Iranian Journal of Mathematical Sciences and Informatics
Vol. 14, No. 1 (2019), pp 35-42
DOI: 10.7508/ijmsi.2019.01.004

Domination and Signed Domination Number of Cayley Graphs

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Abstract. In this paper, we investigate domination number as well as signed domination numbers of $Cay(G:S)$ for all cyclic group G of order n, where $n \in \{p^m, pq\}$ and $S = \{k < n : gcd(k,n) = 1\}$. We also introduce some families of connected regular graphs Γ such that $\gamma_S(\Gamma) \in \{2,3,4,5\}$.

Keywords: Cayley graph, Cyclic group, Domination number, Signed domination number.

2000 Mathematics subject classification: 05C69, 05C25

1. Introduction

By a graph Γ we mean a simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A graph is said to be *connected* if each pair of vertices are joined by a walk. The number of edges of the shortest walk joining v_i and v_j is called the *distance* between v_i and v_j and denoted by $d(v_i, v_j)$. A graph Γ is said to be *regular* of degree k or, *k-regular* if every vertex has degree k. A subset P of vertices of Γ is a $k-$*packing* if $d(x,y) > k$ for all pairs of distinct vertices x and y of P [9].

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Received 20 April 2016; Accepted 14 January 2017

Let G be a non-trivial group, S be an inverse closed subset of G which does not contain the identity element of G, i.e. $S = S^{-1} = \{s^{-1} : s \in S\}$. The *Cayley graph* of G denoted by $Cay(G : S)$, is a graph with vertex set G and two vertices a and b are adjacent if and only if $ab^{-1} \in S$. The Cayley graph $Cay(G : S)$ is connected if and only if S generates G.
A set $D \subseteq V$ of vertices in a graph Γ is a dominating set if every vertex $v \in V$ is an element of D or adjacent to an element of D. The domination number $\gamma(\Gamma)$ of a graph Γ is the minimum cardinality of a dominating set of Γ.

For a vertex $v \in V(\Gamma)$, the closed neighborhood $N[v]$ of v is the set consisting v and all of its neighbors. For a function $f : V(\Gamma) \to \{-1, 1\}$ and a subset W of V we define $f(W) = \sum_{u \in W} f(u)$. A *signed dominating function* of Γ is a function $f : V(\Gamma) \to \{-1, 1\}$ such that $f(N[v]) > 0$ for all $v \in V(\Gamma)$. The *weight* of a function f is $\omega(f) = \sum_{v \in V} f(v)$. The *signed domination number* $\gamma_S(\Gamma)$ is the minimum weight of a signed dominating function of Γ. A signed dominating function of weight $\gamma_S(\Gamma)$ is called a $\gamma_S(\Gamma)-$function. We denote $f(N[v])$ by $f[v]$. Also for $A \subseteq V(\Gamma)$ and signed dominating function f, set $\{v \in A : f(v) = -1\}$ is denoted by A_f^-.
Finding some kinds of domination numbers of graphs is certainly one of the most important properties in any graph. (See for instance [2, 3, 5, 6, 11, 13])

These motivated us to consider on domination and signed domination number of Cayley graphs of cyclic group of orders p^n, pq, where p and q are prime numbers.

2. Cayley Graphs of Order p^n

In this section p is a prime number and $B(1, n) = \{k < n : gcd(k, n) = 1\}$.

Lemma 2.1. *Let G be a group and H be a proper subgroup of G such that $[G : H] = t$. If $S = G \setminus H$, then $Cay(G : S)$ is a complete t-partite graph.*

Proof. One can see $G = \langle S \rangle$ and $e \notin S = S^{-1}$. Let $a \in G$. If $x, y \in Ha$, then $x = h_1 a, y = h_2 a$. Since $xy^{-1} \in H$, $xy \notin E(Cay(G : S))$. So induced subgraph on every coset of H is empty. Let Ha and Hb two disjoint cosets of H and $x \in Ha, y \in Hb$. Hence, $xy^{-1} \in S$. So $xy \in E(Cay(G : S))$. Therefore, $Cay(G : S) = K_{|H|,|H|,\cdots,|H|}$. □

Lemma 2.2. *Let G be a group of order n and $G = \langle S \rangle$, where $S = S^{-1}$ and $0 \notin S$. Then $\gamma(Cay(G : S)) = 1$ if and only if $S = G \setminus \{0\}$.*

Proof. The proof is straightforward.

□

Theorem 2.3. [13] *Let $K_{a,b}$ be a complete bipartite graph with $b \leq a$. Then*

$$\gamma_S(K_{a,b}) = \begin{cases} a+1 & \text{if } b=1, \\ b & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is even}, \\ b+1 & \text{if } 2 \leq b \leq 3 \text{ and } a \text{ is odd }, \\ 4 & \text{if } b \geq 4 \text{ and } a,b \text{ are both even}, \\ 6 & \text{if } b \geq 4 \text{ and } a,b \text{ are both odd}, \\ 5 & \text{if } b \geq 4 \quad \text{and } a,b \text{ have different parity}. \end{cases}$$

Theorem 2.4. *Let $\mathbb{Z}_{2^n} = \langle S \rangle$ and $S = B(1, 2^n)$. Then*

i. $Cay(\mathbb{Z}_{2^n} : S) = K_{2^{n-1}, 2^{n-1}}$
ii. $\gamma(Cay(\mathbb{Z}_{2^n} : S)) = 2$.
iii.
$$\gamma_S(Cay(\mathbb{Z}_{2^n} : S)) = \begin{cases} 2 & \text{if } n = 1, 2, \\ 4 & \text{if } n \geq 3. \end{cases}$$

Proof. i. Let $H = \mathbb{Z}_{2^n} \setminus S$. Then $H = \{i : 2 \mid i\}$. It is not hard to see that H is a subgroup of $\mathbb{Z}_{2^n}$ and $[\mathbb{Z}_{2^n} : H] = 2$. Hence, by Lemma 2.1, $Cay(\mathbb{Z}_{2^n} : S) = K_{2^{n-1}, 2^{n-1}}$.

ii. By part i. $Cay(\mathbb{Z}_{2^n} : S)$ is a complete bipartite graph. So
$$\gamma(Cay(\mathbb{Z}_{2^n} : S)) = 2.$$

iii. The proof is straightforward by Theorem 2.3.

□

Corollary 2.5. *For any integer $n > 2$, there is a $2^{n-1}-$regular graph Γ with 2^n vertices such that $\gamma_S(\Gamma) = 4$.*

Theorem 2.6. *Let $\mathbb{Z}_{p^n} = \langle S \rangle$ (p odd prime) and $S = B(1, p^n)$. Then following statments hold:*

i. $Cay(\mathbb{Z}_{p^n} : S)$ *is a complete p-partite graph.*
ii. $\gamma(Cay(\mathbb{Z}_{p^n} : S)) = 2$.
iii. $\gamma_S(Cay(\mathbb{Z}_{p^n} : S)) = 3$.

Proof. i. Let $H = \mathbb{Z}_{p^n} \setminus S$. Then $H = \{i : p \mid i\}$. H is a subgroup of $\mathbb{Z}_{p^n}$ and $|H| = p^n - \Phi(p^n) = p^{n-1}$. So $[\mathbb{Z}_{p^n} : H] = p$. Hence, by Lemma 2.1, $Cay(\mathbb{Z}_{p^n} : S)$ is a complete p-partite graph of size p^{n-1}.

ii. Since $Cay(\mathbb{Z}_{p^n} : S)$ is a complete p-partite graph, $D = \{a, b\}$ is a minimal dominating set where a, b are not in the same partition.

iii. Let $\Gamma = Cay(\mathbb{Z}_{p^n} : S)$. Let $V(\Gamma) = \bigcup_{i=1}^{p} A_i$ where $A_i = \{v_{ij} : 1 \leq j \leq p^{n-1}\}$. Define $f : V(\Gamma) \to \{-1, 1\}$

$$f(v_{ij}) = \begin{cases} -1 & \text{if } 1 \leq i \leq \lfloor \frac{p}{2} \rfloor - 1 \text{ and } 1 \leq j \leq \lceil \frac{p^{n-1}}{2} \rceil, \\ -1 & \text{if } \lfloor \frac{p}{2} \rfloor \leq i \leq p \text{ and } 1 \leq j \leq \lfloor \frac{p^{n-1}}{2} \rfloor, \\ 1 & \text{otherwise}. \end{cases}$$

Let $v \in \bigcup_{i=1}^{\lfloor \frac{p}{2} \rfloor - 1} A_i$. So $|N(v) \cap V_f^-| = \frac{1}{2}(p^n - p^{n-1} - 4)$. So $f[v] = f(v) + 4 \geq 3$. If $v \in \bigcup_{i=\lfloor \frac{p}{2} \rfloor}^{p} A_i$, then $|N(v) \cap V_f^-| = \frac{1}{2}(p^n - p^{n-1} - 2)$. So $f[v] = f(v) + 2 \geq 1$. Hence, f is a signed dominating function. Since $|V_f^-| = \frac{1}{2}(p^n - 3)$, $\omega(f) = 3$. So $\gamma_S(\Gamma) \leq 3$. On the contrary, suppose $\gamma_S(\Gamma) < 3$. So there is a γ_S-function g such that $\omega(g) < 3$. So $|V_g^-| > \frac{1}{2}(p^n - 3)$. Let $|V_g^-| = \frac{1}{2}(p^n - 1)$. If $A_i \cap V_g^- = \emptyset$ for some $1 \leq i \leq p$, then $g[v] = 1 - p^{n-1}$ for every $v \in A_i$. Hence, $A_i \cap V_g^- \neq \emptyset$ for every $1 \leq i \leq p$. If $|A_i \cap V_g^-| \geq \lceil \frac{p^{n-1}}{2} \rceil$ for every $1 \leq i \leq p$, then $|V_g^-| \geq \frac{1}{2}(p^n + p)$. This is impossible. So there is $j \in \{1, 2, \ldots, p\}$ such that $|A_j \cap V_g^-| \leq \lfloor \frac{p^{n-1}}{2} \rfloor$. Let $u \in A_j \cap V_g^-$. So $g[u] = deg(u) + 1 - 2|N(u) \cap V_g^-| < 0$. This is contradiction. Therefore $\gamma_S(\Gamma) = 3$.

□

Corollary 2.7. *For every integer n, there is a $(p^n - p^{n-1})-$regular graph Γ with p^n vertices such that $\gamma_S(\Gamma) = 3$.*

3. Cayley Graphs of Order pq

In this section p and q are distinct prime numbers where $p < q$. Let $B(1, pq)$ be a generator of $\mathbb{Z}_{pq}$. For $1 \leq i \leq p$ and $1 \leq j \leq q$, set

$$A_i = \{i + kp : 0 \leq k \leq q - 1\}$$

and

$$B_j = \{j + k'q : 0 \leq k' \leq p - 1\}.$$

With these notations in mind we will prove the following results.

Lemma 3.1. *Let $\mathbb{Z}_{pq} = \langle S \rangle$ and $S = B(1, pq)$. Then following statments hold.*

i. $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i$ *and $Cay(\mathbb{Z}_{pq} : S)$ is a p-partite graph.*

ii. $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{j=1}^{q} B_j$ *and $Cay(\mathbb{Z}_{pq} : S)$ is a q-partite graph.*

iii. *Let $1 \leq i \leq p$. For any $x \in A_i$ there is some $1 \leq j \leq q$ such that $x \in B_j$.*

iv. $|A_i \cap B_j| = 1$ *for every i, j.*

Proof. i. Let $s \in V(Cay(\mathbb{Z}_{pq} : S))$. If $p \mid s$, then $s \in A_p$. Otherwise, $s \in A_i$ where $s = kp + i$ for some $1 \leq k \leq (p - 1)$. Thus $V(Cay(\mathbb{Z}_{pq} : S)) = \bigcup_{i=1}^{p} A_i$. Since $1 \leq i \neq j \leq p$, $A_i \cap A_j = \emptyset$. We show that the induced subgraph on A_i is empty. Let $l+t \in E(Cay(\mathbb{Z}_{pq} : S))$. If $l, t \in A_s$ for some $1 \leq s \leq p$, then $l = s + kp, t = s + k'p$. So $p \mid (l-t)$. This is impossible.

ii. The proof is likewise part i.

iii. Let $1 \leq i \leq p$ and let $x \in A_i$. If $x \leq q$, then $x \in B_x$. If not, $x = i + kp > q$ such that $1 \leq k \leq q-1$. Hence, $x \equiv t \pmod{q}$ where $1 \leq t \leq q$, and so $x \in B_t$.

iv. By Case iii and since $|A_i| = q$ and also for every $j \neq j'$, $B_j \cap B_{j'} = \emptyset$, the result reaches.

□

Theorem 3.2. [6] *For any graph* Γ, $\left\lceil \frac{n}{1+\Delta(\Gamma)} \right\rceil \leq \gamma(\Gamma) \leq n - \Delta(\Gamma)$ *where* $\Delta(\Gamma)$ *is the maximum degree of* Γ.

Theorem 3.3. *Let* $\mathbb{Z}_{pq} = \langle S \rangle$ *and* $S = B(1, pq)$. *Then the following is hold.*

$$\gamma(Cay(\mathbb{Z}_{pq} : S)) = \begin{cases} 2 & p = 2; \\ 3 & p > 2. \end{cases}$$

Proof. Let $p = 2$. By Lemma 3.1, $D = \{i, i+q\}$ is a dominating set. Since $Cay(\mathbb{Z}_{pq} : S)$ is a $(q-1)$-regular graph, by Theorem 3.2, $\gamma(Cay(\mathbb{Z}_{pq} : S)) \geq 2$. Thus $\gamma(Cay(\mathbb{Z}_{pq} : S)) = 2$.
Let $p > 2$. We define $D = \{1, 2, s\}$ where $s \in A_1 \setminus N(2)$. Since $1, 2$ are adjacent , $N(1) \cup N(2) = V(Cay(\mathbb{Z}_{pq} : S)) \setminus D$. Thus D is a dominating set. As a consequence, $\gamma(Cay(\mathbb{Z}_{pq} : S)) \leq 2$. It is enough to show that $\gamma(Cay(\mathbb{Z}_{pq} : S)) \neq 2$. Let $D' = \{x, y\}$. We show that D' is not a dominating set. If $x, y \in A_i$ for some $1 \leq i \leq p$, then for every $z \in A_i \setminus D'$, $z \notin N(D')$. If not, $x \in A_i$ and $y \in A_j$ for some $1 \leq i \neq j \leq p$. If x, y are adjacent, then there is $x' \in A_i \setminus \{x\}$ such that $x' \notin N(y)$. Thus D' is not dominating set. If x and y are not adjacent, then there is $z \in A_l$, $l \neq i, j$, such that the induced subgraph on $\{x, y, z\}$ is empty. Hence, D' is not a dominating set and the proof is completed.

□

Theorem 3.4. *Let* $\mathbb{Z}_{pq} = \langle S \rangle$ *where* $p \in \{2, 3, 5\}$ *and* $S = B(1, pq)$. *Then*

$$\gamma_S(Cay(\mathbb{Z}_{pq} : S)) = p.$$

Proof. Let $A = \{1, 1+p, \ldots, 1 + (\lfloor \frac{q}{2} \rfloor - 1)p\}$ and $B = \{i + tq : i \in A \text{ and } 1 \leq t \leq p-1\}$. We define $f : V(Cay(\mathbb{Z}_{pq} : S)) \to \{-1, 1\}$ such that

$$f(x) = \begin{cases} -1 & x \in A \cup B, \\ 1 & \text{otherwise.} \end{cases}$$

Let $v \in V(Cay(\mathbb{Z}_{pq} : S))$. If $f(v) = -1$, then

$$f[v] = -1 + (p-1)(q-1) - 2\left(\left(\lfloor \frac{q}{2} \rfloor - 1\right)(p-1)\right) = 2p - 3.$$

Otherwise,

$$f[v] = 1 + (p-1)(q-1) - 2\left\lfloor \frac{q}{2} \right\rfloor (p-1) = 1.$$

Hence, f is a dominating function. Also

$$\omega(f) = pq - 2\left(|A| + |B|\right) = pq - 2\left(\left\lfloor \frac{q}{2} \right\rfloor + (p-1)\left\lfloor \frac{q}{2} \right\rfloor\right) = p.$$

It is enough to show that f has the minimal wait. Let, to the contrary, g be a dominating function and $\omega(g) < \omega(f)$. So $|V_g^-| > |V_f^-|$. Without lose of generality, suppose that $|V_g^-| = p\lfloor \frac{q}{2} \rfloor + 1$. Let $A_i^- = A_i \cap V_g^-$, $A_i^+ = A_i \setminus A_i^-$ and $B_j^- = B_j \cap V_g^-$. We will reach the contradiction by three steps.

Step 1. For every $1 \leq i \leq p$, $A_i^- \neq \emptyset$.

On the contrary, let $A_s^- = \emptyset$ for some $1 \leq s \leq p$. Let $u \in A_s$. Then by Lemma 3.1, $u \in A_s \cap B_t$ for some $1 \leq t \leq q$. So

$$g[u] = (p-1)(q-1) + 1 - 2(|V_g^-| - |B_t^-|) \geq 1.$$

Thus $|B_t^-| \geq \lceil \frac{q}{2} \rceil$. Hence, $|V_g^-| \geq |A_s| \lceil \frac{q}{2} \rceil$. This imolies $q + (q-p)\lfloor \frac{q}{2} \rfloor < 1$. This is a contradiction. Hence, $A_s^- \neq \emptyset$.

Similar argument applies for B_j. Therefore, $B_j^- \neq \emptyset$ for every $1 \leq j \leq q$.

Step 2. For every $1 \leq i \leq p$, $|A_i^-| \geq \lfloor \frac{q}{2} \rfloor$.

On the contrary, Let $|A_l^-| < \lfloor \frac{q}{2} \rfloor$ for some $1 \leq l \leq p$. Without lose of generality suppose that $|A_l^-| = \lfloor \frac{q}{2} \rfloor - 1$. Let $v \in A_l$. By Lemma 3.1, $v \in A_l \cap B_k$ for some $1 \leq k \leq q$. If $g(v) = -1$, then $g[v] = (p-1)(q-1) - 1 - 2(|V_g^-| - |A_l^-| - |B_k^-| + 2) \geq 1$. Then $|B_k^- \setminus \{v\}| \geq 4$. If $g(v) = 1$, then $|B_k^- \setminus \{v\}| \geq 2$. Hence, $|V_g^-| \geq 4|A_l^-| + |A_l^-| + 2|A_l^+|$. As a consequence $p > 8$. This is impossible.

Therefore, for every $1 \leq i \leq p$, $|A_i^-| \geq \lfloor \frac{q}{2} \rfloor$ and since $|V_g^-| = p\lfloor \frac{q}{2} \rfloor + 1$, we may suppose that $|A_1^-| = \lceil \frac{q}{2} \rceil$ and $|A_i^-| = \lfloor \frac{q}{2} \rfloor$for $2 \leq i \leq p$.

Step 3. For every $1 \leq j \leq q$, $|B_j^-| \geq \lceil \frac{p}{2} \rceil$.

On the contrary, let $|B_h^-| < \lceil \frac{p}{2} \rceil$ for some $1 \leq h \leq q$. Suppose that $|B_h^-| = \lfloor \frac{p}{2} \rfloor$. By Lemma 3.1, $B_h \cap A_i \neq \emptyset$ for any $1 \leq i \leq p$. Let $z \in B_h^- \cap A_i$. Thus

$$\begin{aligned} g[z] =& -1 + (p-1)(q-1) - 2\left(|V_g^-| - |A_i^-| - |B_h^-| + 2\right) \\ \leq& -1 + (p-1)(q-1) - 2\left(p\left\lfloor \frac{q}{2} \right\rfloor + 1 - \left\lceil \frac{q}{2} \right\rceil - \lfloor \frac{p}{2} \rfloor + 2\right) \\ \leq& p - 6 \end{aligned}$$

Since $p \in \{2, 3, 5\}$, $g[z] \leq -1$. This is a contradiction.

By Step 3, $|V_g^-| \geq q\lceil \frac{p}{2} \rceil$. Hence, $p\lfloor \frac{q}{2} \rfloor + 1 \geq q\lceil \frac{p}{2} \rceil$. So $p + q \leq 2$. This is impossible. Therefore $\gamma_S(Cay(G : S)) = \omega(f) = p$. □

Theorem 3.5. *Let $\mathbb{Z}_{pq} = \langle S \rangle$ where $p \geq 7$ and $S = B(1, pq)$. Then*

$$\gamma_S(Cay(\mathbb{Z}_{pq} : S)) = 5.$$

Proof. We define $f : V(Cay(\mathbb{Z}_{pq} : S)) \to \{-1, 1\}$ such that $f(i) = -1$ if and only if $i \in \{1, 2, \ldots, \frac{pq-5}{2}\}$. It is easily seen that $\lfloor \frac{q}{2} \rfloor \leq |A_i^-| \leq \lceil \frac{q}{2} \rceil$ for every $1 \leq i \leq p$. Also $\lfloor \frac{p}{2} \rfloor \leq |B_j^-| \leq \lceil \frac{p}{2} \rceil$ for any $1 \leq j \leq q$. Let $v \in A_t \cap B_s$ such that $1 \leq t \leq p$ and $1 \leq s \leq q$. In the worst situation, $|A_t^-| = \lfloor \frac{q}{2} \rfloor$ and $|B_s^-| = \lfloor \frac{p}{2} \rfloor$. In this case $1 \leq f[v] \leq 5$. Hence, f is a signed dominating function. Also $\omega(f) = pq - 2|V_f^-| = 5$. Thus $\gamma_S(Cay(\mathbb{Z}_{pq} : S)) \leq 5$. What is left is to show that if g is a γ_S-function, then $\omega(g) \geq 5$. On the contrary, suppose that g be a γ_S-function and $\omega(g) < \omega(f)$. Hence, $|V_g^-| < |V_f^-|$. There is no loss of generality in assuming $|V_g^-| = \frac{pq-3}{2}$. Let $A_i^- = A_i \cap V_g^-$ and $B_j^- = B_j \cap V_g^-$. In order to reach the contradiction we use two following steps:

Step 1. $A_i^- \neq \emptyset$ for every $1 \leq i \leq p$.

On the contrary, suppose that for some $1 \leq m \leq p$, $A_m^- = \emptyset$. Let $w \in A_m$. So there is $1 \leq \ell \leq q$ such that $w \in A_m \cap B_\ell$. Hence, $g[w] = (p-1)(q-1)+1-2(|V_g^-|-|B_\ell^-|) \geq 1$. Thus $|B_\ell^-| \geq \frac{p+q-4}{2}$. So $|V_g^-| \geq q(\frac{p+q-4}{2})$. Hence, $pq - 3 \geq q(pq - 4)$. This makes a contradiction.

By similar argument we have $B_j^- \neq \emptyset$ for every $1 \leq j \leq q$.

Step 2. For every $1 \leq i \leq p$, $|A_i^-| \geq \lfloor \frac{q}{2} \rfloor$.

On the contrary, let $|A_l^-| = \lfloor \frac{q}{2} \rfloor - 1$. Let $v \in A_l$. There is $1 \leq l' \leq q$ such that $v \in A_l \cap B_{l'}$. If $g(v) = -1$, then $g[v] = (p-1)(q-1) + 1 - 2(|V_g^-| - |A_l^-| - |B_{l'}^-| + 2) \geq 1$. Hence, $|B_{l'}^- \setminus \{v\}| \geq \lceil \frac{p}{2} \rceil$. If $g(v) = 1$, then $|B_{l'}^-| \geq \lfloor \frac{p}{2} \rfloor$. Therefore, $|V_g^-| \geq |A_l^-|(\lceil \frac{p}{2} \rceil + 1) + |A_l^+|\lfloor \frac{p}{2} \rfloor$. This implies that $q \leq 3$. This is a contradiction.

Likewise Step 2, $|B_j^-| \geq \lfloor \frac{p}{2} \rfloor$ for every $1 \leq j \leq q$. Since $|V_g^-| = \frac{pq-3}{2}$, there is $1 \leq k \leq p$ such that $|A_k^-| = \lfloor \frac{q}{2} \rfloor$. On the other hand, suppose that for $1 \leq t \leq q$, $|B_{l_r}^-| = \lfloor \frac{p}{2} \rfloor$. Let $u \in A_k^- \cap B_s^-$. If $s \in \{l_1, \cdots, l_t\}$, then

$$\begin{aligned} g[u] &= -1 + (p-1)(q-1) - 2\left(|V_g^-| - |A_k^-| - |B_s^-| + 2\right) \\ &= -1 + (p-1)(q-1) - 2\left(\frac{pq-3}{2} - \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{p}{2} \right\rfloor + 2\right) \\ &= -3. \end{aligned}$$

This is a contradiction by g is a signed dominating function. Hence, s is not in $\{l_1, \cdots, l_t\}$. Since $|A_k^-| = \lfloor \frac{q}{2} \rfloor$, $q - t \geq \lfloor \frac{q}{2} \rfloor$ and so $t \leq \lceil \frac{q}{2} \rceil$. As a consequence,

$$|V_g^-| \geq t\lfloor \frac{p}{2} \rfloor + (q-t)\lceil \frac{p}{2} \rceil \geq \lceil \frac{q}{2} \rceil \lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor \lceil \frac{p}{2} \rceil.$$

Since $|V_g^-| = \frac{pq-3}{2}$, this makes a contradiction. Therefore,

$$\gamma_S(Cay(\mathbb{Z}_{pq} : S)) = 5.$$

□

Corollary 3.6. *For any $k-$regular graph Γ on n vertices $\gamma_S(\Gamma) \geq \frac{n}{k+1}$. Hence, $\gamma_S(\Gamma) \geq 1$. It is easy to check that $\gamma_S(\Gamma) = 1$ if and only if Γ is a complete* *graph and n is odd. Furthermore, for any prime numbers $p < q$, there is a $(p-1)(q-1)-$regular graph Γ with pq vertices such that $\gamma_{_S}(\Gamma) \in \{2, 3, 5\}$.*

Acknowledgments

The author is thankful of referees for their valuable comments.

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