Domination and Signed Domination Number of Cayley Graphs

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Abstract. In this paper, we investigate domination number as well as signed domination numbers of \( \text{Cay}(G : S) \) for all cyclic group \( G \) of order \( n \), where \( n \in \{p^m, pq\} \) and \( S = \{k < n : \gcd(k, n) = 1\} \). We also introduce some families of connected regular graphs \( \Gamma \) such that \( \gamma_s(\Gamma) \in \{2, 3, 4, 5\} \).

Keywords: Cayley graph, Cyclic group, Domination number, Signed domination number.

2000 Mathematics subject classification: 05C69, 05C25

1. Introduction

By a graph \( \Gamma \) we mean a simple graph with vertex set \( V(\Gamma) \) and edge set \( E(\Gamma) \). A graph is said to be connected if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining \( v_i \) and \( v_j \) is called the distance between \( v_i \) and \( v_j \) and denoted by \( d(v_i, v_j) \). A graph \( \Gamma \) is said to be regular of degree \( k \) or, \( k \)-regular if every vertex has degree \( k \). A subset \( P \) of vertices of \( \Gamma \) is a \( k \)-packing if \( d(x, y) > k \) for all pairs of distinct vertices \( x \) and \( y \) of \( P \) [9].
Let \( G \) be a non-trivial group, \( S \) be an inverse closed subset of \( G \) which does not contain the identity element of \( G \), i.e. \( S = S^{-1} = \{ s^{-1} : s \in S \} \). The Cayley graph of \( G \) denoted by \( \text{Cay}(G : S) \), is a graph with vertex set \( G \) and two vertices \( a \) and \( b \) are adjacent if and only if \( ab^{-1} \in S \). The Cayley graph \( \text{Cay}(G : S) \) is connected if and only if \( S \) generates \( G \).

A set \( D \subseteq V \) of vertices in a graph \( \Gamma \) is a dominating set if every vertex \( v \in V \) is an element of \( D \) or adjacent to an element of \( D \). The domination number \( \gamma(\Gamma) \) of a graph \( \Gamma \) is the minimum cardinality of a dominating set of \( \Gamma \).

For a vertex \( v \in V(\Gamma) \), the closed neighborhood \( N[v] \) of \( v \) is the set consisting \( v \) and all of its neighbors. For a function \( f : V(\Gamma) \to \{-1, 1\} \) and a subset \( W \) of \( V \) we define \( f(W) = \sum_{u \in W} f(u) \). A signed dominating function of \( \Gamma \) is a function \( f : V(\Gamma) \to \{-1, 1\} \) such that \( f(N[v]) > 0 \) for all \( v \in V(\Gamma) \). The weight of a function \( f \) is \( \omega(f) = \sum_{v \in V} f(v) \). The signed domination number \( \gamma_S(\Gamma) \) is the minimum weight of a signed dominating function of \( \Gamma \). A signed dominating function of weight \( \gamma_S(\Gamma) \) is called a \( \gamma_S(\Gamma) \)-function. We denote \( f(N[v]) \) by \( f[v] \). Also for \( A \subseteq V(\Gamma) \) and signed dominating function \( f \), set \( \{ v \in A : f(v) = -1 \} \) is denoted by \( A^- \).

Finding some kinds of domination numbers of graphs is certainly one of the most important properties in any graph. (See for instance [2, 3, 5, 6, 11, 13])

These motivated us to consider on domination and signed domination number of Cayley graphs of cyclic group of orders \( p^n, pq \), where \( p \) and \( q \) are prime numbers.

2. Cayley Graphs of Order \( p^n \)

In this section \( p \) is a prime number and \( B(1, n) = \{ k < n : \gcd(k, n) = 1 \} \).

**Lemma 2.1.** Let \( G \) be a group and \( H \) be a proper subgroup of \( G \) such that \( |G : H| = t \). If \( S = G \setminus H \), then \( \text{Cay}(G : S) \) is a complete \( t \)-partite graph.

**Proof.** One can see \( G = \langle S \rangle \) and \( e \notin S = S^{-1} \). Let \( a \in G \). If \( x, y \in Ha \), then \( x = h_1a, y = h_2a \). Since \( xy^{-1} \in H, xy \notin E(\text{Cay}(G : S)) \). So induced subgraph on every coset of \( H \) is empty. Let \( Ha \) and \( Hb \) two disjoint cosets of \( H \) and \( x \in Ha, y \in Hb \). Hence, \( xy^{-1} \in S \). So \( xy \in E(\text{Cay}(G : S)) \). Therefore, \( \text{Cay}(G : S) = K_{|H|, |H|, \ldots, |H|} \). \( \square \) 

**Lemma 2.2.** Let \( G \) be a group of order \( n \) and \( G = \langle S \rangle \), where \( S = S^{-1} \) and \( 0 \notin S \). Then \( \gamma(\text{Cay}(G : S)) = 1 \) if and only if \( S = G \setminus \{ 0 \} \).

**Proof.** The proof is straightforward. \( \square \)
Theorem 2.3. [13] Let \( K_{a,b} \) be a complete bipartite graph with \( b \leq a \). Then
\[
\gamma_s(K_{a,b}) = \begin{cases} 
  a + 1 & \text{if } b = 1, \\
  b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even}, \\
  b + 1 & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is odd}, \\
  4 & \text{if } b \geq 4 \text{ and } a, b \text{ are both even}, \\
  6 & \text{if } b \geq 4 \text{ and } a, b \text{ are both odd}, \\
  5 & \text{if } b \geq 4 \text{ and } a, b \text{ have different parity}.
\end{cases}
\]

Theorem 2.4. Let \( \mathbb{Z}_{2n} = \langle S \rangle \) and \( S = B(1, 2^n) \). Then
i. \( \text{Cay}(\mathbb{Z}_{2n} : S) = K_{2^{n-1}, 2^{n-1}} \)
ii. \( \gamma(\text{Cay}(\mathbb{Z}_{2n} : S)) = 2 \).
iii. \( \gamma_s(\text{Cay}(\mathbb{Z}_{2n} : S)) = \begin{cases} 
  2 & \text{if } n = 1, 2, \\
  4 & \text{if } n \geq 3.
\end{cases} \)

Proof. i. Let \( H = \mathbb{Z}_{2n} \setminus S \). Then \( H = \{ i : 2 \mid i \} \). It is not hard to see that \( H \) is a subgroup of \( \mathbb{Z}_{2n} \) and \( [\mathbb{Z}_{2n} : H] = 2 \). Hence, by Lemma 2.1, \( \text{Cay}(\mathbb{Z}_{2n} : S) = K_{2^{n-1}, 2^{n-1}} \).
ii. By part i. \( \text{Cay}(\mathbb{Z}_{2n} : S) \) is a complete bipartite graph. So \( \gamma(\text{Cay}(\mathbb{Z}_{2n} : S)) = 2 \).
iii. The proof is straightforward by Theorem 2.3.

□

Corollary 2.5. For any integer \( n > 2 \), there is a \( 2^{n-1} \)-regular graph \( \Gamma \) with \( 2^n \) vertices such that \( \gamma_s(\Gamma) = 4 \).

Theorem 2.6. Let \( \mathbb{Z}_{p^n} = \langle S \rangle \) (\( p \) odd prime) and \( S = B(1, p^n) \). Then following statements hold:
\[ \text{i. } \text{Cay}(\mathbb{Z}_{p^n} : S) \text{ is a complete } p \text{-partite graph.} \]
\[ \text{ii. } \gamma(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 2. \]
\[ \text{iii. } \gamma_s(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 3. \]

Proof. i. Let \( H = \mathbb{Z}_{p^n} \setminus S \). Then \( H = \{ i : p \mid i \} \). \( H \) is a subgroup of \( \mathbb{Z}_{p^n} \) and \( |H| = p^n - \Phi(p^n) = p^{n-1} \). So \( [\mathbb{Z}_{p^n} : H] = p \). Hence, by Lemma 2.1, \( \text{Cay}(\mathbb{Z}_{p^n} : S) \) is a complete \( p \)-partite graph of size \( p^n-1 \).
ii. Since \( \text{Cay}(\mathbb{Z}_{p^n} : S) \) is a complete \( p \)-partite graph, \( D = \{ a, b \} \) is a minimal dominating set where \( a, b \) are not in the same partition.
iii. Let \( \Gamma = \text{Cay}(\mathbb{Z}_{p^n} : S) \). Let \( V(\Gamma) = \bigcup_{i=1}^{p} A_i \), where \( A_i = \{ v_{ij} : 1 \leq j \leq p^{n-1} \} \). Define \( f : V(\Gamma) \to \{-1, 1\} \)
\[
f(v_{ij}) = \begin{cases} 
  -1 & \text{if } 1 \leq i \leq \frac{p}{2} - 1 \text{ and } 1 \leq j \leq \frac{p^{n-1}}{2}, \\
  -1 & \text{if } \frac{p}{2} \leq i \leq p \text{ and } 1 \leq j \leq \frac{p^{n-1}}{2}, \\
  1 & \text{otherwise}.
\end{cases}
\]
Let $v \in \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor - 1} A_i$. So $|N(v) \cap V^-| = \frac{1}{2}(p^n - p^{n-1} - 4)$. So $f[v] = f(v) + 4 \geq 3$. If $v \in \bigcup_{i=1}^{p} A_i$, then $|N(v) \cap V^-| = \frac{1}{2}(p^n - p^{n-1} - 2)$. So $f[v] = f(v) + 2 \geq 1$. Hence, $f$ is a signed dominating function. Since $|V^-_f| = \frac{1}{2}(p^n - 3)$, $\omega(f) = 3$. So $\gamma_s(\Gamma) \leq 3$. On the contrary, suppose $\gamma_s(\Gamma) < 3$. So there is a $\gamma_s$-function $g$ such that $\omega(g) < 3$. So $|V^-_g| > \frac{1}{2}(p^n - 3)$. Let $|V^-_g| = \frac{1}{2}(p^n - 1)$. If $A_i \cap V^-_g = \emptyset$ for some $1 \leq i \leq p$, then $g[v] = 1 - p^{n-1}$ for every $v \in A_i$. Hence, $A_i \cap V^-_g \neq \emptyset$ for every $1 \leq i \leq p$. If $|A_i \cap V^-_g| = \lceil \frac{p^n-1}{2} \rceil$ for every $1 \leq i \leq p$, then $|V^-_g| \geq \frac{1}{2}(p^n + p)$. This is impossible. So there is $j \in \{1, 2, \ldots, p\}$ such that $|A_j \cap V^-_g| \leq \lfloor \frac{p^n-1}{2} \rfloor$. Let $u \in A_j \cap V^-_g$. So $g[u] = \deg(u) + 1 - 2|N(u) \cap V^-_g| < 0$. This is contradiction. Therefore $\gamma_s(\Gamma) = 3$.

\[\square\]

**Corollary 2.7.** For every integer $n$, there is a $(p^n - p^{n-1})$-regular graph $\Gamma$ with $p^n$ vertices such that $\gamma_s(\Gamma) = 3$.

### 3. Cayley Graphs of Order $pq$

In this section $p$ and $q$ are distinct prime numbers where $p < q$. Let $B(1, pq)$ be a generator of $\mathbb{Z}_{pq}$. For $1 \leq i \leq p$ and $1 \leq j \leq q$, set

$$A_i = \{i + kp : 0 \leq k \leq q - 1\}$$

and

$$B_j = \{j + k'q : 0 \leq k' \leq p - 1\}.$$

With these notations in mind we will prove the following results.

**Lemma 3.1.** Let $\mathbb{Z}_{pq} = \langle S \rangle$ and $S = B(1, pq)$. Then following statements hold.

i. $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i$ and $Cay(\mathbb{Z}_{pq} : S)$ is a $p$-partite graph.

ii. $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{j=1}^{q} B_j$ and $Cay(\mathbb{Z}_{pq} : S)$ is a $q$-partite graph.

iii. Let $1 \leq i \leq p$. For any $x \in A_i$ there is some $1 \leq j \leq q$ such that $x \in B_j$.

iv. $|A_i \cap B_j| = 1$ for every $i, j$.

**Proof.**

i. Let $s \in V(Cay(\mathbb{Z}_{pq} : S))$. If $p \mid s$, then $s \in A_p$. Otherwise, $s \in A_i$ where $s = kp + i$ for some $1 \leq k \leq (p - 1)$. Thus $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i$. Since $1 \leq i \neq j \leq p$, $A_i \cap A_j = \emptyset$. We show that the
induced subgraph on \( A_i \) is empty. Let \( l + t \in E(\text{Cay}(\mathbb{Z}_{pq} : S)) \). If \( l, t \in A_i \) for some \( 1 \leq s \leq p \), then \( l = s + kp, t = s + k'p \). So \( p \mid (l - t) \).

This is impossible.

ii. The proof is likewise part i.

iii. Let \( 1 \leq i \leq p \) and let \( x \in A_i \). If \( x \leq q \), then \( x \in B_z \). If not, \( x = i + kp > q \) such that \( 1 \leq k \leq q - 1 \). Hence, \( x \equiv t \) (mod \( q \)) where \( 1 \leq t \leq q \), and so \( x \in B_t \).

iv. By Case iii and since \( |A_i| = q \) and also for every \( j \neq j' \), \( B_j \cap B_{j'} = \emptyset \), the result reaches.

\[ \square \]

**Theorem 3.2.** [6] For any graph \( \Gamma \), \( \left\lceil \frac{n}{1 + \Delta(\Gamma)} \right\rceil \leq \gamma(\Gamma) \leq n - \Delta(\Gamma) \) where \( \Delta(\Gamma) \) is the maximum degree of \( \Gamma \).

**Theorem 3.3.** Let \( \mathbb{Z}_{pq} = \langle S \rangle \) and \( S = B(1, pq) \). Then the following is hold.

\[
\gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) = \begin{cases} 2 & p = 2; \\ 3 & p > 2. \end{cases}
\]

**Proof.** Let \( p = 2 \). By Lemma 3.1, \( D = \{i, i + q\} \) is a dominating set. Since \( \text{Cay}(\mathbb{Z}_{pq} : S) \) is a (\( q - 1 \))-regular graph, by Theorem 3.2, \( \gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) \geq 2 \). Thus \( \gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) = 2 \).

Let \( p > 2 \). We define \( D = \{1, 2, s\} \) where \( s \in A_1 \setminus N(2) \). Since 1,2 are adjacent, \( N(1) \cup N(2) = \text{V}(\text{Cay}(\mathbb{Z}_{pq} : S)) \setminus D \). Thus \( D \) is a dominating set. As a consequence, \( \gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) \leq 2 \). It is enough to show that \( \gamma(\text{Cay}(\mathbb{Z}_{pq} : S)) \neq 2 \). Let \( D' = \{x, y\} \). We show that \( D' \) is not a dominating set. If \( x, y \in A_i \) for some \( 1 \leq i \leq p \), then for every \( z \in A_i \setminus D', z \notin N(D') \). If not, \( x, y \in A_i \), \( y \in A_j \) for some \( 1 \leq i \neq j \leq p \). If \( x, y \) are adjacent, then there is \( x' \in A_i \setminus \{x\} \) such that \( x' \notin N(y) \). Thus \( D' \) is not dominating set. If \( x \) and \( y \) are not adjacent, then there is \( z \in A_i, l \neq i, j \), such that the induced subgraph on \( \{x, y, z\} \) is empty. Hence, \( D' \) is not a dominating set and the proof is completed.

\[ \square \]

**Theorem 3.4.** Let \( \mathbb{Z}_{pq} = \langle S \rangle \) where \( p \in \{2, 3, 5\} \) and \( S = B(1, pq) \). Then

\[
\gamma_s(\text{Cay}(\mathbb{Z}_{pq} : S)) = p.
\]

**Proof.** Let \( A = \{1, 1 + p, \ldots, 1 + (\left\lfloor \frac{q}{2} \right\rfloor - 1)p\} \) and \( B = \{i + tq : i \in A \text{ and } 1 \leq t \leq p - 1\} \). We define \( f : \text{V}(\text{Cay}(\mathbb{Z}_{pq} : S)) \to \{-1, 1\} \) such that

\[
f(x) = \begin{cases} -1 & x \in A \cup B, \\ 1 & \text{otherwise}. \end{cases}
\]

Let \( v \in \text{V}(\text{Cay}(\mathbb{Z}_{pq} : S)) \). If \( f(v) = -1 \), then

\[
f[v] = -1 + (p - 1)(q - 1) - 2 \left( \left( \left\lfloor \frac{q}{2} \right\rfloor - 1 \right)(p - 1) \right) = 2p - 3.
\]
Otherwise, 
\[ f[v] = 1 + (p - 1)(q - 1) - 2 \left\lfloor \frac{q}{2} \right\rfloor (p - 1) = 1. \]

Hence, \( f \) is a dominating function. Also
\[ \omega(f) = pq - 2(|A| + |B|) = pq - 2 \left( \left\lfloor \frac{q}{2} \right\rfloor + (p - 1)\left\lfloor \frac{q}{2} \right\rfloor \right) = p. \]

It is enough to show that \( f \) has the minimal wait. Let, to the contrary, \( g \) be a dominating function and \( \omega(g) < \omega(f) \). So \( |V_g^-| > |V_f^-| \). Without lose of generality, suppose that \( |V_g^-| = p\left\lfloor \frac{q}{2} \right\rfloor + 1 \). Let \( A_i^- = A_i \cap V_g^- \), \( A_i^+ = A_i \setminus A_i^- \) and \( B_i^- = B_i \cap V_g^- \). We will reach the contradiction by three steps.

Step 1. For every \( 1 \leq i \leq p \), \( A_i^- \neq \emptyset \).
On the contrary, let \( A_i^- = \emptyset \) for some \( 1 \leq i \leq s \). Let \( u \in A_i \). Then by Lemma 3.1, \( u \in A_i \cap B_i \) for some \( 1 \leq t \leq q \). So
\[ g[u] = (p - 1)(q - 1) + 1 - 2(|V_g^-| - |B_i^-|) \geq 1. \]
Thus \( |B_i^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \). Hence, \( |V_g^-| \geq |A_s|\left\lfloor \frac{q}{2} \right\rfloor \). This implies \( q + (q - p)\left\lfloor \frac{q}{2} \right\rfloor < 1 \). This is a contradiction. Hence, \( A_i^- \neq \emptyset \).

Similar argument applies for \( B_j \). Therefore, \( B_j^- \neq \emptyset \) for every \( 1 \leq j \leq q \).

Step 2. For every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \).
On the contrary, Let \( |A_i^-| < \left\lfloor \frac{q}{2} \right\rfloor \) for some \( 1 \leq l \leq p \). Without lose of generality suppose that \( |A_i^-| = \left\lfloor \frac{q}{2} \right\rfloor - 1 \). Let \( v \in A_i \). By Lemma 3.1, \( v \in A_i \cap B_k \) for some \( 1 \leq k \leq q \). If \( g(v) = -1 \), then \( g[v] = (p - 1)(q - 1) - 1 - 2(|V_g^-| - |A_i^-| - |B_k^-| + 2) \geq 1 \). Then \( |B_k^- \setminus \{v\}| \geq 4 \).
If \( g(v) = 1 \), then \( |B_k^- \setminus \{v\}| \geq 2 \). Hence, \( |V_g^-| \geq 4|A_i^-| + |A_i^-| + 2|A_i^-| \).
As a consequence \( p > 8 \). This is impossible.

Therefore, for every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \) and since \( |V_g^-| = p\left\lfloor \frac{q}{2} \right\rfloor + 1 \), we may suppose that \( |A_i^-| = \left\lfloor \frac{q}{2} \right\rfloor \) and \( |A_i^-| = \left\lfloor \frac{q}{2} \right\rfloor \) for \( 2 \leq i \leq p \).

Step 3. For every \( 1 \leq j \leq q \), \( |B_j^-| \geq \left\lfloor \frac{q}{2} \right\rfloor \).
On the contrary, let \( |B_h^-| < \left\lfloor \frac{q}{2} \right\rfloor \) for some \( 1 \leq h \leq q \). Suppose that \( |B_h^-| = \left\lfloor \frac{q}{2} \right\rfloor \). By Lemma 3.1, \( B_h \cap A_i \neq \emptyset \) for any \( 1 \leq i \leq p \). Let \( z \in B_h \setminus A_i \). Thus
\[
g[z] = -1 + (p - 1)(q - 1) - 2(|V_g^-| - |A_i^-| - |B_h^-| + 2)
\leq -1 + (p - 1)(q - 1) - 2\left( p\left\lfloor \frac{q}{2} \right\rfloor + 1 - \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{p}{2} \right\rfloor + 2 \right)
\leq -6.\]

Since \( p \in \{2, 3, 5\} \), \( g[z] \leq -1 \). This is a contradiction.

By Step 3, \( |V_g^-| \geq q\left\lfloor \frac{p}{2} \right\rfloor \). Hence, \( p\left\lfloor \frac{p}{2} \right\rfloor + 1 \geq q\left\lfloor \frac{p}{2} \right\rfloor \). So \( p + q \leq 2 \). This is impossible. Therefore \( \gamma_s(Cay(G : S)) = \omega(f) = p. \]

**Theorem 3.5.** Let \( \mathbb{Z}_{pq} = \langle S \rangle \) where \( p \geq 7 \) and \( S = B(1, pq) \). Then \( \gamma_s(Cay(\mathbb{Z}_{pq} : S)) = 5. \)
Proof. We define \( f : V(Cay(Z_{pq} : S)) \rightarrow \{-1, 1\} \) such that \( f(i) = -1 \) if and only if \( i \in \{1, 2, \ldots, p-1\} \). It is easily seen that \( \frac{q}{2} \leq |A_i^-| \leq \frac{q}{2} \) for every \( 1 \leq i \leq p \). Also \( \frac{q}{2} \leq |B_j^-| \leq \frac{q}{2} \) for any \( 1 \leq j \leq q \). Let \( v \in A_1 \cap B_s \) such that \( 1 \leq t \leq p \) and \( 1 \leq s \leq q \). In the worst situation, \( |A_i^-| = \frac{q}{2} \) and \( |B_j^-| = \frac{q}{2} \). In this case \( 1 \leq f[v] \leq 5 \). Hence, \( f \) is a signed dominating function. Also \( \omega(f) = pq - 2|V_f^-| = 5 \). Thus \( \gamma_s(Cay(Z_{pq} : S)) \leq 5 \). What is left is to show that if \( g \) is a \( \gamma_s \)-function, then \( \omega(g) \geq 5 \). On the contrary, suppose that \( g \) be a \( \gamma_s \)-function and \( \omega(g) < \omega(f) \). Hence, \( |V_g^-| < |V_f^-| \). There is no loss of generality in assuming \( |V_g^-| = \frac{pq - 3}{2} \). Let \( A_i^- = A_i \cap V_g^- \) and \( B_j^- = B_j \cap V_g^- \). In order to reach the contradiction we use two following steps:

Step 1. \( A_i^- \neq \emptyset \) for every \( 1 \leq i \leq p \).

On the contrary, suppose that for some \( 1 \leq m \leq p \), \( A_m^- = \emptyset \). Let \( w \in A_m \). So there is \( 1 \leq \ell \leq q \) such that \( w \in A_m \cap B_{\ell} \). Hence, \( g[w] = (p-1)(q-1) + 1 - 2(|V_g^-| - |B_j^-|) \geq 1 \). Thus \( |B_j^-| \geq \frac{pq - 3}{2} \). So \( |V_g^-| \geq \frac{q}{2} \). Hence, \( pq - 3 \geq q(pq - 4) \). This makes a contradiction.

By similar argument we have \( B_j^- \neq \emptyset \) for every \( 1 \leq j \leq q \).

Step 2. For every \( 1 \leq i \leq p \), \( |A_i^-| \geq \frac{q}{2} \).

On the contrary, let \( |A_i^-| = \frac{q}{2} - 1 \). Let \( v \in A_i \). There is \( 1 \leq \ell \leq q \) such that \( v \in A_i \cap B_{\ell} \). If \( g(v) = -1 \), then \( g[v] = (p-1)(q-1) + 1 - 2(|V_g^-| - |A_i^-| - |B_{\ell}^-| + 2) \geq 1 \). Hence, \( |B_{\ell}^-| \geq \frac{q}{2} \). If \( g(v) = 1 \), then \( |B_{\ell}^-| \geq \frac{q}{2} \). Therefore, \( |V_g^-| \geq |A_i^-|(|\frac{q}{2}| + 1) + |A_i^-|(|\frac{q}{2}|) \). This implies that \( q \leq 3 \). This is a contradiction.

Likewise Step 2, \( |B_j^-| \geq \frac{q}{2} \) for every \( 1 \leq j \leq q \). Since \( |V_g^-| = \frac{pq - 3}{2} \), there is \( 1 \leq k \leq p \) such that \( |A_k^-| = \frac{q}{2} \). On the other hand, suppose that for \( 1 \leq t \leq q \), \( |B_t^-| = \frac{q}{2} \). Let \( u \in A_k \cap B_t \). If \( s \in \{l_1, \ldots, l_t\} \), then

\[
g[u] = -1 + (p-1)(q-1) - 2(|V_g^-| - |A_k^-| - |B_t^-| + 2) = -1 + (p-1)(q-1) - 2\left(\frac{pq - 3}{2} - \frac{q}{2} \right) = -3.\]

This is a contradiction by \( g \) is a signed dominating function. Hence, \( s \) is not in \( \{l_1, \ldots, l_t\} \). Since \( |A_k^-| = \frac{q}{2} \), \( q - t \geq \frac{q}{2} \) and so \( t \leq \frac{q}{2} \). As a consequence,

\[
|V_g^-| \geq t\left(\frac{p}{2}\right) + (q - t)\left(\frac{p}{2}\right) \geq \frac{q}{2}\left(\frac{p}{2}\right) + \frac{q}{2}\left(\frac{q}{2}\right).\]

Since \( |V_g^-| = \frac{pq - 3}{2} \), this makes a contradiction. Therefore,

\[
\gamma_s(Cay(Z_{pq} : S)) = 5.\]

\[\square\]

Corollary 3.6. For any \( k \)-regular graph \( \Gamma \) on \( n \) vertices \( \gamma_s(\Gamma) \geq \frac{n}{k+1} \). Hence, \( \gamma_s(\Gamma) \geq 1 \). It is easy to check that \( \gamma_s(\Gamma) = 1 \) if and only if \( \Gamma \) is a complete graph.

\[\square\]
graph and $n$ is odd. Furthermore, for any prime numbers $p < q$, there is a $(p - 1)(q - 1)$-regular graph $\Gamma$ with $pq$ vertices such that $\gamma_S(\Gamma) \in \{2, 3, 5\}$.

**Acknowledgments**

The author is thankful of referees for their valuable comments.

**References**