Domination and Signed Domination Number of Cayley Graphs

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Abstract. In this paper, we investigate domination number as well as signed domination numbers of $\text{Cay}(G : S)$ for all cyclic group $G$ of order $n$, where $n \in \{p^m, pq\}$ and $S = \{k < n : \gcd(k, n) = 1\}$. We also introduce some families of connected regular graphs $\Gamma$ such that $\gamma_S(\Gamma) \in \{2, 3, 4, 5\}$.

Keywords: Cayley graph, Cyclic group, Domination number, Signed domination number.

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1. Introduction

By a graph $\Gamma$ we mean a simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A graph is said to be connected if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining $v_i$ and $v_j$ is called the distance between $v_i$ and $v_j$ and denoted by $d(v_i, v_j)$. A graph $\Gamma$ is said to be regular of degree $k$ or, $k$-regular if every vertex has degree $k$. A subset $P$ of vertices of $\Gamma$ is a $k$-packing if $d(x, y) > k$ for all pairs of distinct vertices $x$ and $y$ of $P$ [9].

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Let $G$ be a non-trivial group, $S$ be an inverse closed subset of $G$ which does not contain the identity element of $G$, i.e. $S = S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph of $G$ denoted by $\text{Cay}(G : S)$, is a graph with vertex set $G$ and two vertices $a$ and $b$ are adjacent if and only if $ab^{-1} \in S$. The Cayley graph $\text{Cay}(G : S)$ is connected if and only if $S$ generates $G$.

A set $D \subseteq V$ of vertices in a graph $\Gamma$ is a dominating set if every vertex $v \in V$ is an element of $D$ or adjacent to an element of $D$. The domination number $\gamma(\Gamma)$ of a graph $\Gamma$ is the minimum cardinality of a dominating set of $\Gamma$.

For a vertex $v \in V(\Gamma)$, the closed neighborhood $N[v]$ of $v$ is the set consisting $v$ and all of its neighbors. For a function $f : V(\Gamma) \to \{-1, 1\}$ and a subset $W$ of $V$ we define $f(W) = \sum_{u \in W} f(u)$. A signed dominating function of $\Gamma$ is a function $f : V(\Gamma) \to \{-1, 1\}$ such that $f(N[v]) > 0$ for all $v \in V(\Gamma)$. The weight of a function $f$ is $\omega(f) = \sum_{v \in V} f(v)$. The signed domination number $\gamma_s(\Gamma)$ is the minimum weight of a signed dominating function of $\Gamma$. A signed dominating function of weight $\gamma_s(\Gamma)$ is called a $\gamma_s(\Gamma)$–function. We denote $f(N[v])$ by $f[v]$. Also for $A \subseteq V(\Gamma)$ and signed dominating function $f$, set $\{v \in A : f(v) = -1\}$ is denoted by $A_f^−$.

Finding some kinds of domination numbers of graphs is certainly one of the most important properties in any graph. (See for instance \cite{2, 3, 5, 6, 11, 13})

These motivated us to consider on domination and signed domination number of Cayley graphs of cyclic group of orders $p^n, pq$, where $p$ and $q$ are prime numbers.

2. Cayley Graphs of Order $p^n$

In this section $p$ is a prime number and $B(1, n) = \{k < n : \gcd(k, n) = 1\}$.

**Lemma 2.1.** Let $G$ be a group and $H$ be a proper subgroup of $G$ such that $[G : H] = t$. If $S = G \setminus H$, then $\text{Cay}(G : S)$ is a complete $t$-partite graph.

**Proof.** One can see $G = \langle S \rangle$ and $e \notin S = S^{-1}$. Let $a \in G$. If $x, y \in Ha$, then $x = h_1 a, y = h_2 a$. Since $xy^{-1} \in H, xy \notin E(\text{Cay}(G : S))$. So induced subgraph on every coset of $H$ is empty. Let $Ha$ and $Hb$ two disjoint cosets of $H$ and $x \in Ha, y \in Hb$. Hence, $xy^{-1} \in S$. So $xy \in E(\text{Cay}(G : S))$. Therefore, $\text{Cay}(G : S) = K_{|H| \times |H| \times \cdots \times |H|}$.

**Lemma 2.2.** Let $G$ be a group of order $n$ and $G = \langle S \rangle$, where $S = S^{-1}$ and $0 \notin S$. Then $\gamma(\text{Cay}(G : S)) = 1$ if and only if $S = G \setminus \{0\}$.

**Proof.** The proof is straightforward.
Theorem 2.3. [13] Let \( K_{a,b} \) be a complete bipartite graph with \( b \leq a \). Then
\[
\gamma_s(K_{a,b}) = \begin{cases} 
    a + 1 & \text{if } b = 1, \\
    b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even,} \\
    b + 1 & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is odd,} \\
    4 & \text{if } b \geq 4 \text{ and } a, b \text{ are both even,} \\
    6 & \text{if } b \geq 4 \text{ and } a, b \text{ are both odd,} \\
    5 & \text{if } b \geq 4 \text{ and } a, b \text{ have different parity.} 
\end{cases}
\]

Theorem 2.4. Let \( \mathbb{Z}_{2^n} = \langle S \rangle \) and \( S = B(1,2^n) \). Then
1. \( \text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2n-1,2n-1} \)
2. \( \gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2. \)
3. \( \gamma_s(\text{Cay}(\mathbb{Z}_{2^n} : S)) = \begin{cases} 
    2 & \text{if } n = 1,2, \\
    4 & \text{if } n \geq 3. 
\end{cases} \)

Proof. 
1. Let \( H = \mathbb{Z}_{2^n} \setminus S \). Then \( H = \{ i : 2 \mid i \} \). It is not hard to see that \( H \) is a subgroup of \( \mathbb{Z}_{2^n} \) and \( [\mathbb{Z}_{2^n} : H] = 2 \). Hence, by Lemma 2.1, \( \text{Cay}(\mathbb{Z}_{2^n} : S) = K_{2n-1,2n-1} \).
2. By part i. \( \text{Cay}(\mathbb{Z}_{2^n} : S) \) is a complete bipartite graph. So \( \gamma(\text{Cay}(\mathbb{Z}_{2^n} : S)) = 2. \)
3. The proof is straightforward by Theorem 2.3.

\( \square \)

Corollary 2.5. For any integer \( n > 2 \), there is a \( 2^{n-1} \)-regular graph \( \Gamma \) with \( 2^n \) vertices such that \( \gamma_s(\Gamma) = 4 \).

Theorem 2.6. Let \( \mathbb{Z}_{p^n} = \langle S \rangle \) (\( p \) odd prime) and \( S = B(1,p^n) \). Then following statements hold:
1. \( \text{Cay}(\mathbb{Z}_{p^n} : S) \) is a complete \( p \)-partite graph.
2. \( \gamma(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 2. \)
3. \( \gamma_s(\text{Cay}(\mathbb{Z}_{p^n} : S)) = 3. \)

Proof. 
1. Let \( H = \mathbb{Z}_{p^n} \setminus S \). Then \( H = \{ i : p \mid i \} \). \( H \) is a subgroup of \( \mathbb{Z}_{p^n} \) and \( |H| = p^n - \Phi(p^n) = p^{n-1} \). So \( [\mathbb{Z}_{p^n} : H] = p \). Hence, by Lemma 2.1, \( \text{Cay}(\mathbb{Z}_{p^n} : S) \) is a complete \( p \)-partite graph of size \( p^{n-1} \).
2. Since \( \text{Cay}(\mathbb{Z}_{p^n} : S) \) is a complete \( p \)-partite graph, \( D = \{a,b\} \) is a minimal dominating set where \( a, b \) are not in the same partition.
3. Let \( \Gamma = \text{Cay}(\mathbb{Z}_{p^n} : S) \). Let \( V(\Gamma) = \bigcup_{i=1}^{p} A_i \) where \( A_i = \{v_{ij} : 1 \leq j \leq p^{n-1}\} \). Define \( f : V(\Gamma) \rightarrow \{-1,1\} \)
\[
f(v_{ij}) = \begin{cases} 
    -1 & \text{if } 1 \leq i \leq \lfloor \frac{p}{2} \rfloor - 1 \text{ and } 1 \leq j \leq \lfloor \frac{p^{n-1}}{2} \rfloor, \\
    -1 & \text{if } \lfloor \frac{p}{2} \rfloor \leq i \leq p \text{ and } 1 \leq j \leq \lfloor \frac{p^{n-1}}{2} \rfloor, \\
    1 & \text{otherwise.} 
\end{cases} 
\]
Let $v \in \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor - 1} A_i$. So $|N(v) \cap V^-| = \frac{1}{2}(p^n - p^{n-1} - 4)$. So $f[v] = f(v) + 4 \geq 3$. If $v \in \bigcup_{i=1}^{\frac{p}{2}} A_i$, then $|N(v) \cap V^-| = \frac{1}{2}(p^n - p^{n-1} - 2)$.

So $f[v] = f(v) + 2 \geq 1$. Hence, $f$ is a signed dominating function. Since $|V^-| = \frac{1}{2}(p^n - 3)$, $\omega(f) = 3$. So $\gamma_S(\Gamma) \leq 3$. On the contrary, suppose $\gamma_S(\Gamma) < 3$. So there is a $\gamma_S$-function $g$ such that $\omega(g) < 3$. So $|V^-_g| > \frac{1}{2}(p^n - 3)$. Let $|V^-_g| = \frac{1}{2}(p^n - 1)$. If $A_i \cap V^-_g = \emptyset$ for some $1 \leq i \leq p$, then $g[v] = 1 - p^{n-1}$ for every $v \in A_i$. Hence, $A_i \cap V^-_g \neq \emptyset$ for every $1 \leq i \leq p$. If $|A_i \cap V^-_g| \geq \lceil \frac{p^n-3}{2} \rceil$ for every $1 \leq i \leq p$, then $|V^-_g| \geq \frac{1}{2}(p^n + p)$. This is impossible. So there is $j \in \{1, 2, \ldots, p\}$ such that $|A_j \cap V^-_g| \leq \lceil \frac{p^n-3}{2} \rceil$. Let $u \in A_j \cap V^-_g$. So $g[u] = \deg(u) + 1 - 2|N(u) \cap V^-_g| < 0$. This is contradiction. Therefore $\gamma_S(\Gamma) = 3$.

\[ \square \]

**Corollary 2.7.** For every integer $n$, there is a $(p^n - p^{n-1})$-regular graph $\Gamma$ with $p^n$ vertices such that $\gamma_S(\Gamma) = 3$.

3. **Cayley Graphs of Order $pq$**

In this section $p$ and $q$ are distinct prime numbers where $p < q$. Let $B(1, pq)$ be a generator of $\mathbb{Z}_{pq}$. For $1 \leq i \leq p$ and $1 \leq j \leq q$, set

$$A_i = \{i + kp : 0 \leq k \leq q - 1\}$$

and

$$B_j = \{j + k'q : 0 \leq k' \leq p - 1\}.$$  

With these notations in mind we will prove the following results.

**Lemma 3.1.** Let $\mathbb{Z}_{pq} = \langle S \rangle$ and $S = B(1, pq)$. Then following statements hold.

i. $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i$ and $Cay(\mathbb{Z}_{pq} : S)$ is a $p$-partite graph.

ii. $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{j=1}^{q} B_j$ and $Cay(\mathbb{Z}_{pq} : S)$ is a $q$-partite graph.

iii. Let $1 \leq i \leq p$. For any $x \in A_i$ there is some $1 \leq j \leq q$ such that $x \in B_j$.

iv. $|A_i \cap B_j| = 1$ for every $i, j$.

**Proof.**

i. Let $s \in V(Cay(\mathbb{Z}_{pq} : S))$. If $p \mid s$, then $s \in A_p$. Otherwise, $s \in A_i$ where $s = kp + i$ for some $1 \leq k \leq (p - 1)$. Thus $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i$. Since $1 \leq i \neq j \leq p$, $A_i \cap A_j = \emptyset$. We show that the
induced subgraph on $A_i$ is empty. Let $l + t \in E(Cay(Z_{pq} : S))$. If $l, t \in A_i$ for some $1 \leq s \leq p$, then $l = s + kp, t = s + k'p$. So $p \mid (l - t)$. This is impossible.

ii. The proof is likewise part i.

iii. Let $1 \leq i \leq p$ and let $x \in A_i$. If $x \leq q$, then $x \in B_z$. If not, $x = i + kp > q$ such that $1 \leq k \leq q - 1$. Hence, $x \equiv t \mod q$ where $1 \leq t \leq q$, and so $x \in B_t$.

iv. By Case iii and since $|A_i| = q$ and also for every $j \neq j'$, $B_j \cap B_{j'} = \emptyset$, the result reaches.

\[ \square \]

**Theorem 3.2.** [6] For any graph $\Gamma$, \[ \frac{n}{\Gamma + \Delta(\Gamma)} \leq \gamma(\Gamma) \leq n - \Delta(\Gamma) \] where $\Delta(\Gamma)$ is the maximum degree of $\Gamma$.

**Theorem 3.3.** Let $Z_{pq} = \langle S \rangle$ and $S = B(1, pq)$. Then the following is hold.

\[ \gamma(Cay(Z_{pq} : S)) = \begin{cases} 2 & p = 2; \\ 3 & p > 2. \end{cases} \]

**Proof.** Let $p = 2$. By Lemma 3.1, $D = \{i, i + q\}$ is a dominating set. Since $Cay(Z_{pq} : S)$ is a $(q - 1)$-regular graph, by Theorem 3.2, $\gamma(Cay(Z_{pq} : S)) \geq 2$. Thus $\gamma(Cay(Z_{pq} : S)) = 2$.

Let $p > 2$. We define $D = \{1, 2, s\}$ where $s \in A_1 \setminus N(2)$. Since 1,2 are adjacent, $N(1) \cup N(2) = V(Cay(Z_{pq} : S)) \setminus D$. Thus $D$ is a dominating set. As a consequence, $\gamma(Cay(Z_{pq} : S)) \leq 2$. It is enough to show that $\gamma(Cay(Z_{pq} : S)) \neq 2$. Let $D' = \{x, y\}$. We show that $D'$ is not a dominating set. If $x, y \in A_i$ for some $1 \leq i \leq p$, then for every $z \in A_i \setminus D'$, $z \notin N(D')$. If not, $x \in A_i$ and $y \in A_j$ for some $1 \leq i \neq j \leq p$. If $x, y$ are adjacent, then there is $x' \in A_i \setminus \{x\}$ such that $x' \notin N(y)$. Thus $D'$ is not dominating set. If $x$ and $y$ are not adjacent, then there is $z \in A_i, l \neq i, j$, such that the induced subgraph on $\{x, y, z\}$ is empty. Hence, $D'$ is not a dominating set and the proof is completed.

\[ \square \]

**Theorem 3.4.** Let $Z_{pq} = \langle S \rangle$ where $p \in \{2, 3, 5\}$ and $S = B(1, pq)$. Then

\[ \gamma_s(Cay(Z_{pq} : S)) = p. \]

**Proof.** Let $A = \{1, 1 + p, \ldots, 1 + \left(\left\lfloor \frac{q}{2} \right\rfloor - 1\right)p\}$ and $B = \{i + tq : i \in A$ and $1 \leq t \leq p - 1\}$. We define $f : V(Cay(Z_{pq} : S)) \rightarrow \{-1, 1\}$ such that

\[ f(x) = \begin{cases} -1 & x \in A \cup B, \\ 1 & \text{otherwise}. \end{cases} \]

Let $v \in V(Cay(Z_{pq} : S))$. If $f(v) = -1$, then

\[ f[v] = -1 + (p - 1)(q - 1) - 2 \left(\left\lfloor \frac{q}{2} \right\rfloor - 1\right)(p - 1) = 2p - 3. \]
Otherwise, \[ f[v] = 1 + (p - 1)(q - 1) - 2 \left\lfloor \frac{q}{2} \right\rfloor (p - 1) = 1. \]

Hence, \( f \) is a dominating function. Also
\[
\omega(f) = pq - 2 (|A| + |B|) = pq - 2 \left( \left\lfloor \frac{q}{2} \right\rfloor + (p - 1) \left\lfloor \frac{q}{2} \right\rfloor \right) = p.
\]

It is enough to show that \( f \) has the minimal wait. Let, to the contrary, \( g \) be a dominating function and \( \omega(g) < \omega(f) \). So \( |V_g^-| > |V_f^-| \). Without lose of generality, suppose that \( |V_g^-| = p\left\lceil \frac{q}{2} \right\rceil + 1 \). Let \( A_i^- = A_i \cap V_g^- \), \( A_i^+ = A_i \setminus A_i^- \) and \( B_j^- = B_j \cap V_g^- \). We will reach the contradiction by three steps.

Step 1. For every \( 1 \leq i \leq p \), \( A_i^- \neq \emptyset \).
On the contrary, let \( A_i^- = \emptyset \) for some \( 1 \leq s \leq p \). Let \( u \in A_s \). Then by Lemma 3.1, \( u \notin A_t \cap B_t \) for some \( 1 \leq t \leq q \). So
\[
g[u] = (p - 1)(q - 1) + 2(|V_g^-| - |B_i^-|) \geq 1.
\]
Thus \( |B_f^-| \geq \left\lceil \frac{q}{2} \right\rceil \). Hence, \( |V_g^-| \geq |A_s| \left\lceil \frac{q}{2} \right\rceil \). This implies \( q + (q - p)\left\lceil \frac{q}{2} \right\rceil < 1 \).

This is a contradiction. Hence, \( A_i^- \neq \emptyset \).
Similar argument applies for \( B_j \). Therefore, \( B_j^- \neq \emptyset \) for every \( 1 \leq j \leq q \).

Step 2. For every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lceil \frac{q}{2} \right\rceil \).
On the contrary, let \( |A_i^-| < \left\lceil \frac{q}{2} \right\rceil \) for some \( 1 \leq l \leq p \). Without lose of generality suppose that \( |A_l^-| = \left\lceil \frac{q}{2} \right\rceil - 1 \). Let \( v \in A_l \). By Lemma 3.1, \( v \notin A_t \cap B_t \) for some \( 1 \leq k \leq q \). If \( g(v) = -1 \), then \( g[v] = (p - 1)(q - 1) - 2(|V_g^-| - |A_l^-| - |B_l^-| + 2) \geq 1 \). Then \( |B_l^- \setminus \{v\}| \geq 4 \). If \( g(v) = 1 \), then \( |B_l^- \setminus \{v\}| \geq 2 \). Hence, \( |V_g^-| \geq |A_l^-| + |A_l^+| + 2|A_l^-| \).

As a consequence \( p > 8 \). This is impossible. Therefore, for every \( 1 \leq i \leq p \), \( |A_i^-| \geq \left\lceil \frac{q}{2} \right\rceil \) and since \( |V_g^-| = p\left\lceil \frac{q}{2} \right\rceil + 1 \), we may suppose that \( |A_i^-| = \left\lceil \frac{q}{2} \right\rceil \) and \( |A_i^+| = \left\lfloor \frac{q}{2} \right\rfloor \) for \( 2 \leq i \leq p \).

Step 3. For every \( 1 \leq j \leq q \), \( |B_j^-| \geq \left\lceil \frac{q}{2} \right\rceil \).
On the contrary, let \( |B_h^-| < \left\lceil \frac{q}{2} \right\rceil \) for some \( 1 \leq h \leq q \). Suppose that \( |B_h^-| = \left\lfloor \frac{q}{2} \right\rfloor \). By Lemma 3.1, \( B_h \cap A_i \neq \emptyset \) for any \( 1 \leq i \leq p \). Let \( z \in B_h \cap A_i \). Thus
\[
g[z] = -1 + (p - 1)(q - 1) - 2 (|V_g^-| - |A_i^-| - |B_h^-| + 2)
\leq -1 + (p - 1)(q - 1) - 2 \left( p \left\lfloor \frac{q}{2} \right\rfloor + 1 - \left\lfloor \frac{q}{2} \right\rfloor - 1 \right)
\leq -6
\]

Since \( p \in \{2, 3, 5\} \), \( g[z] \leq -1 \). This is a contradiction.

By Step 3, \( |V_g^-| \geq q\left\lceil \frac{q}{2} \right\rceil \). Hence, \( p\left\lceil \frac{q}{2} \right\rceil + 1 \geq q\left\lceil \frac{q}{2} \right\rceil \). So \( p + q \leq 2 \). This is impossible. Therefore \( \gamma_S(\text{Cay}(G : S)) = \omega(f) = p \). \( \Box \)

**Theorem 3.5.** Let \( \mathbb{Z}_{pq} = \langle S \rangle \) where \( p \geq 7 \) and \( S = B(1, pq) \). Then
\[
\gamma_S(\text{Cay}(\mathbb{Z}_{pq} : S)) = 5.
\]
Proof. We define \( f : V(Cay(Z_{pq} : S)) \to \{-1, 1\} \) such that \( f(i) = -1 \) if and only if \( i \in \{1, 2, \ldots, \frac{pq - 3}{2}\} \). It is easily seen that \( \frac{q}{2} \leq |A^+_r| \leq \frac{p}{2} \) for every \( 1 \leq i \leq p \). Also \( \frac{q}{2} \leq |B^+_j| \leq \frac{p}{2} \) for any \( 1 \leq j \leq q \). Let \( v \in A_i \cap B_s \) such that \( 1 \leq t \leq p \) and \( 1 \leq s \leq q \). In the worst situation, \( |A^+_r| = \frac{q}{2} \) and \( |B^+_s| = \frac{p}{2} \).

In this case \( 1 \leq f(v) \leq 5 \). Hence, \( f \) is a signed dominating function. Also \( \omega(f) = pq - 2|V^-_g| = 5 \). Thus \( \gamma_s(Cay(Z_{pq} : S)) \leq 5 \). What is left is to show that if \( g \) is a \( \gamma_s \)-function, then \( \omega(g) \geq 5 \). On the contrary, suppose that \( g \) be a \( \gamma_s \)-function and \( \omega(g) < \omega(f) \). Hence, \( |V^-_g| < |V^-_f| \). There is no loss of generality in assuming \( |V^-_g| = \frac{pq - 3}{2} \). Let \( A^-_r = A_i \cap V^-_g \) and \( B^-_j = B_j \cap V^-_g \).

In order to reach the contradiction we use two following steps:

Step 1. \( A^-_r \neq \emptyset \) for every \( 1 \leq i \leq p \).

On the contrary, suppose that for some \( 1 \leq m \leq p \), \( A^-_m = \emptyset \). Let \( w \in A_m \). So there is \( 1 \leq \ell \leq q \) such that \( w \in A_m \cap B_{t} \). Hence, \( g[w] = (p - 1)(q - 1) + 1 - 2(|V^-_g| - |B^-_j|) \geq 1 \). Thus \( |B^-_j| \geq \frac{pq - 4}{2} \). So \( |V^-_j| \geq q(\frac{pq - 4}{2}) \). Hence, \( pq - 3 \geq q(pq - 4) \). This makes a contradiction.

By similar argument we have \( B^-_j \neq \emptyset \) for every \( 1 \leq j \leq q \).

Step 2. For every \( 1 \leq i \leq p \), \( |A^-_r| \geq \frac{q}{2} \).

On the contrary, let \( |A^-_r| = \frac{q}{2} - 1 \). Let \( v \in A_i \). There is \( 1 \leq \ell' \leq q \) such that \( v \in A_i \cap B_{\ell'} \). If \( g(v) = -1 \), then \( g[v] = (p - 1)(q - 1) + 1 - 2(|V^-_g| - |A^+_r| - |B^+_s| + 2) \geq 1 \). Hence, \( |B^-_j| \geq \frac{q}{2} \). If \( g(v) = 1 \), then \( |B^-_j| \geq \frac{q}{2} \). Therefore, \( |V^-_g| \geq |A^-_r|(|\frac{q}{2} | + 1) + |A^+_r|(|\frac{q}{2} | + 2) \). This implies that \( q \leq 3 \). This is a contradiction.

Likewise Step 2, \( |B^-_j| \geq \frac{q}{2} \) for every \( 1 \leq j \leq q \). Since \( |V^-_g| = \frac{pq - 3}{2} \), there is \( 1 \leq k \leq p \) such that \( |A^-_r| = \frac{k}{2} \). On the other hand, suppose that for \( 1 \leq t \leq q \), \( |B^-_j| = \frac{q}{2} \). Let \( u \in A^+_r \cap B^-_s \). If \( s \in \{l_1, \ldots, l_t\} \), then \( g[u] = -1 + (p - 1)(q - 1) - 2\left(|V^-_g| - |A^-_r| - |B^-_s| + 2\right) \)

\[ = -1 + (p - 1)(q - 1) - 2\left(\frac{pq - 3}{2} - \frac{q}{2} - \frac{p}{2} + 2\right) \]

\[ = -3 \]

This is a contradiction by \( g \) is a signed dominating function. Hence, \( s \) is not in \( \{l_1, \ldots, l_t\} \). Since \( |A^-_r| = \frac{k}{2} \), \( q - t \geq \frac{k}{2} \) and so \( t \leq \frac{q}{2} \). As a consequence,

\[ |V^-_g| \geq t\left(\frac{p}{2}\right) + (q - t)\left(\frac{q}{2}\right) \geq \frac{q}{2}\left(\frac{p}{2}\right) + \frac{q}{2}\left(\frac{q}{2}\right) \]

Since \( |V^-_g| = \frac{pq - 3}{2} \), this makes a contradiction. Therefore,

\( \gamma_s(Cay(Z_{pq} : S)) = 5 \).

\( \square \)

Corollary 3.6. For any \( k \)-regular graph \( \Gamma \) on \( n \) vertices \( \gamma_s(\Gamma) \geq \frac{n}{k + 1} \). Hence, \( \gamma_s(\Gamma) \geq 1 \). It is easy to check that \( \gamma_s(\Gamma) = 1 \) if and only if \( \Gamma \) is a complete graph.
graph and \( n \) is odd. Furthermore, for any prime numbers \( p < q \), there is a \((p - 1)(q - 1)\)-regular graph \( \Gamma \) with \( pq \) vertices such that \( \gamma_s(\Gamma) \in \{2, 3, 5\} \).

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