_S(\text{Cay}(\mathbb{Z}_{pq} : S)) = 5.$$

*Proof.* We define  $f : V(\text{Cay}(\mathbb{Z}_{pq} : S)) \rightarrow \{-1, 1\}$  such that  $f(i) = -1$  if and only if  $i \in \{1, 2, \dots, \frac{pq-5}{2}\}$ . It is easily seen that  $\lfloor \frac{q}{2} \rfloor \leq |A_i^-| \leq \lceil \frac{q}{2} \rceil$  for every  $1 \leq i \leq p$ . Also  $\lfloor \frac{p}{2} \rfloor \leq |B_j^-| \leq \lceil \frac{p}{2} \rceil$  for any  $1 \leq j \leq q$ . Let  $v \in A_t \cap B_s$  such that  $1 \leq t \leq p$  and  $1 \leq s \leq q$ . In the worst situation,  $|A_t^-| = \lfloor \frac{q}{2} \rfloor$  and  $|B_s^-| = \lfloor \frac{p}{2} \rfloor$ . In this case  $1 \leq f[v] \leq 5$ . Hence,  $f$  is a signed dominating function. Also  $\omega(f) = pq - 2|V_f^-| = 5$ . Thus  $\gamma_s(\text{Cay}(\mathbb{Z}_{pq} : S)) \leq 5$ . What is left is to show that if  $g$  is a  $\gamma_s$ -function, then  $\omega(g) \geq 5$ . On the contrary, suppose that  $g$  be a  $\gamma_s$ -function and  $\omega(g) < \omega(f)$ . Hence,  $|V_g^-| < |V_f^-|$ . There is no loss of generality in assuming  $|V_g^-| = \frac{pq-3}{2}$ . Let  $A_i^- = A_i \cap V_g^-$  and  $B_j^- = B_j \cap V_g^-$ . In order to reach the contradiction we use two following steps:

Step 1.  $A_i^- \neq \emptyset$  for every  $1 \leq i \leq p$ .

On the contrary, suppose that for some  $1 \leq m \leq p$ ,  $A_m^- = \emptyset$ . Let  $w \in A_m$ . So there is  $1 \leq \ell \leq q$  such that  $w \in A_m \cap B_\ell$ . Hence,  $g[w] = (p-1)(q-1) + 1 - 2(|V_g^-| - |B_\ell^-|) \geq 1$ . Thus  $|B_\ell^-| \geq \frac{p+q-4}{2}$ . So  $|V_g^-| \geq q(\frac{p+q-4}{2})$ . Hence,  $pq - 3 \geq q(pq - 4)$ . This makes a contradiction.

By similar argument we have  $B_j^- \neq \emptyset$  for every  $1 \leq j \leq q$ .

Step 2. For every  $1 \leq i \leq p$ ,  $|A_i^-| \geq \lfloor \frac{q}{2} \rfloor$ .

On the contrary, let  $|A_l^-| = \lfloor \frac{q}{2} \rfloor - 1$ . Let  $v \in A_l$ . There is  $1 \leq l' \leq q$  such that  $v \in A_l \cap B_{l'}$ . If  $g(v) = -1$ , then  $g[v] = (p-1)(q-1) + 1 - 2(|V_g^-| - |A_l^-| - |B_{l'}^-| + 2) \geq 1$ . Hence,  $|B_{l'}^- \setminus \{v\}| \geq \lceil \frac{p}{2} \rceil$ . If  $g(v) = 1$ , then  $|B_{l'}^-| \geq \lfloor \frac{p}{2} \rfloor$ . Therefore,  $|V_g^-| \geq |A_l^-|(\lceil \frac{p}{2} \rceil + 1) + |A_l^+| \lfloor \frac{p}{2} \rfloor$ . This implies that  $q \leq 3$ . This is a contradiction.

Likewise Step 2,  $|B_j^-| \geq \lfloor \frac{p}{2} \rfloor$  for every  $1 \leq j \leq q$ . Since  $|V_g^-| = \frac{pq-3}{2}$ , there is  $1 \leq k \leq p$  such that  $|A_k^-| = \lfloor \frac{q}{2} \rfloor$ . On the other hand, suppose that for  $1 \leq t \leq q$ ,  $|B_t^-| = \lfloor \frac{p}{2} \rfloor$ . Let  $u \in A_k^- \cap B_t^-$ . If  $s \in \{l_1, \dots, l_t\}$ , then

$$\begin{aligned} g[u] &= -1 + (p-1)(q-1) - 2(|V_g^-| - |A_k^-| - |B_t^-| + 2) \\ &= -1 + (p-1)(q-1) - 2\left(\frac{pq-3}{2} - \lfloor \frac{q}{2} \rfloor - \lfloor \frac{p}{2} \rfloor + 2\right) \\ &= -3. \end{aligned}$$

This is a contradiction by  $g$  is a signed dominating function. Hence,  $s$  is not in  $\{l_1, \dots, l_t\}$ . Since  $|A_k^-| = \lfloor \frac{q}{2} \rfloor$ ,  $q-t \geq \lfloor \frac{q}{2} \rfloor$  and so  $t \leq \lceil \frac{q}{2} \rceil$ . As a consequence,

$$|V_g^-| \geq t \lfloor \frac{p}{2} \rfloor + (q-t) \lceil \frac{p}{2} \rceil \geq \lceil \frac{q}{2} \rceil \lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor \lceil \frac{p}{2} \rceil.$$

Since  $|V_g^-| = \frac{pq-3}{2}$ , this makes a contradiction. Therefore,

$$\gamma_s(\text{Cay}(\mathbb{Z}_{pq} : S)) = 5.$$

□

**Corollary 3.6.** For any  $k$ -regular graph  $\Gamma$  on  $n$  vertices  $\gamma_s(\Gamma) \geq \frac{n}{k+1}$ . Hence,  $\gamma_s(\Gamma) \geq 1$ . It is easy to check that  $\gamma_s(\Gamma) = 1$  if and only if  $\Gamma$  is a complete

graph and  $n$  is odd. Furthermore, for any prime numbers  $p < q$ , there is a  $(p-1)(q-1)$ -regular graph  $\Gamma$  with  $pq$  vertices such that  $\gamma_s(\Gamma) \in \{2, 3, 5\}$ .

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